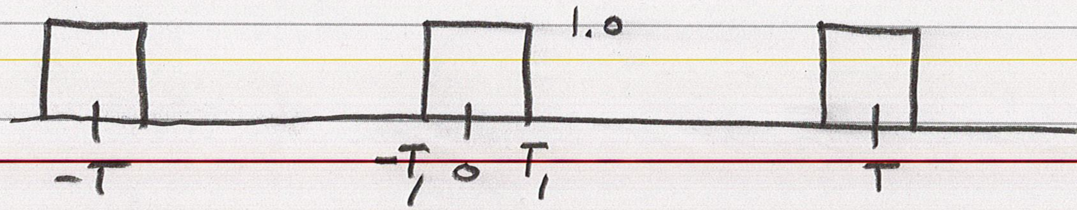
Continuous Time Fourier Transform (Week 9 – Monday)

Approach: Start with F.S. and look at case: period $\to \infty$

Remember the example from before

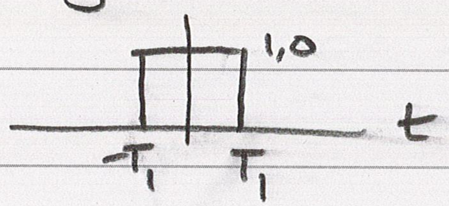


Found the F.S. coeffs: $a_k = \dfrac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$ $\quad k \in \mathbb{Z}$

$\omega_0 = 2\pi/T$

- An aperiodic signal is the limit of a periodic signal as $T \to \infty$

$\Rightarrow$



this single pulse might be thought of as limit $T \to \infty$ of the pulse train

- F.S. coeffs are samples of

$$\left.\frac{2\sin \omega T_1}{\omega}\right\} \text{ this is indep. of } T$$

- $a_k T = \left.\dfrac{2\sin \omega T_1}{\omega}\right|_{\omega = k\omega_0}$

Two ways of looking at

$Ta_k = \dfrac{2\sin(k\omega_0 T_1)}{k\omega_0}$

as T grows

$(\omega_0 = 2\pi/T)$

Plotted as function of index k

(this is actually a plot of a_k not Ta_k)

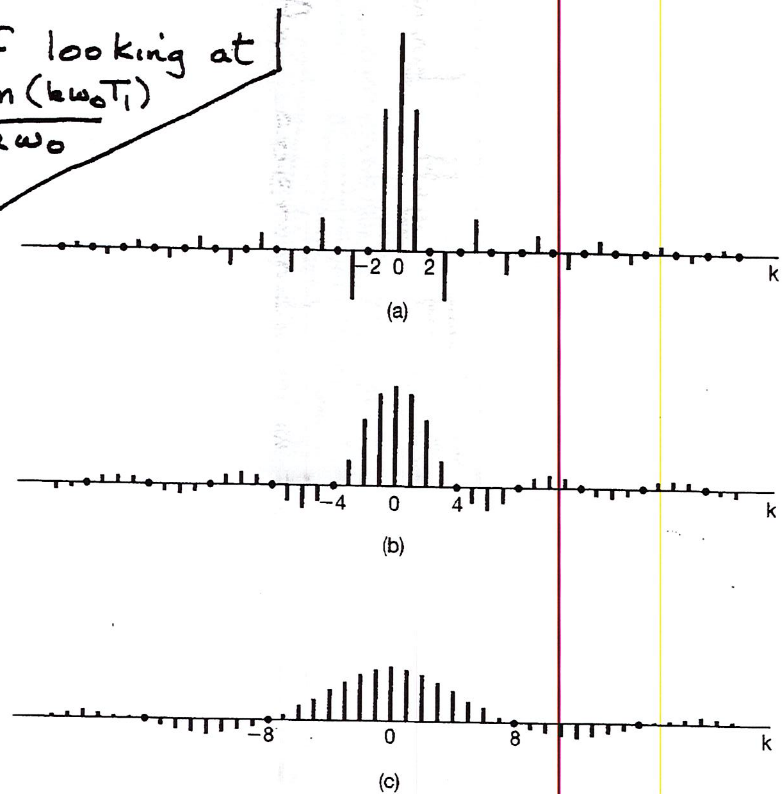


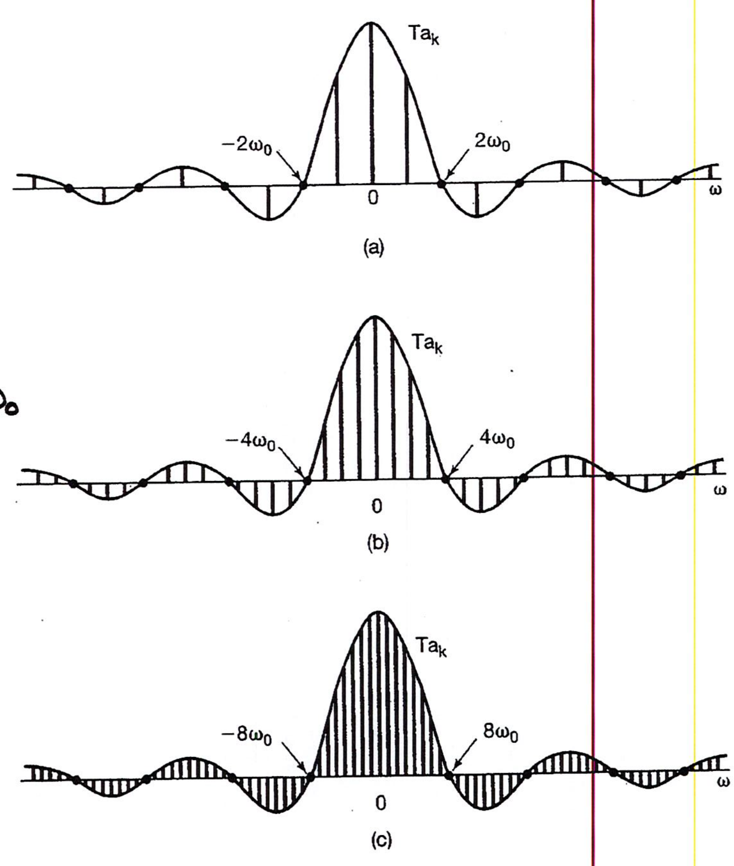
Figure 3.7 Plots of the scaled Fourier series coefficients Ta_k for the periodic square wave with T_1 fixed and for several values of T: (a) $T = 4T_1$; (b) $T = 8T_1$; (c) $T = 16T_1$. The coefficients are regularly spaced samples of the envelope $(2\sin\omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as T increases.

Plotted as samples of envelope

$Ta_k = \left.\dfrac{2\sin\omega T_1}{\omega}\right|_{\omega = k\omega_0}$

sample spacing is $\omega_0 = \dfrac{2\pi}{T}$

ie. samples get closer together as T increases.



Look at the matlab files from 10/5 Lect.

Figure 4.2 The Fourier series coefficients and their envelope for the periodic square wave in Figure 4.1 for several values of T (with T_1 fixed): (a) $T = 4T_1$; (b) $T = 8T_1$; (c) $T = 16T_1$.

Can use the prev. approx and idea of Riemann Sum as def. of an integral to derive form of F.T. $\dashrightarrow$ see Text.

For our purposes: $x(t)$ $t \in \mathbb{R}$ aperiodic

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt$$

this is the CTFT
also called the spectrum of $x(t)$.

Think of this as a function of ω.

The inverse F.T.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{+j\omega t} \, d\omega$$

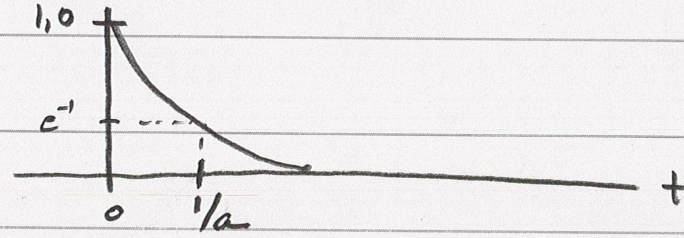
$$x(t) \longleftrightarrow X(j\omega)$$

Freq variable is ω

The relationship between radian freq and Hertzian freq is

$$2\pi f = \omega$$

Example $x(t) = e^{-at}\,u(t) \quad a > 0$



$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}\,u(t)\,e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-at}\,e^{-j\omega t}\,dt$$

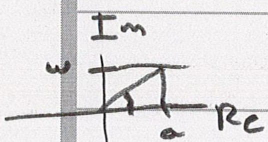
$$= \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt = \frac{1}{-(a+j\omega)}\,e^{-(a+j\omega)t}\Bigg|_{t=0}^{\infty}$$

$$= \frac{1}{a+j\omega} \Longrightarrow$$

If want a plot, then two ways to do so

① plot real, imag vs. ω separately

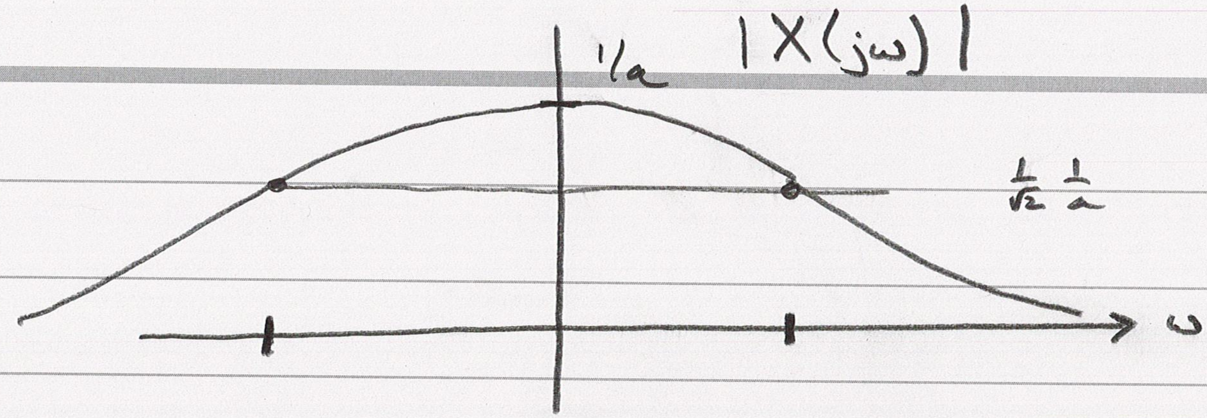
② plot $|X(j\omega)|$, $\angle X(j\omega)$ vs. ω sep.

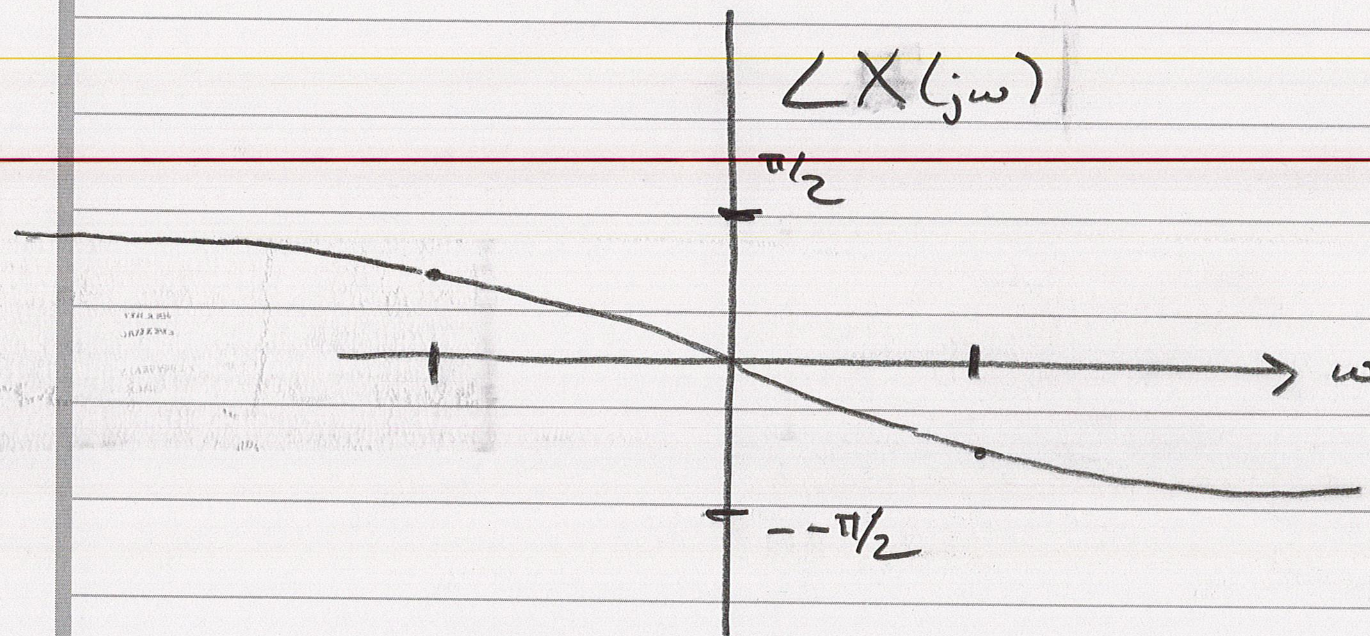


$$|X(j\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}$$

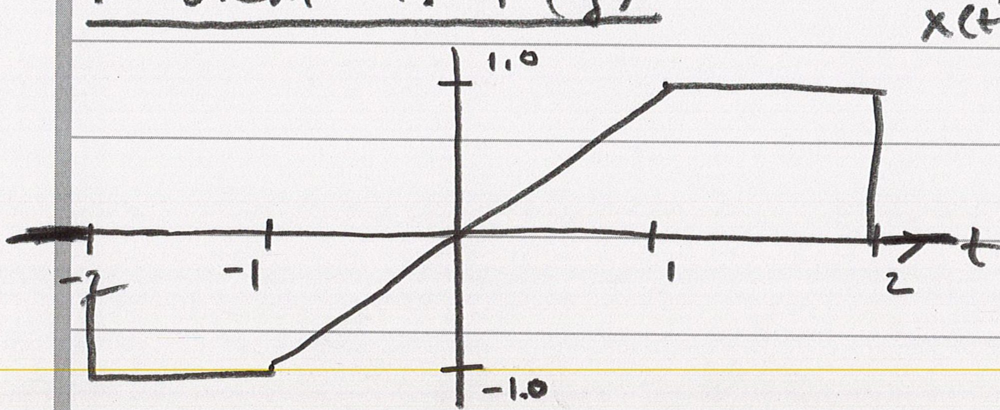
$$\angle X(j\omega) = -\tan^{-1}\left(\omega/a\right)$$

$\omega = a \qquad \frac{1}{\sqrt{2a^2}} = \frac{1}{\sqrt{2}}\,\frac{1}{a}$


$|X(j\omega)|$
$1/a$
$\frac{1}{\sqrt{2}}\frac{1}{a}$
ω


$\angle X(j\omega)$
$\pi/2$
$-\pi/2$
ω

Problem 4.21 (g)



$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \underbrace{-\int_{-2}^{-1} e^{-j\omega t}\, dt}_{\frac{1}{j\omega}\left[e^{j\omega} - e^{j2\omega}\right]} + \int_{-1}^{1} t\, e^{-j\omega t}\, dt + \underbrace{\int_{1}^{2} e^{-j\omega t}\, dt}_{\frac{1}{-j\omega}\left[e^{-j2\omega} - e^{-j\omega}\right]}$$

use integration by parts.

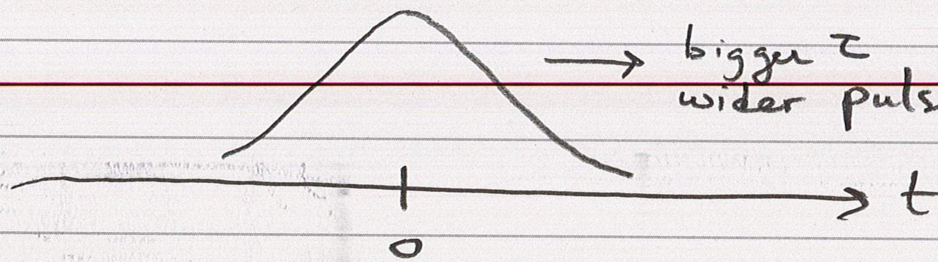
$$= \frac{1}{-j\omega}\left[e^{-j\omega} + e^{j\omega}\right] + \left(\frac{1}{\omega^2}\right)\left[e^{-j\omega} - e^{+j\omega}\right]$$

If combine + simplify:

$$X(j\omega) = \frac{j2}{\omega}\left[\cos(2\omega) - \frac{\sin\omega}{\omega}\right]$$

Think about the Gaussian Pulse

$$x(t) = \frac{1}{\sqrt{2\pi}\,\tau} e^{-t^2/2\tau^2} \qquad t \in \mathbb{R}$$



$\longrightarrow$ bigger τ
wider pulse.

Compute $X(j\omega)$??

$$X(j\omega) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\tau} e^{-t^2/2\tau^2} e^{-j\omega t}\, dt$$

use "completion of the square"