(Week 9 – Friday) 1/

CTFT Notation:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{+j\omega t}\,d\omega = x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$\Longrightarrow \quad X(j\omega) = \mathcal{F}\{x(t)\} \qquad x(t) = \mathcal{F}^{-1}\{X(j\omega)\}$$

Example

$$X(j\omega) = \cos(4\omega + \pi/3) = \cos\left(4\left(\omega + \frac{\pi}{12}\right)\right)$$

$$= \frac{e^{j(4\omega+\pi/3)} + e^{-j(4\omega+\pi/3)}}{2}$$

$$= \frac{1}{2}e^{j\pi/3}e^{j4\omega} + \frac{1}{2}e^{-j\pi/3}e^{-j4\omega}$$

Plug in to the inverce transf. integral & see what happens

$$\int_{-\infty}^{\infty} e^{j4\omega}e^{j\omega t}\,d\omega = \int_{-\infty}^{\infty} e^{j(4+t)\omega}\,d\omega = \frac{1}{j(4+t)}e^{j(4+t)\omega}\Big|_{\omega=-\infty}^{\infty}$$

$\Longrightarrow$ What to Do ?

We think there will be some $\delta(t)$ like thing in time domain since the $\mathcal{F}^{-1}$ blew up !!!

$$x(t) = \delta(t)$$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} \, dt = \underline{1} \quad \therefore \quad \delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1$$

New function

$$x(t) = \delta(t - t_0)$$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} \, dt = e^{-j\omega t_0}$$

$$\therefore \quad e^{-j\omega 4} \longleftrightarrow \delta(t-4) \implies \frac{1}{2} e^{-j\pi/3} e^{-j4\omega} \longleftrightarrow \frac{1}{2} e^{-j\pi/3} \delta(t-4)$$

$$e^{+j4\omega} \longleftrightarrow \delta(t+4) \implies \frac{1}{2} e^{+j\pi/2} e^{+j4\omega} \longleftrightarrow \frac{1}{2} e^{+j\pi/3} \delta(t+4)$$

Together: $\cos\left(4\omega + \pi/3\right) \longleftrightarrow \frac{1}{2} e^{-j\pi/3} \delta(t-4) + \frac{1}{2} e^{+j\pi/3} \delta(t+4)$

In response to Student question:

There is a Modulation Property

$$x(t) \longleftrightarrow X(j\omega)$$

$$\underbrace{e^{j\omega_0 t}\, x(t)}\; \longleftrightarrow$$

$$\int_{-\infty}^{\infty} e^{j\omega_0 t}\, x(t)\, e^{-j\omega t}\, dt = X(j(\omega-\omega_0))$$

$$e^{-j(\omega-\omega_0)t}$$

what would the analogous property be for freq domain ?

$$e^{jt_0\omega} X(j\omega) \longleftrightarrow ??$$

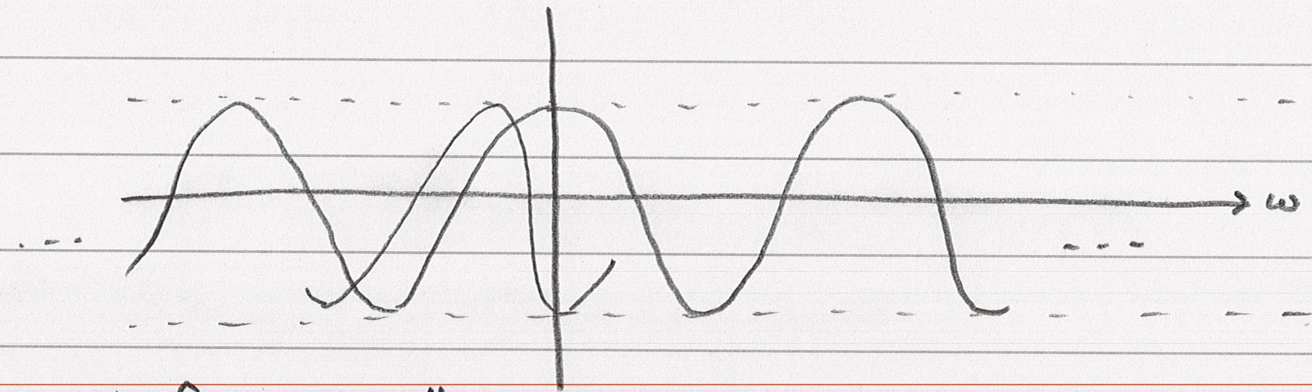
It's going to be a shift in time

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{e^{jt_0\omega} X(j\omega)}\, e^{+j\omega t}\, d\omega = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{+j\omega(t+t_0)}\, d\omega}_{= x(t+t_0)}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Student Question (cont'd).

$\cos(4\omega)$



Think as a "freq shift"?

Promised to return to this.

Fourier Transforms of Periodic Signals

$$x(t) \longleftrightarrow X(j\omega) = 2\pi\,\delta(\omega-\omega_o)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\,\delta(\omega-\omega_o)\,e^{+j\omega t}\,d\omega = \frac{1}{2\pi}\,2\pi\,e^{j\omega_o t}$$

$$(*)\quad e^{j\omega_o t} \longrightarrow 2\pi\,\delta(\omega-\omega_o) \implies$$ If want F.T.s for $\cos\omega_o t$ and $\sin\omega_o t$, its easy.

Now think about a periodic wave $x(t) = x(t+T)$ $\omega_o = 2\pi/T$

$$\implies x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

$\implies$ Use linearity of F.T. and the transf pair $(*)$ to get

$$X(j\omega) = \mathfrak{F}\left\{\sum_k a_k e^{jk\omega_o t}\right\} = \sum_k a_k \underbrace{\mathfrak{F}\left\{e^{jk\omega_o t}\right\}}_{=2\pi\,\delta(\omega-k\omega_o)}$$

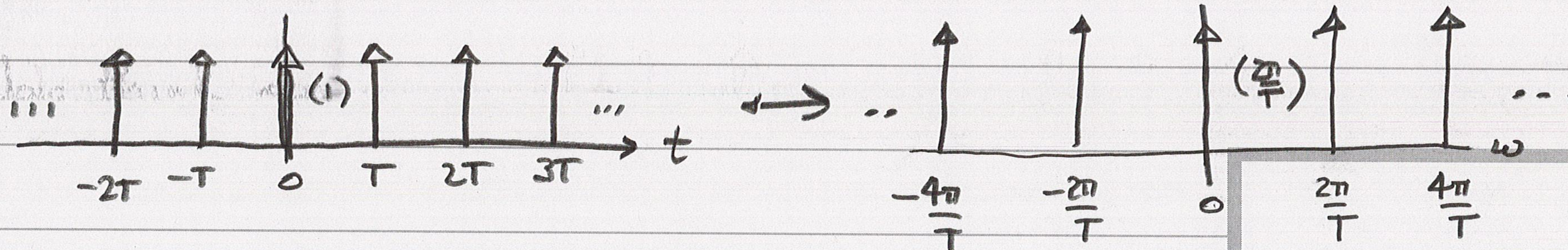
$$\implies \therefore \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t} \longleftrightarrow 2\pi\sum_{k=-\infty}^{\infty} a_k\,\delta(\omega-k\omega_o)$$

Special Case: Important for "Sampling Theorem"

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} \qquad \omega_0 = 2\pi/T$$

$$\updownarrow$$

$$X(j\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} 2\pi\,\delta(\omega - n\omega_0) = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$



(Poisson's Sum Formula)