Week 8 - Wednesday

$$x[n] = x[n+N] \quad \forall n \in \mathbb{Z}$$

$$\left.\begin{aligned} a_k &= \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-jk(2\pi/N)n} \\ &\updownarrow \\ x[n] &= \sum_{k=0}^{N-1} a_k\, e^{+jk(2\pi/N)n} \end{aligned}\right\}$$

The discrete-time Fourier Series.

<u>Examples</u> $x[n] = \sin(\omega_o n)$ where $\omega_o = 2\pi \cdot \left(\text{a rational } \#\right)$

(a) $\omega_o = 2\pi/N$

$$\sin \omega_o n = \frac{e^{+j\omega_o n} - e^{-j\omega_o n}}{j2}$$

$$= \frac{1}{j2} e^{j(2\pi/N)n} - \frac{1}{j2} e^{-j(2\pi/N)n}$$

$k=1$

$k=-1$, $k=N-1$

$$\underset{\text{in gen.}}{=} \sum_{k=0}^{N-1} a_k\, e^{+jk\left(\frac{2\pi}{N}\right)n}$$

$\Longrightarrow$ $k=1$ term $\quad a_1 = \frac{1}{j2}$

$k=-1$, $k=N-1$ $\quad a_{-1} = a_{N-1} = -\frac{1}{j2}$

and all other a_k's zero.

(b) $\omega_0 = \frac{2\pi M}{N} \longrightarrow$ where $\gcd(M,N) = 1$

$$\Rightarrow \sin \omega_0 n = \frac{1}{j2} e^{jM\left(\frac{2\pi}{N}\right)n} - \frac{1}{j2} e^{-jM\left(\frac{2\pi}{N}\right)n}$$

$$\Rightarrow a_M = \frac{1}{j2} \qquad a_{-M} = -\frac{1}{j2} = a_{N-M}$$ all others zero.

(c) $\omega_0 = \frac{2\pi M}{N}$ but say they have factors in common.

Say $N =$ even and say $M = 2$

$$\sin \omega_0 n = \frac{1}{j2} e^{j2\left(\frac{2\pi}{N}\right)n} - \frac{1}{j2} e^{-j2\left(\frac{2\pi}{N}\right)n} \Longrightarrow$$ the N periodic DFS has $a_2 = \frac{1}{j2}$, $a_{-2} = \frac{-1}{j2}$

but $N = 2K$

$$\sin \omega_0 n = \frac{1}{j2} e^{j\left(\frac{2\pi}{K}\right)n} - \frac{1}{j2} e^{-j\left(\frac{2\pi}{K}\right)n}$$

$\Longrightarrow$ Its period of period K and its DFS for K has

$$b_1 = \frac{1}{j2} \qquad b_{-1} = -\frac{1}{j2}$$

Example

$$x[n] = e^{\overbrace{j0.1\pi n}^{N=20}} + e^{\overbrace{j0.2\pi n}^{N=10}} \qquad n \in \mathbb{Z}$$

ⓐ Find the DTFS.

$\Rightarrow$ $N = 20$ since this is the LCM {periods of two sigs summed to give $x[n]$}.

$$x[n] = 1 \cdot e^{j\frac{\pi}{10}n} + 1 \cdot e^{j\frac{2\pi}{10}n}$$

$$= \underbrace{1}_{a_1} \cdot e^{j1\cdot\frac{2\pi}{20}n} + \underbrace{1}_{a_2} e^{j2(2\pi/20)n}$$

and all other coeffs are zero.

Question 5: [8%, Work-out question, Learning Objective 4] Consider a DT signal

$$x[n] = \begin{cases} 2\pi & \text{if } 0 \le n \le 2 \\ 0 & \text{if } 3 \le n \le 49\,. \\ x[n] \text{ is periodic with period 50} & \end{cases} \tag{8}$$

Find the DTFS representation of $x[n]$.

$$a_k = \frac{1}{50} \sum_{n=0}^{49} x[n] e^{-jk(2\pi/50)n}$$

$$= \frac{2\pi}{50} \left\{ 1 + e^{-jk(\pi/25)} + e^{-jk(\pi/25)2} \right\}$$

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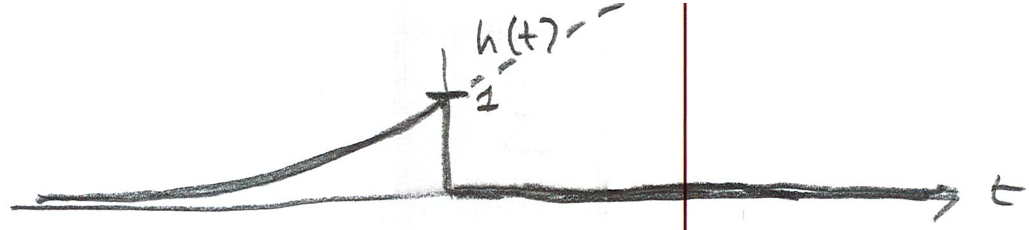

Question 2: [14%, Work-out question, Learning Objectives 2, 3, 4, and 5] Consider a CT-LTI system with impulse response:

$$h(t) = e^{t}\mathcal{U}(-t). \tag{2}$$

If the input signal is $x(t) = e^{jt} + \cos(\sqrt{3}t - 1)$, find the corresponding output $y(t)$.

$$x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t) = \int x(\tau)\,h(t-\tau)\,d\tau = \int x(t-\tau)\,h(\tau)\,d\tau$$

$$= \int_t^{\infty} \left[e^{j\tau} + \frac{e^{j(\sqrt{3}\tau-1)} + e^{-j(\sqrt{3}\tau-1)}}{2} \right] e^{t-\tau}\, d\tau$$

$$= e^{t} \int_t^{\infty} \left\{ e^{(j-1)\tau} + \frac{1}{2} e^{-j} e^{(-1+j\sqrt{3})\tau} + \frac{1}{2} e^{+j} e^{(-1-j\sqrt{3})\tau} \right\} d\tau$$

$$x(t) = \underbrace{e^{jt}}_{\omega_0 = 1} + \frac{1}{2} e^{-j} \underbrace{e^{j\sqrt{3}t}}_{\omega_0 = \sqrt{3}} + \frac{1}{2} e^{+j} \underbrace{e^{-j\sqrt{3}t}}_{\omega_0 = -\sqrt{3}}$$

$$\Rightarrow e^{jt} H(1) + \frac{1}{2} e^{-j}\, e^{j\sqrt{3}t} H(\sqrt{3}) + \frac{1}{2} e^{+j}\, e^{-j\sqrt{3}t} H(-\sqrt{3})$$

$$= y(t)$$

where

$$H(\omega) = \int_{-\infty}^{0} h(t)\, e^{-j\omega t}\, dt = \frac{1}{1-j\omega}$$

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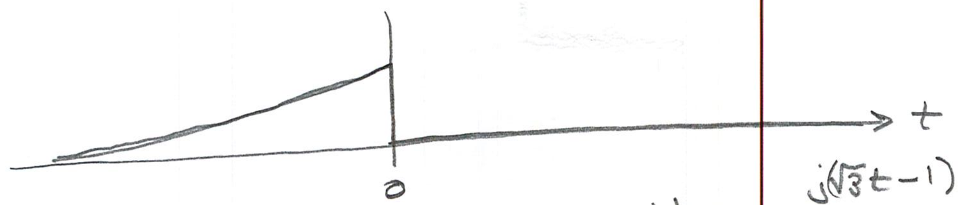
The "Harder Way"

Q2: $h(t) = e^{t} u(-t)$



$$x(t) = e^{jt} + \cos(\sqrt{3}t - 1) = e^{jt} + \frac{e^{j(\sqrt{3}t-1)} + e^{-j(\sqrt{3}t-1)}}{2}$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \int x(s)h(t-s)\,ds = \int h(s)\,x(t-s)\,ds$$

$$\Longrightarrow y(t) = \int_{-\infty}^{0} e^{s} \cdots \, ds$$

$$= \int_{t}^{\infty} \left\{ e^{js} + \frac{1}{2} e^{-j} e^{j\sqrt{3}s} + \frac{1}{2} e^{+j} e^{-j\sqrt{3}s} \right\} e^{t-s} \, ds$$

$$= e^{t} \int_{t}^{\infty} e^{s(-1+j)} + \frac{1}{2} e^{-j} e^{s(-1+\sqrt{3})} + \frac{1}{2} e^{j} e^{s(-1-\sqrt{3})} \, ds$$

$$= e^{t} \left\{ \frac{-e^{t(-1+j)}}{-1+j} - \frac{1}{2} e^{-j} \frac{e^{t(-1+\sqrt{3})}}{-1+\sqrt{3}} - \frac{1}{2} e^{j} \frac{e^{t(-1-\sqrt{3})}}{-1-\sqrt{3}} \right\}$$

In class

$$x(t) = e^{st} \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h(t)\end{matrix}} \longrightarrow y(t) = h * x(t)$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, \underbrace{e^{s(t-\tau)}}_{e^{st}\, e^{-s\tau}}\, d\tau$$

$$= e^{st} \cdot \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau}_{\triangleq H(s)}$$

Previous example: Apply to $s = 1, +j\sqrt{3}, -j\sqrt{3}$