# D.T. Fourier Series

$x[n] = x[n+N] \quad \forall n \in \mathbb{Z} \qquad \omega_0 = 2\pi/N$

analysis equation: $$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk(2\pi/N)n}$$ $k \in \mathbb{Z}$, but $N$-periodic

synthesis equation: $$x[n] = \sum_{k=0}^{N-1} X_k\, e^{+jk(2\pi/N)n}$$

# C.T. Fourier Series

$x(t) = x(t+T) \quad \forall t \in \mathbb{R} \qquad \omega_0 = 2\pi/T$

analysis equation: $$X_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$ $k \in \mathbb{Z}$

synthesis equation: $$x(t) = \sum_{k=-\infty}^{\infty} X_k\, e^{+jk\omega_0 t}$$

# Eigenfunctions of LTI Systems

## DT:

$$z^n;\ n \in \mathbb{Z} \longrightarrow \boxed{\begin{array}{c}\text{LTI} \\ h[n]\end{array}} \longrightarrow H(z)\,z^n \quad n \in \mathbb{Z}$$

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]\, z^{-k}$$

$z = e^{j\omega}$ most important case

## CT:

$$e^{st};\ t \in \mathbb{R} \longrightarrow \boxed{\begin{array}{c}\text{LTI} \\ h(t)\end{array}} \longrightarrow H(s)\,e^{st} \quad t \in \mathbb{R}$$

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$$

$s = j\omega$ most important case

# Fourier Series & LTI Systems

True Fact for DT or CT: periodic signal $\longrightarrow$ $\boxed{\text{LTI System}}$ $\longrightarrow$ periodic output of same period as input

DT case

$$x[n] = \sum_{k=0}^{N-1} X_k \, e^{+jk(2\pi/N)n}$$

$\omega_k$ = freq. of a complex sinusoid at input to LTI syst.

$$e^{j\omega_k n} \longrightarrow \boxed{h[n]} \longrightarrow H(z)\Big|_{z=e^{j\omega_k}} \cdot e^{j\omega_k n}$$

$\longleftarrow$ This holds for the ks of interest

$\Longrightarrow$ Use linearity:

$$y[n] = \sum_{k=0}^{N-1} \underbrace{X_k H\left(e^{jk2\pi/N}\right)}_{Y_k} e^{jk(2\pi/N)n}$$

This means there are two ways to compute output

$$\begin{tikzcd} x[n] \arrow[r, "* h[n]"] \arrow[d, leftrightarrow] \arrow[r, bend right=90, looseness=2] & y[n] \arrow[d, leftrightarrow] \\ X_k \arrow[r, "\cdot H(e^{jk2\pi/N})"'] & Y_k \end{tikzcd}$$

Sometimes the bottom path is preferred:

① Can be computationally faster.

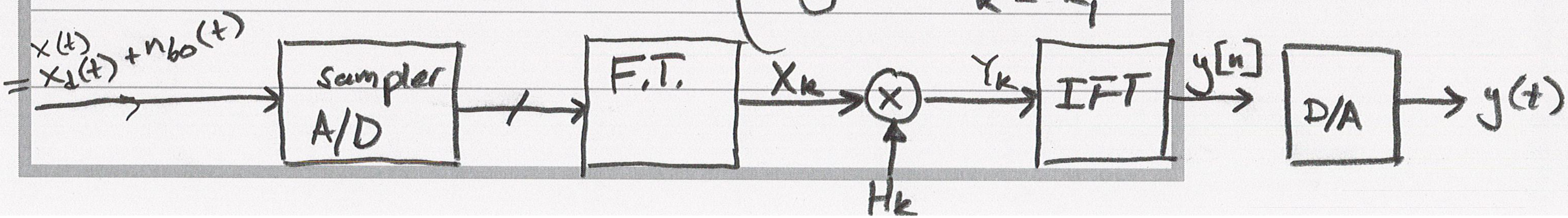
② Design is easier.

Re: 60 Hz nuller?

one of the k's might "correspond" to 60 Hz, say $k = k_1$

Then

$$H\left(e^{jk2\pi/N}\right) = \begin{cases} 1 & \text{if } k = 0, 1, \ldots N-1 \\ & \quad k \neq k_1 \\ 0 & \quad k = k_1 \end{cases}$$



# CT Periodic Convolution

$$x(t) = x(t+T) \quad t \in \mathbb{R}$$
$$y(t) = y(t+T)$$

$$z(t) = \int_0^T x(\tau)\, y(t-\tau)\, d\tau \quad \overset{F.S.}{\underset{T}{\longleftrightarrow}} \quad T X_k Y_k = Z_k$$

# CT Multiplication

$$w(t) = x(t)\, y(t) \quad \overset{F.S.}{\underset{T}{\longleftrightarrow}} \quad \sum_{n=-\infty}^{\infty} X_n Y_{k-n} = W_k$$

# D.T. Periodic Convolution

$x[n] = x[n+N]$
$y[n] = y[n+N]$
$\forall n \in \mathbb{Z}$

$$z[n] = \sum_{k=0}^{N-1} x[k]\, y[n-k] \qquad n \in \mathbb{Z}$$

Then:

① $z[\cdot]$ is $N$-periodic

② The sum in the definition can be extended over any complete period.

③ The DTFS of $z[\cdot]$ is given by

$$Z_k = N\, X_k Y_k$$

$\boxed{\text{Re ②}}$ $F(k,n) = x[k]\, y[n-k]$

$\boxed{\text{Re ①}}$ $z[\cdot]$ has to be periodic.

$\boxed{\text{Re } ③}$ Want to compute the DTFS of $z[n]$

$$Z_k = \frac{1}{N} \sum_{n=0}^{N-1} z[n]\, e^{-jk(2\pi/N)n}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \left( \sum_{\ell=0}^{N-1} x[\ell]\, y[n-\ell] \right) e^{-jk(2\pi/N)n}$$

$$= \frac{1}{N} \sum_{\ell=0}^{N-1} x[\ell] \sum_{n=0}^{N-1} y[n-\ell]\, e^{-jk(2\pi/N)n}$$

$N \cdot \frac{1}{N}$

$$= \frac{1}{N} \sum_{\ell=0}^{N-1} x[\ell] \cdot N \cdot \left( \frac{1}{N} \sum_{n=0}^{N-1} y[n-\ell]\, e^{-jk(2\pi/N)n} \right)$$

$n' = n - \ell$ $\qquad$ $n = n' + \ell$

$$= \frac{1}{N} \sum_{\ell=0}^{N-1} x[\ell] \cdot N \cdot \left( \frac{1}{N} \sum_{n'=-\ell}^{N-1-\ell} y[n']\, e^{-jk(2\pi/N)n'} \right) e^{-jk(2\pi/N)\ell}$$

$$= N \underbrace{\left( \frac{1}{N} \sum_{\ell=0}^{N-1} x[\ell]\, e^{-jk(2\pi/N)\ell} \right)}_{X_k} \underbrace{\left( \frac{1}{N} \sum_{n'=-\ell}^{N-1-\ell} y[n']\, e^{-jk(2\pi/N)n'} \right)}_{Y_k}$$

FFT

Cooley Tukey

Gauss.