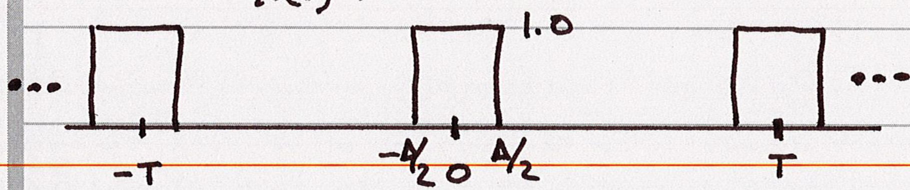
(Week 7 – Wednesday)

Periodic Pulse Train

$$x(t) \overset{F.S.}{\underset{T}{\longleftrightarrow}} X_k$$

$$x(t) = \sum_k X_k e^{jk\omega_0 t} \qquad X_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

$x(t)$:

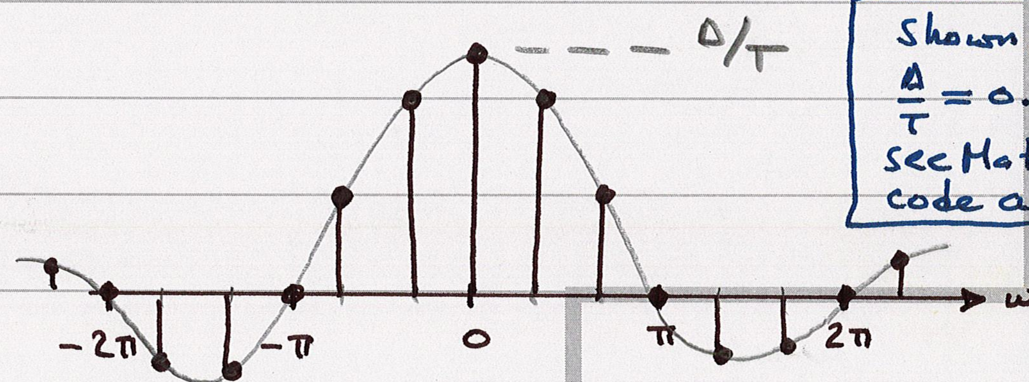


$$\overset{F.S.}{\underset{T}{\longleftrightarrow}} \quad X_k = \frac{\Delta}{T}\,\frac{\sin(\pi k \Delta/T)}{(\pi k \Delta/T)}$$

From Parseval:

$$\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt = \frac{1}{T}\int_{-\Delta/2}^{\Delta/2} dt = \frac{\Delta}{T}$$

$$= \sum_{k=-\infty}^{\infty} |X_k|^2$$



Shown for $\frac{\Delta}{T} = 0.25$ see Matlab code also

Last time in the Matlab experiments

$\Delta \longrightarrow 0$

then looked at plot of X_k vs. k $\Longrightarrow$ Saw that it looked flatter and flatter, but amplitudes were also going to zero. Here see that power $\longrightarrow 0$ as $\Delta \rightarrow 0$.

Scaling the Amplitudes

Consider $\tilde{x}(t) = \alpha x(t)$
whence

- $\frac{1}{T}\int_{-T/2}^{T/2} |\tilde{x}(t)|^2 dt = \alpha^2 \frac{\Delta}{T}$
- $\tilde{X}_k = \alpha X_k$

Case 1 Scale s.t. Energy per pulse stays constant

$$\alpha^2 \Delta = K \longrightarrow \alpha = K/\sqrt{\Delta}$$

(say $K = 1$)

$$\tilde{x}(t) = \frac{1}{\sqrt{\Delta}} x(t) \xleftrightarrow[T]{FS} \tilde{X}_k = \frac{1}{\sqrt{\Delta}} X_k = \frac{\sqrt{\Delta}}{T} \frac{\sin(\pi k \Delta/T)}{(\pi k \Delta/T)}$$

now power stays constant as $\Delta \to 0$ although amplitude does fall.

Case 2 Scale s.t. area per pulse remains const.: $\alpha \Delta = 1$

$$\tilde{x}(t) = \frac{1}{\Delta} x(t) \longleftrightarrow \tilde{X}_k = \frac{1}{\Delta} X_k$$

Here

$$\frac{1}{T}\int_{-T/2}^{T/2} |\tilde{x}(t)|^2 dt = \frac{1}{\Delta T} \longrightarrow \infty \text{ as } \Delta \to 0.$$

Case 2 $\alpha = \frac{1}{\Delta}$ $\qquad \tilde{X}_k = \frac{1}{T} \dfrac{\sin(\pi k \Delta / T)}{(\pi k \Delta / T)}$

Area per pulse stays constant.

For $\pi k \frac{\Delta}{T} \approx 0$ then $\sin(\pi k \Delta / T) \approx \pi k \Delta / T$

$\Longrightarrow \tilde{X}_k \approx \frac{1}{T}$ is roughly const.

$$\lim_{\Delta \to 0} \tilde{X}_k = \frac{1}{T}$$

Note:

$$\tilde{\tilde{x}}(t) = \sum_k \tilde{\tilde{X}}_k e^{jk\omega_0 t}$$

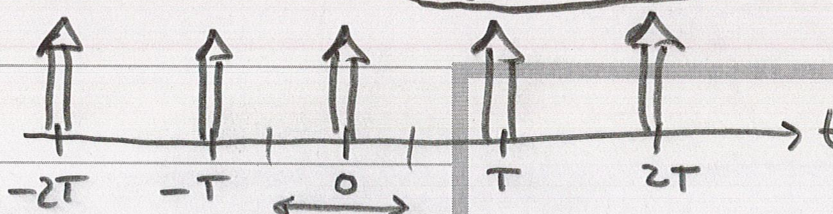
$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}$$

$$= \sum_{\ell=-\infty}^{\infty} \delta(t - \ell T)$$

We will use this identity later.

Thus might compare above to:

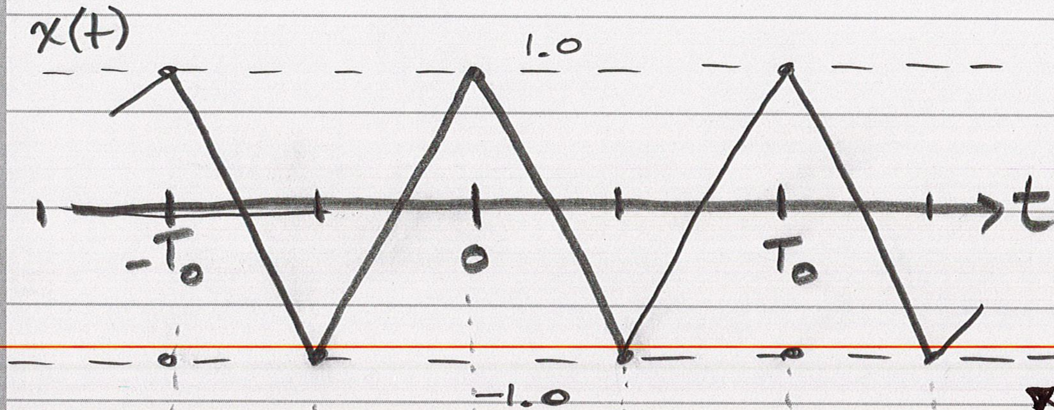
$$\tilde{\tilde{x}}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$



I can simply apply the formula to compute F.S. coeffs of delta train

$$\tilde{\tilde{X}}_k = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{\tilde{x}}(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T} \quad \forall k.$$

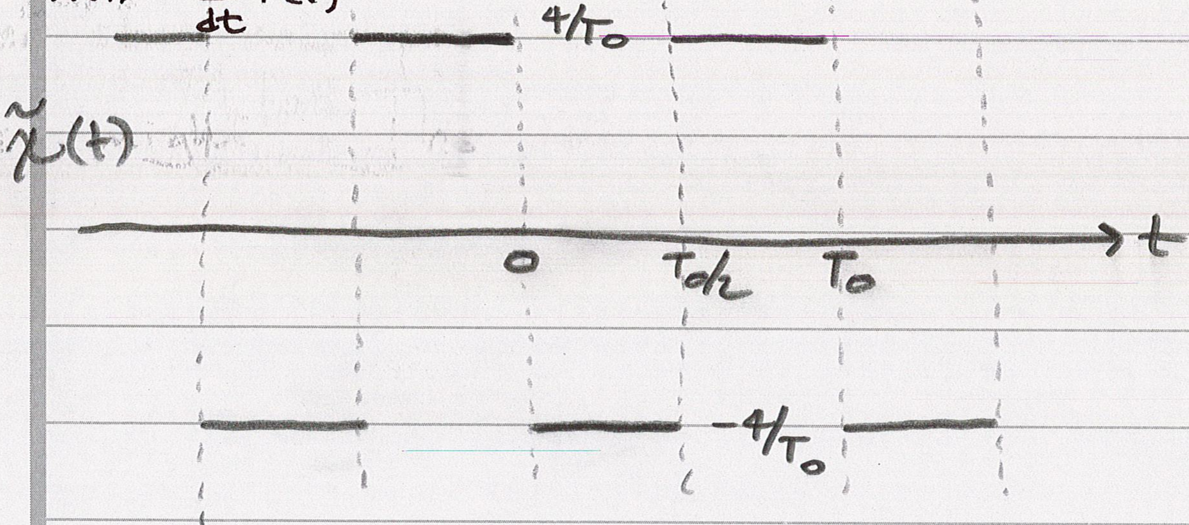
Example: Use F.S. Props and the "Sinc" F.S. we found for pulse train to compute another F.S.



$$\overset{F.S.}{\underset{T_0}{\longleftrightarrow}} \begin{cases} 0 & k \text{ even} \\ \dfrac{4}{\pi^2 k^2} & k \text{ odd} \end{cases} = X_k$$

$$\left(\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{2}{T_0/2} = 4/T_0\right)$$

$\tilde{x}(t) \triangleq \frac{d}{dt} x(t)$

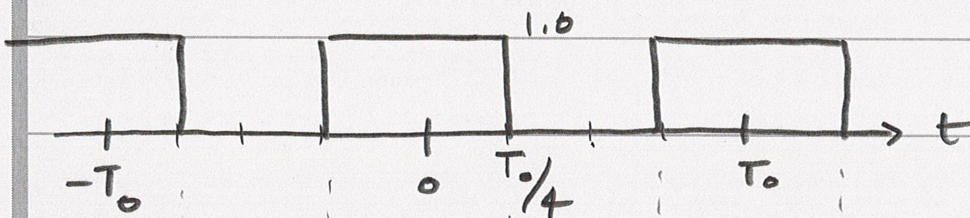


From the derivative prop have

$$x(t) \overset{FS}{\underset{T_0}{\longleftrightarrow}} X_k$$

$$\underbrace{\tilde{X}_k}_{\text{in a prev. example we found a FS for a flat topped pulse train.}} = jk\omega_0 \underbrace{X_k}_{\text{want this}} \qquad \omega_0 = \frac{2\pi}{T_0}$$

Going back to 1st example, set duty cycle to $\frac{\Delta}{T_0} = 1/2$

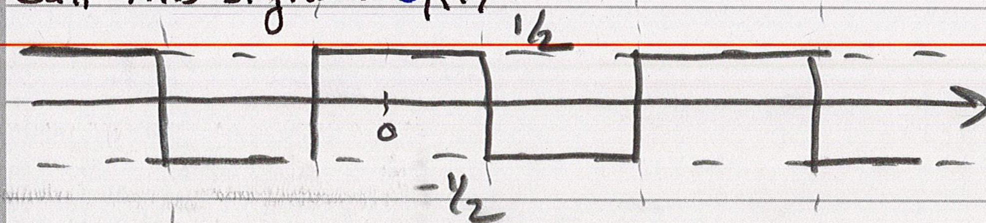


$$\xleftrightarrow[T_0]{F.S.} \quad \frac{1}{2}\,\frac{\sin(\pi k/2)}{\pi k/2}$$

To make this look like $\tilde{x}(t)$ I need to
1. have zero dc
2. a new amplitude

Step 1 subtract $1/2$

Call this signal $s_1(t)$:



Call the F.S. coeffs $S_{1,k}$:

$$\xleftrightarrow[T_0]{F.S.} \begin{cases} 0 & k=0 \\ \frac{1}{2}\,\frac{\sin(\pi k/2)}{\pi k/2} & k \neq 0. \end{cases}$$

Step 2 Scale up amplitude $\alpha \frac{1}{2} = \frac{4}{T_0} \Longrightarrow \alpha = 8/T_0$

$$\frac{8}{T_0}\cdot\left\{\text{signal just above}\right\} \xleftrightarrow[T_0]{F.S.} \underbrace{\begin{cases} 0 & k=0 \\ \frac{4}{T}\,\frac{\sin(\pi k/2)}{\pi k/2} & k \neq 0. \end{cases}}$$

$$= \begin{cases} 0 & k=2\ell \\ \frac{8}{T_0}\,\frac{(-1)^{\ell}}{\pi(2\ell+1)} & k=2\ell+1. \end{cases}$$

this is the signal from step 2.

Call this signal $s_2(t)$

Call the corresp. F.S. coeffs $S_{2,k}$

Step 3 Time shift left: Call this $s_3(t) \overset{FS}{\underset{T_0}{\longleftrightarrow}} S_{3,k}$

$$\text{Signal from step 2}\left(t+\frac{T_0}{4}\right) \longleftrightarrow \underset{T_0}{\text{F.S.}}\left\{\begin{matrix}\text{signal}\\ \text{from step}\\ 2\end{matrix}\right\}_k e^{+jk\omega_0 T_0/4}$$

$$\therefore \quad s_3(t) = s_2\left(t+\frac{T_0}{4}\right) \overset{F.S.}{\underset{T_0}{\longleftrightarrow}} S_{2,k}\, e^{+jk\omega_0 T_0/4} = S_{3,k}$$

Note: $s_3(t) = \tilde{x}(t)$ from p. 4. $\Longleftrightarrow$ $S_{3,k} = \tilde{X}_k$