Finishing Example:

Look at the phase term: $e^{jk\omega_0 T_0/4}$

$$= e^{jk\pi/2} \quad \longleftarrow \omega_0 T_0 = 2\pi, \quad \omega_0 = \frac{2\pi}{T_0}$$

$$S_{3,k} = S_{2,k}\, e^{jk\pi/2}$$

$$= \left\{\begin{array}{ll} 0 & k=2\ell \\ \frac{8}{T_0}\frac{(-1)^\ell}{\pi(2\ell+1)} & k=2\ell+1 \end{array}\right\} e^{jk\pi/2} = \left\{\begin{array}{ll} 0 & k=2\ell \\ \frac{8}{T_0}\frac{(-1)^\ell}{\pi(2\ell+1)} e^{j(2\ell+1)\pi/2} & k=2\ell+1 \end{array}\right.$$

$\ell=-1 \longrightarrow e^{-j\pi/2} = -j$

$\ell=0 \longrightarrow e^{j\pi/2} = +j$

$\ell=1 \longrightarrow e^{j3\pi/2} = -j$

$\vdots$

$(-1)^\ell j$

$$\Rightarrow S_{3,k} = \left\{\begin{array}{ll} 0 & k=2\ell \\ \frac{8}{T_0}\frac{(-1)^\ell(-1)^\ell j}{\pi(2\ell+1)} \longrightarrow \frac{8}{\pi T_0}\frac{j}{2\ell+1} & \end{array}\right.$$

$= \tilde{X}_k$ as defined originally.

Wanted $X_k = \frac{1}{jk\omega_0}\tilde{X}_k \quad k\neq 0$ (but $X_0 = 0$)

From this

$$X_k = \begin{cases} 0 & \text{if } k = \text{even} \\ \underbrace{\frac{\cancel{8}^{\,4}}{\pi \cancel{T_0}} \frac{\cancel{j}}{2l+1} \frac{1}{\cancel{j}(2l+1)} \frac{\cancel{T_0}}{\cancel{2}\pi}}_{\displaystyle \frac{4}{\pi^2 (2l+1)^2}} & k = \text{odd} \\ & \quad = 2l+1 \end{cases}$$

is what we wanted to show.

In response to a Question about the proper form of shift property:

Shifting Property

$$x(t) \underset{T}{\overset{FS}{\longleftrightarrow}} X_k$$

$$\text{"} \sum_k X_k e^{+jk\omega_0 t} \qquad = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

Now let $z(t) \triangleq x(t-\tau_0)$

$$z(t) = x(t-\tau_0) = \sum_k X_k e^{+jk\omega_0 (t-\tau_0)}$$

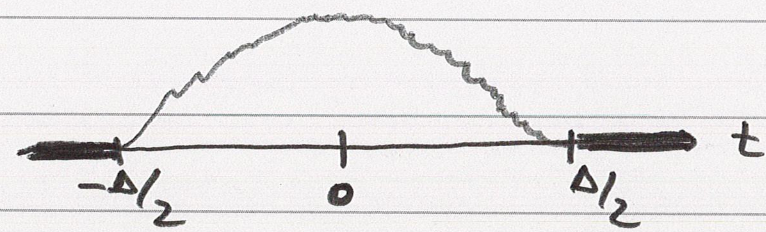
$$= \sum_k \underbrace{\left(X_k e^{-jk\omega_0 \tau_0}\right)}_{Z_k} e^{jk\omega_0 t}$$

(Proof of Shift Property).

Generic Pulse Train

Let $p(t)$ be a finite duration pulse

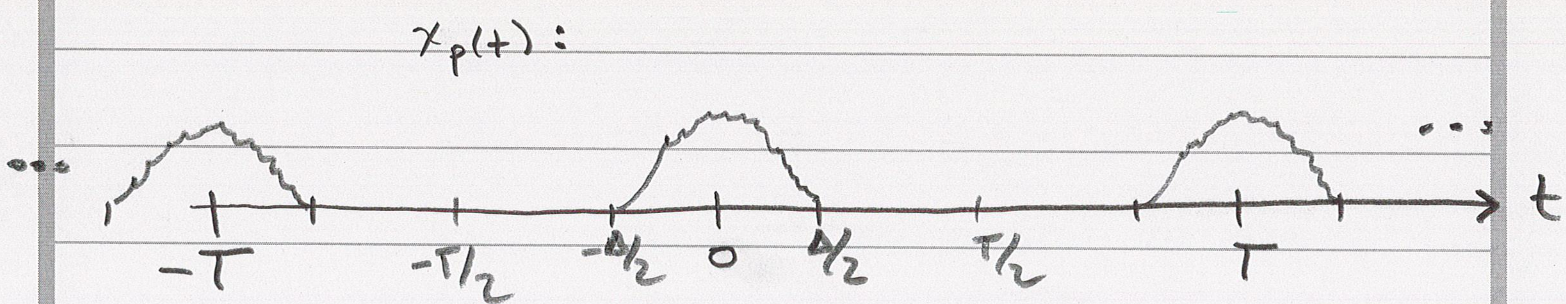
ie $p(t) = 0 \quad |t| > \Delta/2$



Construct a periodic wave via repeating the pulse over and over:

$$x_p(t) = \sum_{n=-\infty}^{\infty} p(t-nT) \qquad \text{where} \quad T > \Delta$$

Then the picture looks like:



$x_p(t)$ is periodic with period T. Thus has a F.S.

$$x_p(t) = \sum_{n=-\infty}^{\infty} p(t-nT) = \sum_{k=-\infty}^{\infty} X_{p,k}\, e^{jk\omega_0 t} \qquad \omega_0 = 2\pi/T$$

where

$$X_{p,k} = \frac{1}{T}\int_{\text{one period}} x_p(t)\, e^{-jk\omega_0 t}\, dt$$

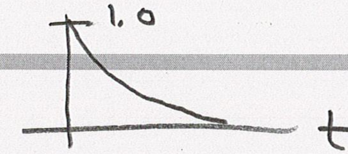
$$= \frac{1}{T}\int_{-\Delta/2}^{\Delta/2} p(t)\, e^{-jk\omega_0 t}\, dt \;\triangleq\; \frac{1}{T} P(\omega)\Big|_{\omega = k\omega_0}$$

where

$$P(\omega) \triangleq \int_{-\infty}^{\infty} p(t)\, e^{-j\omega t}\, dt$$

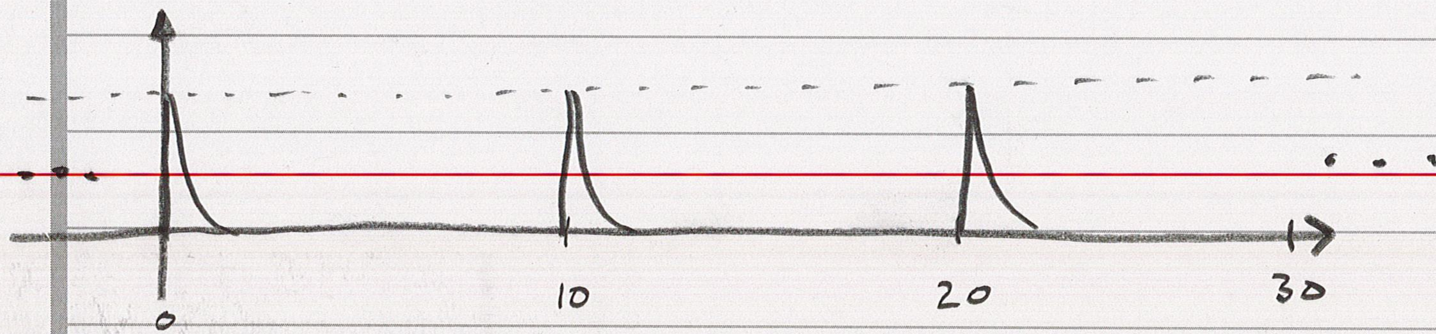
is the cont. time Fourier Transf. of the pulse.

Example define $w(t) = e^{-3t} u(t)$



and use it to make a periodic wave

$$z(t) = \sum_{k=-\infty}^{\infty} w(t-10k) \qquad \text{spaced by 10 sec.}$$



Compute the F.S. for $z(t)$: Z_k

$$Z_k = \frac{1}{10}\int_0^{10} \underbrace{z(t)}_{e^{-3t}} e^{-jk\pi t/5}\,dt \approx \frac{1}{10}\int_0^{\infty} e^{-3t} e^{-jk\pi t/5}\,dt$$

$$= \frac{1/10}{-(3+jk\pi/5)}\, e^{-(3+jk\pi/5)t}\Big|_{t=0}^{\infty} = \frac{1/10}{3+jk\pi/5} = \frac{1}{10} W(\gamma)\Big|_{\gamma = k\pi/5}$$

$$w(t) \xleftrightarrow{\text{CTFT}} W(\gamma) = \frac{1}{3+j\gamma}$$

$$= \int_{-\infty}^{\infty} w(t)\, e^{-j\gamma t}\,dt$$

There was some confusion about the freq. variable ω since it looks like the name of pulse w .... so I changed $\omega \mapsto \gamma$

Generic F.S.

$$x(t) = x(t+T)$$

$$x(t) \overset{FS}{\underset{T}{\longleftrightarrow}} X_k$$

$$\sum_k X_k e^{jk\omega_0 t} \qquad = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

Generic cont. variable: Replace t with γ

T with Γ

$x(t) \longrightarrow Z(\gamma)$

$X_k \longrightarrow z[k]$

$\Longrightarrow$ This gives the DTFT.

(Come back to this)