(Week 6 — Wednesday)

Summary of Previous Episode

① Eigenfunctions of LTI Systems :

$$e^{st};\ t\in\mathbb{R},\ s\in\mathbb{C} \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h(t)\end{matrix}} \longrightarrow \underbrace{H(s)}_{\text{eigenvalue}}\ \underbrace{e^{st}}_{\text{eigenfunction}};\ t\in\mathbb{R}$$

$$z^{n};\ n\in\mathbb{Z},\ z\in\mathbb{C} \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h[n]\end{matrix}} \longrightarrow \overbrace{H(z)}\ \overbrace{z^{n}};\ n\in\mathbb{Z}$$

② $x(t) = x(t+T)\ \forall t\in\mathbb{R}$

$\omega_0 = 2\pi/T$

⊛ A Fourier Series Representation for $x(t)$ is

$$x(t) \sim \sum_{k} X_k e^{jk\omega_0 t}$$

Today: How to compute X_k; what sense equality in ⊛

How to Calculate FS Representation

Suppose $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t} \quad t \in \mathbb{R} \quad \omega_0 = 2\pi/T$ ⊛

Want: Method to compute the X_k's from the signal $x(t)$---

Assume ⊛ holds --- mult by $e^{-jn\omega_0 t}$ --- integrate over a period

$$\int_0^T x(t) e^{-jn\omega_0 t} dt = \int_0^T e^{-jn\omega_0 t} \sum_k X_k e^{jk\omega_0 t} dt = \int_0^T \sum_k X_k e^{j(k-n)\omega_0 t} dt$$

$$= \sum_k X_k \int_0^T e^{j(k-n)\omega_0 t} dt \longrightarrow = \begin{cases} T & \text{if } k=n \\ 0 & \text{if } k \neq n \end{cases}$$

$$= T X_n$$

Solve for X_n: $\quad X_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt \quad n \in \mathbb{Z}$

A F.S. representation:

$$x(t) = x(t+T) \ \forall t$$

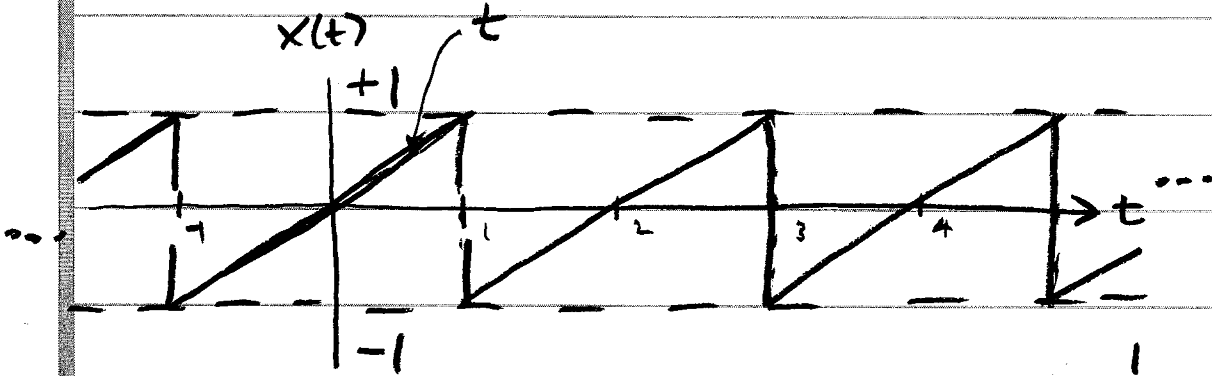
$$\omega_0 = 2\pi/T$$

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t} \quad \xleftrightarrow{F.S.} \quad X_n = \frac{1}{T}\int_0^T x(t) e^{-jn\omega_0 t}\, dt$$

$t \in R$ $\qquad\qquad n \in \mathbb{Z}$

$$x(t) \underset{T,\,\omega_0}{\overset{F.S.}{\longleftrightarrow}} X_n$$

Example: Periodic Sawtooth



$T = 2$

$\omega_0 = \frac{2\pi}{T} = \pi$

$$X_n = \frac{1}{T} \int_{\text{one period}} x(t) e^{-jn\omega_0 t} \, dt = \frac{1}{2} \int_{-1}^{1} t e^{-jn\pi t} \, dt$$

Parts:

$\int u\,dv = uv - \int v\,du$

$u = t \quad du = dt$

$dv = e^{-jn\pi t} dt \rightarrow v = \frac{-1}{jn\pi} e^{-jn\pi t}$

Since $n=0$ corresp. to average value of $x(t) \Rightarrow X_0 = 0$

Plug in:

$$X_n = \frac{1}{2} \left[\frac{-t}{jn\pi} e^{-jn\pi t} \Big|_{t=-1}^{+1} - \int_{-1}^{+1} \left(-\frac{1}{jn\pi}\right) e^{-jn\pi t} \, dt \right]$$

$$= \frac{1}{2} \left[-\frac{1}{jn\pi} e^{-jn\pi} - \frac{1}{jn\pi} e^{+jn\pi} + \left(\frac{1}{jn\pi}\right)\left(\frac{-1}{jn\pi}\right) e^{-jn\pi t} \Big|_{-1}^{+} \right]$$

$$X_n = \frac{1}{2}\frac{1}{n^2\pi^2}\underbrace{\left[e^{-jn\pi} - e^{+jn\pi}\right]}_{0} + \frac{1}{2}\left(\frac{-1}{jn\pi}\right)\left[e^{-jn\pi} + e^{+jn\pi}\right]$$

But $e^{jn\pi} = e^{-jn\pi} = (-1)^n$

$$\Rightarrow X_n = -\frac{1}{jn\pi}\left[\frac{(-1)^n + (-1)^n}{2}\right] = \frac{j}{n\pi}(-1)^n \qquad n \neq 0$$

Convergence of the Fourier Series

Mean-Square Convergence:

$x(t)$ is of finite energy over one period $\qquad x(t) \overset{F.S.}{\underset{T}{\longleftrightarrow}} X_k$

$$x_N(t) = \sum_{k=-N}^{N} X_k e^{jk\omega_0 t}$$

Want to understand $\quad x(t) \overset{?}{=} \lim_{N\to\infty} x_N(t)$

Under the finite energy ass. the limit exist in mean-square in sense that

$$E_N \triangleq \int_T |x(t) - x_N(t)|^2 \, dt \longrightarrow 0 \quad \text{as } N \to \infty$$

But (Weirstrauss) showed that this might not conv. ptwise anywhere.

Dirichlet Conditions

$$x(t) = x(t+T) \quad \forall t \in \mathbb{R}$$

$$X_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$$

D.C.s: Assume

(i) $x(t)$ is absolutely integrable over one period

(ii) $x(t)$ is of bounded variation over any finite interval.

(iii) $x(t)$ has at most a finite # of discontinuities of finite jump size in any finite interval.

Then

$$\lim_{N\to\infty} \sum_{k=-N}^{N} X_k e^{jk\omega_0 t} = x(t)$$

at any point t where $x(t)$ is continuous.