(Week 6 – Monday)

Response of LTI systs to Complex Exp Inputs

1. Continuous Time

$$x(t) \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h(t)\end{matrix}} \longrightarrow y(t) = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

$x(t) = e^{st}$, $s \in \mathbb{C}$ $\Longrightarrow$

$$= \int h(\tau)\,\underbrace{e^{s(t-\tau)}}_{e^{st}e^{-s\tau}}\,d\tau \;=\; \underbrace{e^{st}}_{\text{input signal}}\;\underbrace{\int h(\tau)\,e^{-s\tau}\,d\tau}_{\substack{\text{a complex \# indep. of } t\\ H(s)}}$$

Aside on Linear Algebra:

H is a $N \times N$ matrix

x an $N \times 1$ vector is called an eigenvector if $x \neq 0$ and $\exists$ a complex number λ s.t.

$$Hx = \lambda x$$

λs for which the above holds are called eigen values

Signals e^{st} with above prop. are called eigenfunctions.

All signals e^{st} $s \in \mathbb{C}$ are eigenfunctions of any LTI system.

2. Discrete Time

$$x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = \sum_k h[k]\, x[n-k]$$

$$\|$$

$$z^n \longrightarrow y[n] = \sum_k h[k] \underbrace{z^{n-k}}_{z^n z^{-k}} = \underbrace{z^n}_{\text{input}} \underbrace{\sum_k h[k] z^{-k}}_{H(z)}$$

$$y[n] = \underbrace{H(z)}_{\text{eigenvalue}} \underbrace{z^n}_{\text{eigenfunction}}$$

Most important cases:

1. CT $\quad s = j\omega \qquad e^{j\omega t} \longrightarrow y(t) = H(j\omega)\, e^{j\omega t}$

$$= |H(j\omega)|\, e^{j\angle H(j\omega)}\, e^{j\omega t}$$

"If input is sinusoid of freq ω $\Longrightarrow$ output is sinusoid of same freq. with only amplitude, phase changed"

2. DT $\quad z = e^{j\omega} \quad e^{j\omega n} \rightarrow H(e^{j\omega})\, e^{j\omega n}$

Main use of fact

Can decompose an arbitrary input to an LTI system as a superposition of complex exponential sigs

$\Longrightarrow$ output is just a superposition of responses to them.

This is the Fourier Series.

Fourier Series: CT Periodic Sigs

$x(\cdot)$ is periodic of period T

$$x(t) = x(t+T) \quad \forall t \in \mathbb{R}$$

Examples ① $\cos \omega_0 t$ $\quad T = 2\pi/\omega_0$ is fund. period

② $\phi_1(t) = e^{j\omega_0 t}$

⋮

③ harmonically related $\quad \phi_k(t) = e^{jk\omega_0 t} \quad k \in \mathbb{Z}$

④ Any weight L.C.

$$(*) \begin{cases} x(t) = \displaystyle\sum_k X_k e^{jk\omega_0 t} \quad \left(\sum_k = \sum_{k=-\infty}^{\infty}\right) \\ = \underbrace{X_0}_{\text{DC or zero freq. term}} + \underbrace{\left(X_1 e^{j\omega_0 t} + X_{-1} e^{-j\omega_0 t}\right)}_{\text{fundamental or 1st harmonic}} + \overbrace{\left(X_2 e^{j2\omega_0 t} + X_{-2} e^{-j2\omega_0 t}\right)}^{\text{2nd harmonic.}} \\ \quad + \cdots \end{cases}$$

(*) is called a Fourier Series Representation

Special Case

$x(t)$ T periodic and real-valued

$\Longrightarrow$ (*) must have complex conj. symm. $X_k = X_{-k}^*$

Then if write $X_k = |X_k| e^{j\theta_k}$ polar form

and plug into (*) we get

$$x(t) = X_0 + 2\sum_{k=1}^{\infty} |X_k| \cos(k\omega_0 t + \theta)$$

Also if $X_k = B_k + jC_k$ (rect. form)

$$x(t) = X_0 + 2\sum_{k=1}^{\infty} \left[B_k \cos k\omega_0 t - C_k \sin k\omega_0 t \right]$$

sketch of how to derive

$$x(t) = \sum_k X_k e^{jk\omega_0 t} \rightarrow \cdots + \underbrace{X_k e^{jk\omega_0 t} + X_{-k} e^{-jk\omega_0 t}}_{} + \cdots$$

$$= (B_k + jC_k)(\cos k\omega_0 t + j\sin k\omega_0 t) + (B_k - jC_k)(\cos(-k\omega_0 t) + j\sin(-k\omega_0 t))$$