

(Week 6 - Monday)

Response of LTI systems to Complex Exp Inputs

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1. Continuous Time

Block diagram: Input $x(t)$ enters a box labeled "LTI $h(t)$ ". The output is $y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$.

Below the input, it is noted: $x(t) \parallel e^{st}$, $s \in \mathbb{C}$.

The derivation shows:

$$y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau = \int_{-\infty}^{\infty} h(\tau) \underbrace{e^{s(t-\tau)}}_{e^{st} e^{-s\tau}} d\tau = e^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau}_{\substack{\text{a complex \#} \\ \text{indep. of } t} = H(s)}$$

An arrow points from the e^{st} term in the final expression to the text "input signal".

Aside on Linear Algebra:

H is a $N \times N$ matrix
 x an $N \times 1$ vector is called an eigenvector if $x \neq 0$ and $\exists$ a complex number λ st.

$$Hx = \lambda x$$

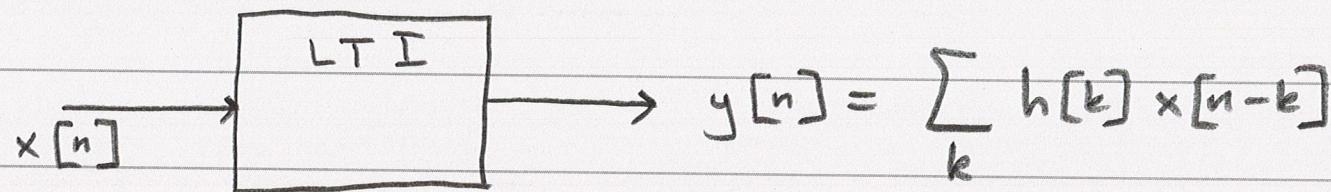
λ s for which the above holds are called eigen values

Signals e^{st} with above prop. are called eigenfunctions.

All signals e^{st} $s \in \mathbb{C}$ are eigenfunctions of any LTI system.

2. Discrete Time

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$$z^n \longrightarrow y[n] = \sum_k h[k] \underbrace{z^{n-k}}_{z^n z^{-k}} = z^n \underbrace{[h[k] z^{-k}]_k}_{H(z)}$$

input.

$$y[n] = H(z) z^n$$

$\swarrow$ eigenvalue.
 $\searrow$ eigenfunction

Most important cases:

1. CT $s = j\omega$ $e^{j\omega t} \longrightarrow y(t) = H(j\omega) e^{j\omega t}$

"If input is sinusoid of freq $\omega \Rightarrow$ output is sinusoid of same freq. with only amplitude, phase changed"

$= |H(j\omega)| e^{j\angle H(j\omega)} e^{j\omega t}$

2. DT $x = e^{j\omega}$ $e^{j\omega n} \rightarrow H(e^{j\omega}) e^{j\omega n}$

Main use of fact

Can decompose an arbitrary input
to an LTI system as a superposition
of complex exponential sigs

$\Rightarrow$ output is just a superposition
of responses to them.

✓ This is the Fourier Series.

Fourier Series: CT Periodic Sigs

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$x(\cdot)$ is periodic of period T

$$x(t) = x(t+T) \quad \forall t \in \mathbb{R}$$

Examples ① $\cos \omega_0 t$ $T = 2\pi/\omega_0$ is fund. period

② $\phi_1(t) = e^{j\omega_0 t}$

③ harmonically related $\phi_k(t) = e^{jk\omega_0 t} \quad k \in \mathbb{Z}$

④ Any weight L.C.

$$\sum_{k=-\infty}^{\infty}$$

$$x(t) = \sum_k X_k e^{jk\omega_0 t}$$

2nd harmonic.

$$= X_0 + \left(X_1 e^{j\omega_0 t} + X_{-1} e^{-j\omega_0 t} \right) + \left(X_2 e^{j2\omega_0 t} + X_{-2} e^{-j2\omega_0 t} \right) + \dots$$

DC or zero freq. term

fundamental or 1st harmonic

⊛ is called a Fourier Series Representation

Special Case

$x(t)$ T periodic and real-valued

$\Rightarrow (*)$ must have complex conj. symm. $X_k = X_{-k}^*$

Then if write $X_k = |X_k| e^{j\theta_k}$ polar form
and plug into $(*)$ we get

$$x(t) = X_0 + 2 \sum_{k=1}^{\infty} |X_k| \cos(k\omega_0 t + \theta_k)$$

Also if $X_k = B_k + jC_k$ (rect. form)

$$\rightarrow x(t) = X_0 + 2 \sum_{k=1}^{\infty} [B_k \cos k\omega_0 t - C_k \sin k\omega_0 t]$$

$$\begin{aligned} x(t) &= \sum_k X_k e^{jk\omega_0 t} \rightarrow \dots + X_k e^{jk\omega_0 t} + X_{-k} e^{-jk\omega_0 t} + \dots \\ &= (B_k + jC_k)(\cos k\omega_0 t + j \sin k\omega_0 t) + (B_k - jC_k)(\cos(-k\omega_0 t) + j \sin(-k\omega_0 t)) \end{aligned}$$

sketch of how
to derive