(Week 6 – Friday)

Fourier Series Properties

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TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Proving the Multiplication Property

$$x(t) \overset{FS}{\underset{T}{\longleftrightarrow}} X_k$$

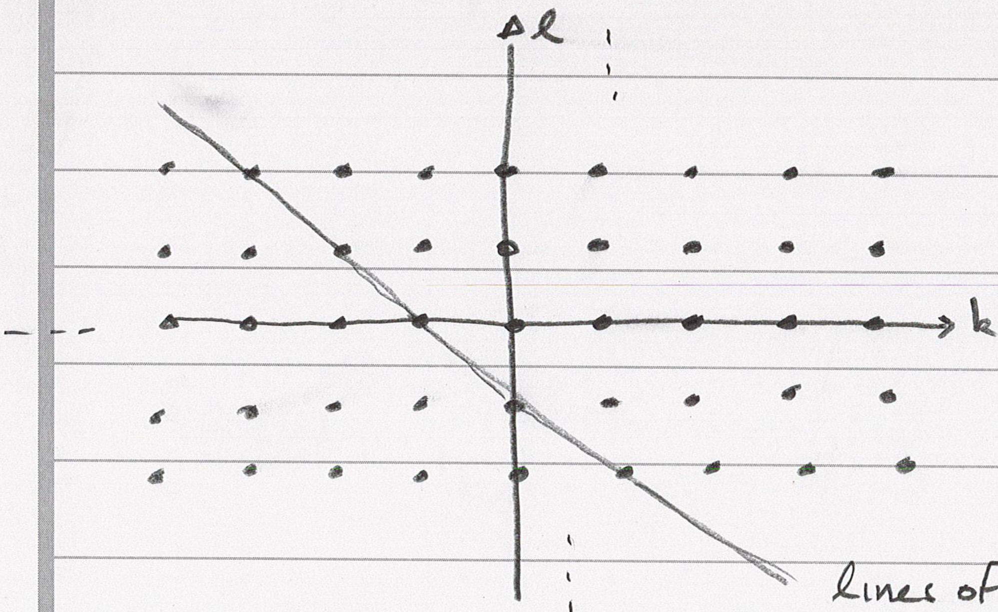
$$y(t) \overset{FS}{\underset{T}{\longleftrightarrow}} Y_k$$

$$z(t) = x(t)\,y(t) \overset{FS}{\underset{T}{\longleftrightarrow}} [X * Y]_k = Z_k$$

$$x(t) = \sum_k X_k e^{jk\omega_0 t} \qquad y(t) = \sum_l Y_l e^{jl\omega_0 t}$$

$$\Longrightarrow \quad x(t)y(t) = \left(\sum_k X_k e^{jk\omega_0 t}\right)\left(\sum_l Y_l e^{jl\omega_0 t}\right)$$

$$= \sum_k \sum_l X_k Y_l e^{j(k+l)\omega_0 t}$$



$m = k + l$

$l = -k + m$

lines of slope -1

Thus

$$\sum_k \sum_\ell X_k Y_\ell \, e^{j(k+\ell)\omega_0 t} \qquad k+\ell=m$$

$$= \sum_m \left(\sum_\ell X_{m-\ell} Y_\ell \right) e^{jm\omega_0 t}$$

$$Z_m = [X * Y]_m$$

Parseval $\quad x(t) \xleftrightarrow[T]{FS} X_k \implies \frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_k |X_k|^2$

$x(t) \xleftrightarrow[T]{FS} X_k$

ie $\quad x(t) = \sum_k X_k e^{jk\omega_0 t} \; ; \; X_k = \frac{1}{T}\int_0^T x(t) e^{-jk\omega_0 t} dt$

$z(t) = |x(t)|^2 = x(t)x^*(t)$

$\Rightarrow \frac{1}{T}\int_0^T |x(t)|^2 dt = $ $k=0$ F.S. coeff of $z(t)$.

note that $z(t)$ is a product of $x(t)$ and $\tilde{x}(t) = x^*(t)$

From the mult. prop: $Z_k = [X * \tilde{X}]_k$ so need F.S. $\tilde{X}_k$ of $\tilde{x}(t)$

$$\tilde{x}(t) = x^*(t) = \left(\sum_k X_k e^{jk\omega_0 t}\right)^* = \sum_k X_k^* e^{-jk\omega_0 t}$$

c.o.v. $\ell = -k$

$$= \sum_\ell X_{-\ell}^* e^{j\ell\omega_0 t} \implies \tilde{X}_\ell = X_{-\ell}^*$$

Thus $\quad Z_k = [X * \tilde{X}]_k = \sum_m X_{k-m}\tilde{X}_m = \sum_n X_{k-m} X_{-m}^*$

Now evaluate @ $k=0$:

$$Z_0 = \sum_m X_{-m} X_{-m}^* = \sum_n X_n X_n^* = \sum_n |X_n|^2$$

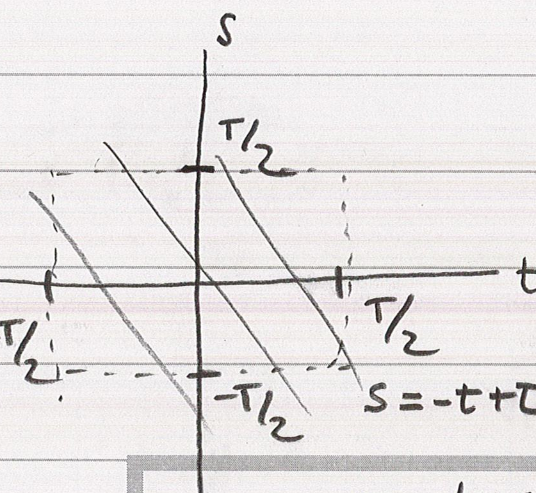
Periodic Conv. Prop

$$\int_{\text{one period}} x(\tau)\,y(t-\tau)\,d\tau \;\overset{FS}{\underset{T}{\longleftrightarrow}}\; T X_k Y_k$$

Start with

$$T X_k Y_k = T\left(\frac{1}{T}\int_T x(t)\,e^{-jk\omega_0 t}\,dt\right)\left(\frac{1}{T}\int_T y(s)\,e^{-jk\omega_0 s}\,ds\right)$$

$$= \frac{1}{T}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} x(t)\,y(s)\,e^{-jk\omega_0(t+s)}\,dt\,ds$$



Let $I(t,s) = \text{integrand} \triangleq x(t)\,y(s)\,e^{-jk\omega_0(t+s)}$

this is periodic in either variable of period T.

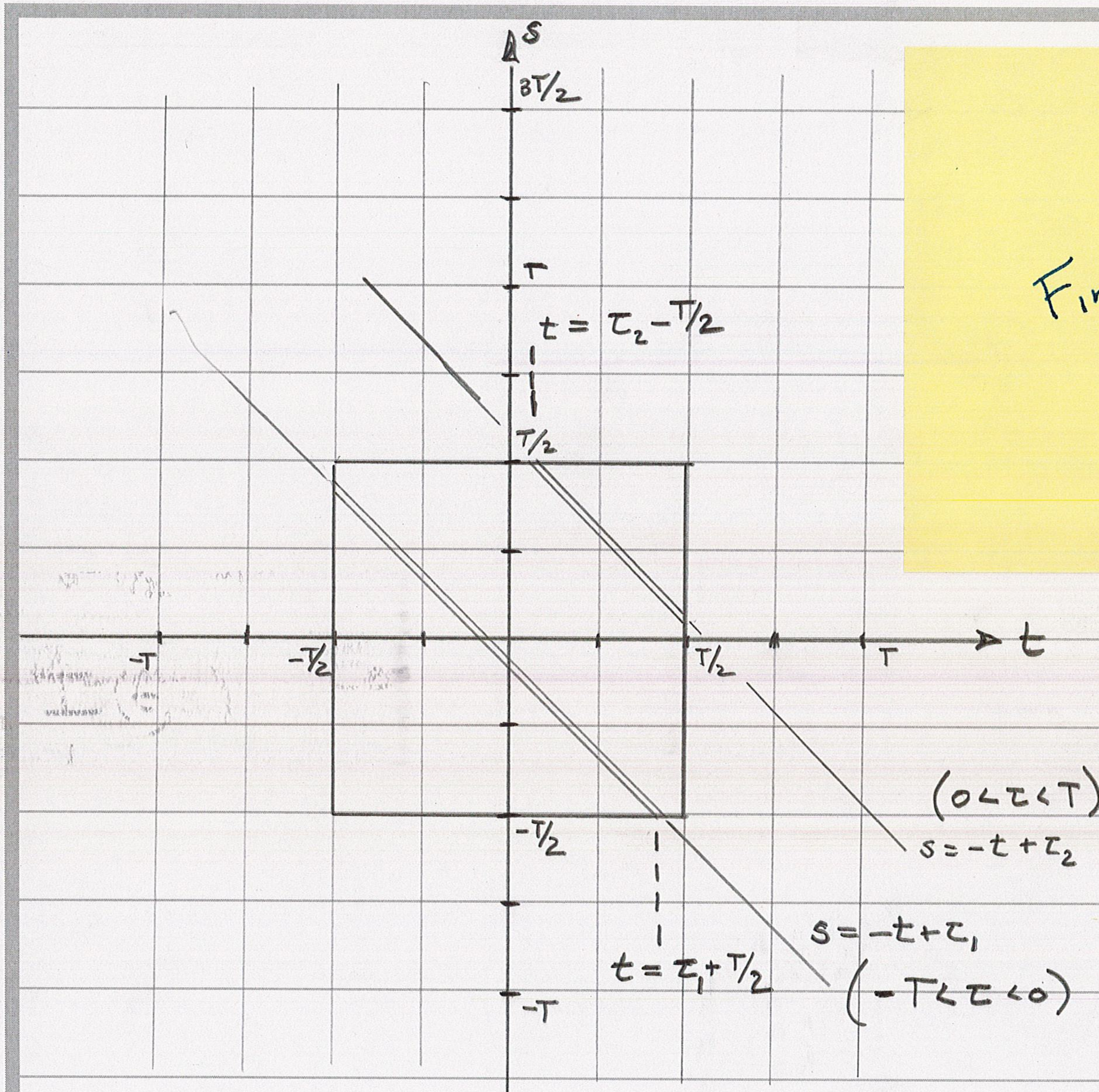
$\tau = t + s$

$s = -t + \tau$

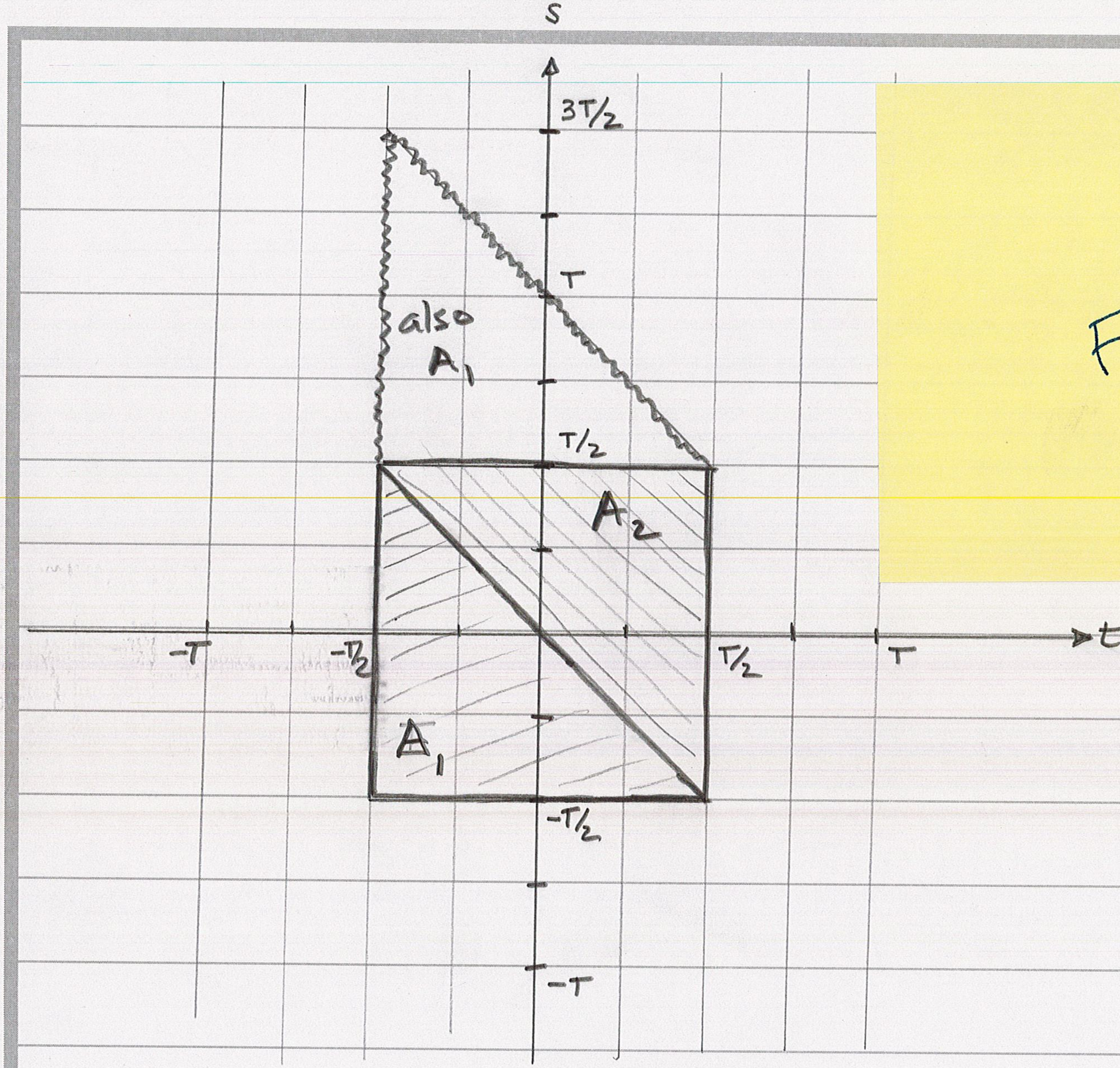
Trying to ...

$$\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} I(t,s)\,dt\,ds = \int_{t=-T/2}^{T/2}\int_{\tau} I(t,-t+\tau)\,dt\,d\tau$$

↳ must be careful in computing the range for τ.



Finish Monday.



Finish Monday.