(Wednesday)

# Summary of Week 5

DT $\quad x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \implies y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$

impulse $\qquad$ impulse response

CT $\quad x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \implies y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$

Computational Examples

Properties of LTI systems

$$x * h = h * x$$
$$x * (h_1 + h_2) = x * h_1 + x * h_2$$
$$x * (y * z) = (x * y) * z$$

LTI memoryless $\iff h[n] = K\delta[n]$; $h(t) = K\delta(t)$

LTI $h$ invertible $\iff \exists\, h_{inv}$ st $h * h_{inv} = \delta$

LTI causal $\iff h[n] = 0 \;\; n<0$; $h(t) = 0 \;\; t<0$

Stability means: Bounded inputs result in bounded outputs

BIBO means $\forall\, K_{out} > 0 \;\exists\, K_{in} > 0$ st if $|x[n]| < K_{in} \;\forall n \implies |y[n]| < K_{out}$

Thm: BIBO $\iff \sum_n |h[n]| < \infty$

$\int$

# Solution: Homog. nth order linear diff. eqn. with constant real coeffs

$$y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_0 y(t) = 0 \quad (*) \qquad y^{(k)} = y^{(k)}(t) = \frac{d^k}{dt^k} y(t)$$

Let the characteristic polynomial of (*) be:

$$s^n + a_{n-1} s^{n-1} + \cdots + a_0 = (s-s_1)^{r_1} (s-s_2)^{r_2} \cdots (s-s_k)^{r_k}$$

To get char poly assume soln of form $y(t) = e^{st}$ and substitute into (*)

and on the right hand side we indicate its factorization into distinct roots $s_i$. Then

(a) If $s_i$ is real, then the functions

$$e^{s_i t}, \; t e^{s_i t}, \; \ldots, \; t^{r_i - 1} e^{s_i t}$$

are linearly independent solutions of (*).

(b) If $s_i = a_i + j b_i$ ($a_i, b_i$ real; $b_i \neq 0$) then the functions

$$e^{a_i t} \sin b_i t, \quad t e^{a_i t} \sin b_i t, \quad \ldots, \quad t^{r_i - 1} e^{a_i t} \sin b_i t$$
$$e^{a_i t} \cos b_i t, \quad t e^{a_i t} \cos b_i t, \quad \ldots, \quad t^{r_i - 1} e^{a_i t} \cos b_i t$$

are linearly independent solutions of (*)

roots $s_i$ come in complex conjugate pairs

(c) Every solution of (*) can be written uniquely as a linear combination of solutions in (a) and (b).

# Soln: Homog. Nth order linear difference eqn. with constant real coeffs.

$$y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = 0 \quad (*)$$

To get char. poly. assume soln. of form $y[n] = r^n$ and plug into (*)

Let the characteristic polynomial of (*) be:

$$r^N + a_1 r^{N-1} + \cdots + a_0 = (r - r_1)^{m_1} (r - r_2)^{m_2} \cdots (r - r_k)^{m_k}$$

where on the right hand side we indicate its factorization into distinct roots $r_i$. Then

(a) If $r_i$ is real, then the functions

$$r_i^n, \; n r_i^n, \; \ldots, \; n^{m_i - 1} r_i^n$$

are linearly independent solutions of (*).

(b) For complex-conjugate roots $r_i = \rho_i e^{\pm j\phi_i}$ then the functions

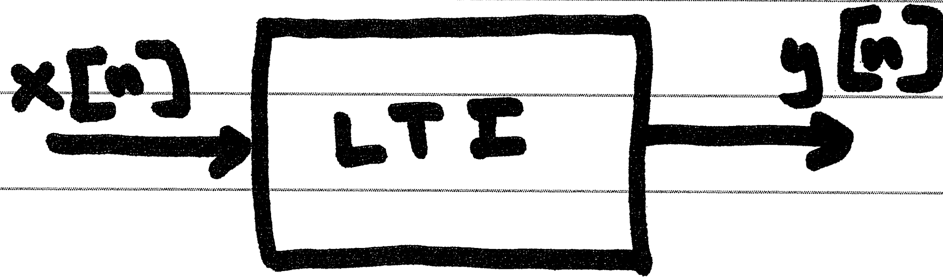
$$\rho_i^n \cos n\phi_i, \quad n\rho_i^n \cos n\phi_i, \quad \ldots, \quad n^{m_i-1} \rho_i^n \cos n\phi_i$$

$$\rho_i^n \sin n\phi_i, \quad n\rho_i^n \sin n\phi_i, \quad \ldots, \quad n^{m_i-1} \rho_i^n \sin n\phi_i$$

are linearly independent solutions of (*).

(c) Every solution of (*) can be written uniquely as a linear combination of solutions in (a) and (b).

# Example: Find impulse response from difference equation




Syst. initially at rest + satisfies:

$$y[n] - \frac{1}{2} y[n-1] = x[n]$$

$$n \geq 0$$

Suppose that $x[n] = \delta[n] \Longrightarrow y[n] = h[n]$. Must assume "at rest" before the application $\delta[n]$. Thus: $y[-1] = h[-1] = 0$

$\Longrightarrow h[n] - \frac{1}{2} h[n-1] = \delta[n]$ for $n \geq 0$ with $h[-1] = 0$

$n=0$ $\quad h[0] - \frac{1}{2} h[-1] = 1 \Longrightarrow h[0] = 1$

$n>0$ $\quad h[n] - \frac{1}{2} h[n-1] = 0$ for $n > 0$ $\rightarrow$ solve in general using char. eqN idea.

Char eqN is: $r - \frac{1}{2} = 0 \Longrightarrow$ root $= 1/2$

$\therefore h[n] = c\left(\frac{1}{2}\right)^n \quad n \geq 0$ with $h[0] = 1 \Longrightarrow c = 1.$

Char. EqN. Assume $h[n] = r^n$ & plug in

$$h[n] - \frac{1}{2} h[n-1] = 0 \quad n > 0.$$

$$r^n - \frac{1}{2} r^{n-1} = 0$$

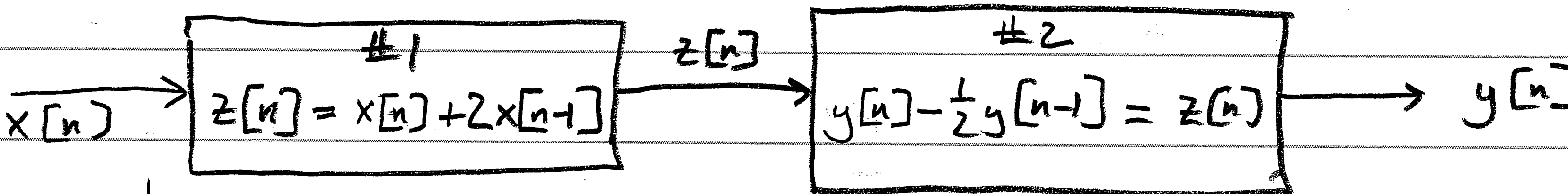
$$r^{n-1}\left(r - \frac{1}{2}\right) = 0$$

# Change the system:

$$y[n] - \frac{1}{2} y[n-1] = x[n] + 2x[n-1] \quad n \geq 0$$

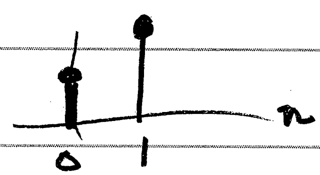
(initially at rest)

Can think of above as a cascade of two systems:




This (by inspection)

$$h_1[n] = \delta[n] + 2\delta[n-1]$$




$$h_1[n] = \left(\delta[n] + 2\delta[n-1]\right) u[n]$$

impulse response found in prev. example

$$h_2[n] = \left(\frac{1}{2}\right)^n u[n]$$

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$= \begin{cases} \left(\frac{1}{2}\right)^n & n \geq 0 \\ 0 & n < 0. \end{cases}$$

Goal: Find impulse resp. of cascade.

$$h_{overall}[n] = h_1[n] * h_2[n]$$

$$= \left(\delta[n] + 2\delta[n-1]\right) * \left\{\left(\tfrac{1}{2}\right)^n u[n]\right\}$$

$$= \delta[n] * \left\{\left(\tfrac{1}{2}\right)^n u[n]\right\} + 2\delta[n-1] * \left\{\left(\tfrac{1}{2}\right)^n u[n]\right\}$$

$$= \left(\tfrac{1}{2}\right)^n u[n] + 2\left(\tfrac{1}{2}\right)^{n-1} u[n-1]$$

$$= \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ \underbrace{\left(\tfrac{1}{2}\right)^n + 2\left(\tfrac{1}{2}\right)^{n-1}}_{5\left(\tfrac{1}{2}\right)^n} & n \geq 1 \end{cases}$$

# Impulse Response From Differential Eqn:

$$\sum_{k=0}^{N} a_k y^{(k)}(t) = x(t) \qquad a_N \neq 0$$

Initially at rest

$$y(0^-) = 0 \quad y^{(1)}(0^-) = 0 \ \ldots \ y^{(N-1)}(0^-) = 0$$

$$x(t) \longrightarrow \boxed{\begin{matrix} \text{LTI} \\ h(t) \end{matrix}} \longrightarrow y(t)$$