(Friday)
Summary of Week 5

DT $\quad x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\underbrace{\delta[n-k]}_{\text{impulse}} \Longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]\,\underbrace{h[n-k]}_{\text{impulse response}}$

CT $\quad x(t) = \int_{-\infty}^{\infty} x(\tau)\,\overbrace{\delta(t-\tau)}^{\text{impulse}}\,d\tau \Longrightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\,\overbrace{h(t-\tau)}^{\text{impulse response}}\,d\tau$

Computational Examples → See Monday's Lecture

Properties of LTI systems
- commutative: $x*h = h*x$
- distributes over $+$: $x*(h_1+h_2) = (x*h_1)+(x*h_2)$
- associative: $x*y*z = (x*y)*z = x*(y*z)$

LTI memoryless $\Longleftrightarrow h[n] = K\delta[n]$; $h(t) = K\delta(t)$

$\forall$

LTI h invertible $\Longleftrightarrow \exists\, h_{inv}$ st. $h*h_{inv} = \delta$

LTI causal $\Longleftrightarrow h[k] = 0$ for $k<0$; $h(t)=0$ for $t<0$.

Stability means: Bounded inputs result in bounded outputs

BIBO $\Longleftrightarrow \sum_{k=-\infty}^{\infty} |h[k]| < \infty$; $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$

Solution: Homog. nth order linear diff. eqn. with constant real coeffs

$$y^{(n)}(t) + a_{n-1}\, y^{(n-1)}(t) + \cdots + a_0\, y(t) = 0 \quad (*) \qquad y^{(k)} = y^{(k)}(t) = \frac{d^k}{dt^k} y(t)$$

Let the characteristic polynomial of $(*)$ be:

$$s^n + a_{n-1} s^{n-1} + \cdots + a_0 = (s-s_1)^{r_1}(s-s_2)^{r_2} \cdots (s-s_k)^{r_k}$$

To get char poly assume soln of form $y(t) = e^{st}$ and substitute into $(*)$

and on the right hand side we indicate its factorization into distinct roots s_i. Then

(a) If s_i is real, then the functions

$$e^{s_i t},\; t e^{s_i t},\; \ldots,\; t^{r_i - 1} e^{s_i t}$$

are linearly independent solutions of $(*)$.

(b) If $s_i = a_i + j b_i$ (a_i, b_i real; $b_i \neq 0$) then the functions

$$e^{a_i t} \sin b_i t,\; t e^{a_i t} \sin b_i t,\; \ldots,\; t^{r_i - 1} e^{a_i t} \sin b_i t$$
$$e^{a_i t} \cos b_i t,\; t e^{a_i t} \cos b_i t,\; \ldots,\; t^{r_i - 1} e^{a_i t} \cos b_i t$$

are linearly independent solutions of $(*)$

roots s_i come in complex conjugate pairs

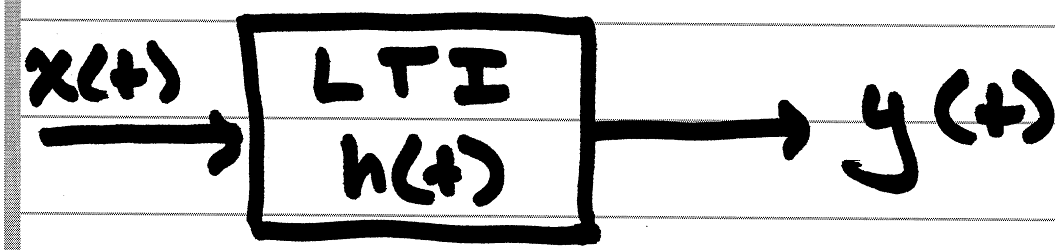
(c) Every solution of $(*)$ can be written uniquely as a linear combination of solutions in (a) and (b).

Impulse Response From Differential Eqn :

$$\sum_{k=0}^{N} a_k y^{(k)}(t) = x(t) \qquad a_N \neq 0$$

Initially at rest

$$y(0-) = 0 \quad y^{(1)}(0^-) = 0 \ \dots \ y^{(N-1)}(0-) = 0$$



Apply $x(t) = \delta(t)$ to the system ⊛ at rest at $t = 0^-$. Solve for output $y(t) = h(t)$ $t > 0$

$$\sum_{k=0}^{N} a_k \underbrace{y^{(k)}(t)}_{h^{(k)}(t)} = \delta(t)$$

where the "at rest" assumpt. @ $t = 0^-$ means
$h(0^-) = 0$ $h^{(1)}(0^-) = 0$
$\ldots$ $h^{(N-1)}(0^-) = 0$

Above is not a homog. eqN. However if $t > 0$ then $\delta(t) = 0$ and can rewrite above as

$$\sum_{k=0}^{N} a_k h^{(k)}(t) = 0 \quad \text{for } t > 0$$

with new initial conditions

$$h(0^+) = ? \quad h^{(1)}(0^+) = ? \quad \ldots \quad h^{(N-1)}(0^+) = ?$$

What ICs to Use in:

$$\sum_{k=0}^{N} a_k h^{(k)}(t) = 0 \quad \text{for } t > 0$$

i.e. $h(0^+) = ?\quad h^{(1)}(0^+) = ? \cdots h^{(N-1)}(0^+) = ?$

Need to go from $t=0^-$ to $t=0^+$ in some way.

$$(*) \quad \sum_{k=0}^{N} a_k h^{(k)}(t) = \delta(t) \qquad t \geq 0^-$$

Technique is to integrate both sides from 0^- to 0^+: $\int_{0^-}^{0^+} (\cdots)\, dt$

$$\Longrightarrow \sum_{k=0}^{N} a_k \int_{0^-}^{0^+} h^{(k)}(t) = \int_{0^-}^{0^+} \delta(t)\, dt = 1$$

Note: Indefinite integral (ie antiderivative)

$$\int h^{(k)}(t)\, dt = h^{(k-1)}(t)$$

$$1 = a_0 \int_{0^-}^{0^+} h(t)\,dt + \sum_{k=1}^{N} a_k \underbrace{\int_{0^-}^{0^+} h^{(k)}(t)\,dt}_{h^{(k-1)}(t)\Big|_{t=0^-}^{0^+}}$$

$$= a_0 \int_{0^-}^{0^+} h(t)\,dt + \sum_{k=1}^{N} a_k \left[h^{(k-1)}(0^+) - \underbrace{h^{(k-1)}(0^-)}_{=0} \right]$$

$= 0$ by the at rest assumption.

$$\Longrightarrow$$

$$1 = \underbrace{a_0 \int_{0^-}^{0^+} h(t)\,dt}_{} + a_N h^{(N-1)}(0^+) + \underbrace{a_{N-1} h^{(N-2)}(0^+) + \cdots + a_1 h(0^+)}_{}$$

We will argue/convince you into believing that all these equal zero.

If you come to believe as I do ..

$$1 = a_N h^{(N-1)}(0^+) \Longrightarrow h^{(N-1)}(0^+) = 1/a_N$$

all the rest are zero.

Smoothness Observation

Everytime we take a derivative we get "less smooth"

$$h(t) \quad h^{(1)}(t) \quad h^{(2)}(t) \quad \ldots \quad h^{(N-1)}(t) \quad h^{(N)}(t)$$

$\longrightarrow$ smoothness decreases going this way

Other observation is that a differential equation requires derivatives to exist.

Conclude: Solution $h(t)$ to (*) with $x(t) = \delta(t)$ will have

$h^{(N)}(t)$ has a delta @ $t=0$ to match $\delta(t)$

$h^{(N-1)}(t)$ has a jump discontinuity at $t=0$

$h^{(N-2)}(t)$ ... $h^{(1)}(t)$ $h(t)$ are continuous at $t=0$

$$\Longrightarrow \quad h(0^+) = h(0^-) = 0$$

$$h^{(1)}(0^+) = h^{(1)}(0^-) = 0$$

$$\vdots$$

$$h^{(N-2)}(0^+) = h^{(N-2)}(0^-) = 0$$

$$h^{(N-1)}(0^+) \neq h^{(N-1)}(0^-)$$ because of the jump discontinuity.

$$\Longrightarrow \quad \int_{0^-}^{0^+} h(t)\,dt = 0$$

Extension for $\sum_{k=0}^{N} a_k y^{(k)}(t) = \sum_{\ell=0}^{M} b_\ell x^{(\ell)}(t)$

at rest at $t = 0^-$

$$x(t) \longrightarrow \boxed{\sum_{\ell=0}^{M} b_\ell x^{(\ell)}(t) = z(t)} \xrightarrow{z(t)} \underbrace{\boxed{\sum_{k=0}^{N} a_k y^{(k)}(t) = z(t)}}_{\text{just did this}} \longrightarrow y(t).$$

Equiv. to:

$$\underset{\underset{\delta(t)}{\parallel}}{z(t)} \longrightarrow \boxed{\sum_{k=0}^{N} a_k y^{(k)}(t) = z(t)} \xrightarrow[h_{before}(t)]{} \boxed{\sum_{\ell=0}^{M} b_\ell \frac{d^\ell}{dt^\ell}(\cdot)} \longrightarrow \underset{\underset{h_{overall}(t)}{\parallel}}{y(t)}$$

$$\therefore \quad h_{overall}(t) = \sum_{\ell=0}^{M} b_\ell \, h_{before}^{(\ell)}(t)$$

Lets do this part now while we are in theory mode.
Let $h(t)$ be solution to

$$\sum_{k=0}^{N} a_k h^{(k)}(t) = \delta(t) \qquad t \geq 0 \qquad h(0^-)=0,\ h^{(1)}(0^-)=0 \ldots \ h^{(N-1)}(0^-)=0.$$

here $h(t) = h_{before}(t)$

As we argued (non rigorously) in part (b):

$h^{(N)}(t)$ has delta at $t=0$

$h^{(N-1)}(t)$ has step discont. at $t=0$

$h^{(N-2)}(t) \ldots h(t)$ are continuous at $t=0$.

Thus if in (**)

$M < N \implies h_{overall}(t)$ will have no deltas

$M = N \implies$ " " " a delta

$M > N \implies$ " " " a delta, and derivatives of delta up to order $M-N$.

Example

$$y^{(2)}(t) + 3y^{(1)}(t) + 2y(t) = x(t)$$

Find $h(t)$.

$\Longrightarrow$ Solve $h^{(2)}(t) + 3h^{(1)}(t) + 2h(t) = 0 \qquad t \geq 0^+$

with $h^{(1)}(0^+) = 1/1 = 1 \qquad h(0^+) = 0$

$\Rightarrow$ Char. eqN : $s^2 + 3s + 2 = 0$

$(s+1)(s+2)$

$$h(t) = \underbrace{c_1}_{1} e^{-t} + \underbrace{c_2}_{-1} e^{-2t} \qquad t \geq 0$$

$$\Longrightarrow h(t) = \left(e^{-t} - e^{-2t}\right)u(t)$$

What if change the system to:

$$y^{(2)}(t) + 3y^{(1)}(t) + 2y(t) = x^{(2)}(t) + 2x^{(1)}(t) + 3x(t)$$

$$\Rightarrow h_{overall}(t) = h^{(2)}_{before}(t) + 2h^{(1)}_{before}(t) + 3h_{before}(t)$$

where $h_{before}(t) = h(t) = \underbrace{\left(e^{-t} - e^{-2t}\right)u(t)}$

when taking derivatives be sure to use the product rule to include the unit step and its derivatives.

$$h(t) = \left(e^{-t} - e^{-2t}\right)u(t)$$

$$h'(t) = \left(\frac{d}{dt}\left\{e^{-t} - e^{-2t}\right\}\right)u(t) + \left(e^{-t} - e^{-2t}\right)\frac{du(t)}{dt}$$

$$= \left(-e^{-t} + 2e^{-2t}\right)u(t) + \underbrace{\left(e^{-t} - e^{-2t}\right)\delta(t)}_{\text{note that this is actually } 0.}$$

$$h''(t) = \left(\frac{d}{dt}\left\{-e^{-t} + 2e^{-2t}\right\}\right)u(t) + \left(-e^{-t} + 2e^{-2t}\right)\delta(t)$$

$$= \left(e^{-t} - 4e^{-2t}\right)u(t) + \underbrace{\left(-e^{-t} + 2e^{-2t}\right)}_{\text{now equals } -1+2 = 1}\delta(t)$$

$$\therefore\ h''(t) = \left(e^{-t} - 4e^{-2t}\right)u(t) + \delta(t).$$