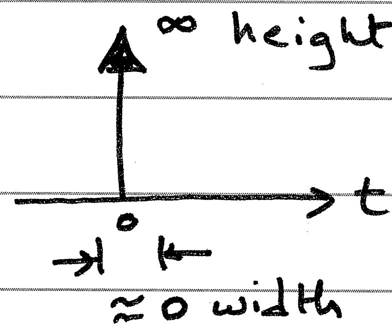
Unit Impulse/Unit Step Functions

① Discrete-Time (Trivial)

$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases} \qquad u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$

② Continuous-Time

$$\delta(t) = ?$$



$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

What is this Dirac delta function?

Dirac Delta

Technically $\delta(t)$ is not a function (can't have unit area, $+\infty$ @ $t=0$, 0 @ $t \neq 0$, a normal function)

Instead define it to be a

functional ie a mapping from functions to real #

$\left\{ \begin{array}{l} \text{some space or} \\ \text{set of signals} \end{array} \right\} \longrightarrow \mathbb{R}$ or $\mathbb{C}$

For our purposes we give $\delta(t)$ a meaning only inside integral ...

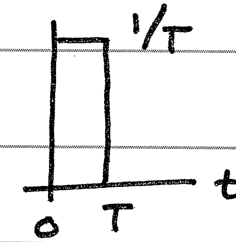
$$\int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(0)$$

$\forall$ functions f cont. @ $t=0$

Note: This functional is <u>linear</u>.

Dirac Delta as a Limit

$$\delta(t) = \lim_{T \to 0} p_T(t)$$ where



is $p_T(t)$

The meaning of limit used here:

$$\lim_{T \to 0} \int_{-\infty}^{\infty} p_T(t)\, f(t)\, dt = \lim_{T \to 0} \frac{1}{T} \int_{0}^{T} f(t)\, dt$$

for f continuous at $t = 0$.

Recall Fundamental Theorem of Calculus

Assume $f:[a,b] \longrightarrow \mathbb{R}$ is continuous

Define
$$F(x) \triangleq \int_a^x f(t)\,dt \qquad a \leq x \leq b$$

Let c be any point such that $a \leq c \leq b$

Then:

F has a derivative at c

and $F'(c) = f(c)$

Re: $\int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(0)$ $\forall$ functions f cont. @ $t=0$

Consider

$$\int_{-\infty}^{\infty} p_T(t) f(t)\, dt = \int_0^T \frac{1}{T} f(t)\, dt = \frac{1}{T}\int_0^T f(t)\, dt$$

for $f(t)$ continuos @ $t=0$. As prev. def $F(s) = \int_{-\infty}^{s} f(t)\,dt$

$$\therefore \frac{\int_0^T f(t)\, dt}{T} = \frac{F(T) - F(0)}{T}$$

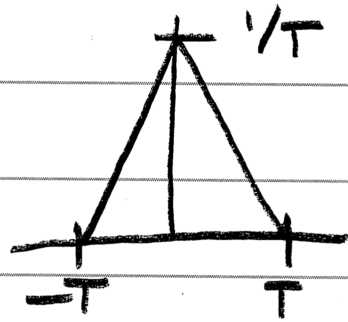
Take limit of above a $T \to 0$. $F'(0) \triangleq \lim_{T\to 0} \frac{F(T)-F(0)}{T}$

and is $= f(0)$ by F.T.C.

In this sense $\lim_{T\to 0} p_T(t) = \delta(t)$

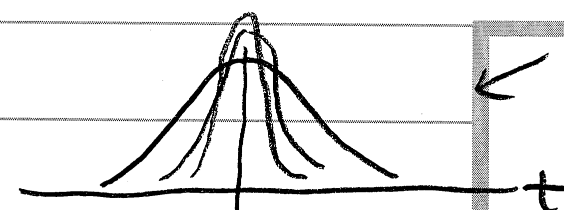
Other tall, skinny Pulse Families

Try



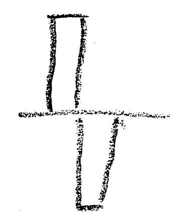
Any family of integrable functions that get "tall and skinny" and have area = 1 for each family member will work to make $\delta(t)$ as $T \to 0$.

$$\frac{1}{\sqrt{2\pi}\,T} e^{-t^2/2T^2} \qquad t \in \mathbb{R}$$

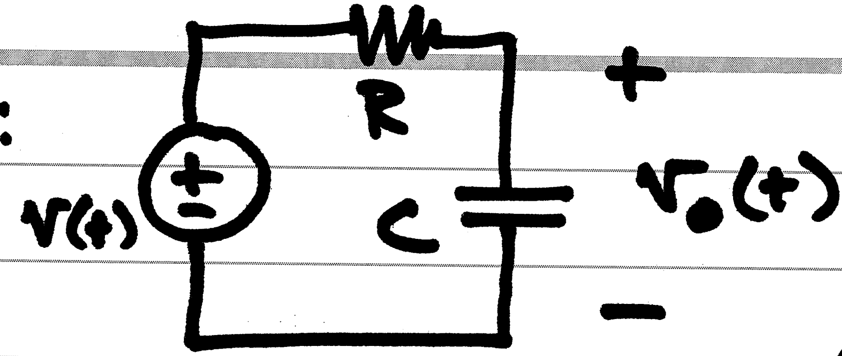


get taller & narrower as $T \to 0$

<u>Question for You</u> Can we define $\delta'(t)$? What should it be?



Example:



$\tau = RC$

Unit step response

$v(t) = V_o u(t)$; $v_o(0^-) = 0$

$\Rightarrow v_o(t) = V_o\left(1 - e^{-t/\tau}\right) \quad t \geq 0$

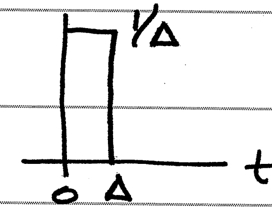
zero input response

$v(t) = 0$; $v_o(0^-) = V_1$

$\Rightarrow v_o(t) = V_1 e^{-t/\tau} \quad t \geq 0$

Use these to find solutions to

$$v(t) = p_\Delta(t) = \begin{cases} 1/\Delta & 0 \leq t \leq \Delta \\ 0 & \text{else} \end{cases}$$

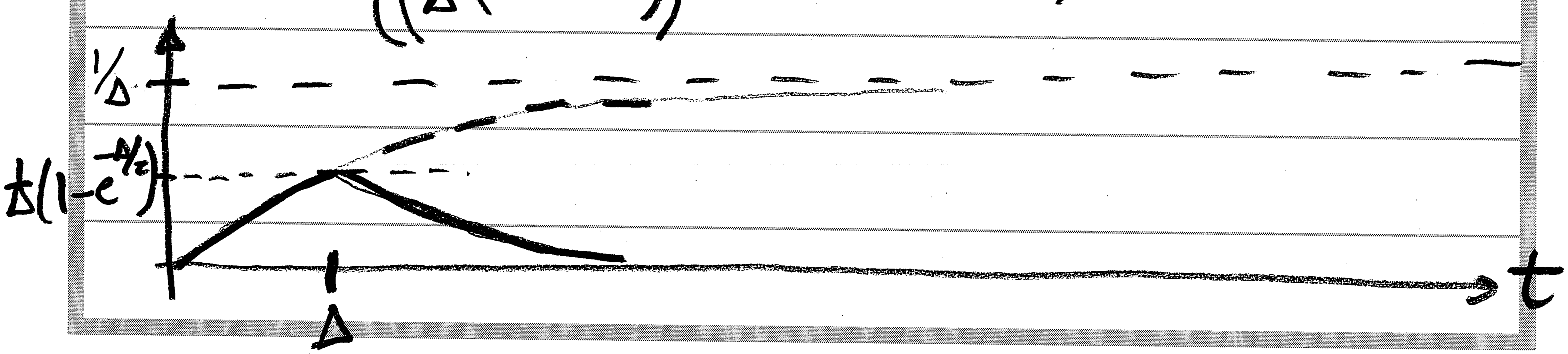


To solve break time up into intervals ① $0 \leq t \leq \Delta$; ② $t \geq \Delta$

① $0 \leq t \leq \Delta \longrightarrow$ use unit step resp. solution

② $t \geq \Delta \longrightarrow$ use the zero input resp.

$$v_o(t) = \begin{cases} \frac{1}{\Delta}\left(1 - e^{-t/\tau}\right) & 0 \leq t \leq \Delta \\ \left(\frac{1}{\Delta}\left(1 - e^{-\Delta/\tau}\right)\right) e^{-(t-\Delta)/\tau} & t \geq \Delta \end{cases}$$



Case of interest: $\Delta << \tau$

$\Delta << \tau \iff \frac{\Delta}{\tau} << 1 \implies e^{-\Delta/\tau} \approx 1 - \frac{\Delta}{\tau}$

Look a e^x Taylor series at $x=0$.

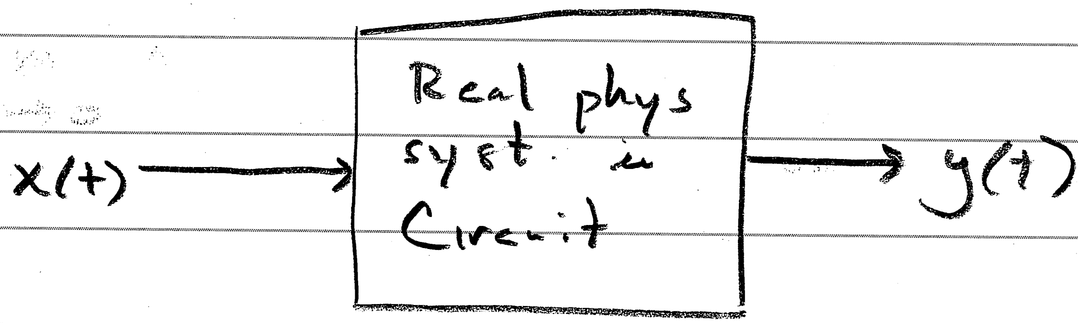
Term in the orig. $t > \Delta$ solution

$$\frac{1 - e^{-\Delta/\tau}}{\Delta} \approx \frac{1 - \left(1 - \frac{\Delta}{\tau}\right)}{\Delta} = \frac{1}{\tau}$$

$$\therefore \lim_{\Delta \to 0} v_o(t) = \underbrace{\frac{1}{\tau} e^{-t/\tau}}_{\text{this is the impulse response of the RC circuit.}} \quad 0 < t < \infty$$

Physical Systems & Smoothing

Inputs that are appox zero outside a common & suff. short interal have essentially the same effect at the output as long as the areas of inputs are equal.



Sifting Property of Delta

$$\int_{-\infty}^{\infty} \delta(t)\, f(t)\, dt = f(0) \quad \text{if } f(\cdot) \text{ cont @ } t=0.$$

$$\Longrightarrow \int_{-\infty}^{\infty} \delta(t-s)\, f(t)\, dt = \int_{-\infty}^{\infty} \delta(\tau)\, f(\tau+s)\, d\tau = f(s)$$

$\tau = t-s$
$d\tau = dt$

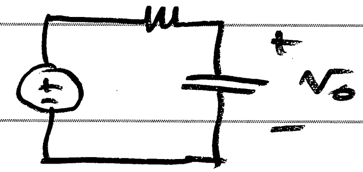
prov. cont. at $t=s$.

This is really a statement of fact that any continuos signal $f(t)$ is a "linear combination" of delta.

We had define "coordinate sigs" in discrete time for same purpose.

Topic: Now System Theory

① CT example



→ equations are differential equations.

② DT → Equations are difference equations

<u>Example</u>: Borrowing a sum of money, interest rate, how to determine payment?

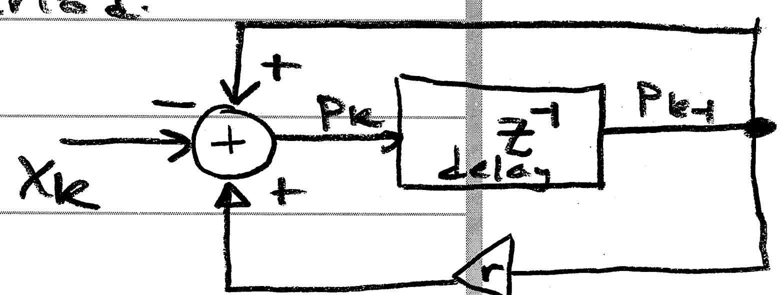
$p_k \triangleq$ principal owed @ end of period k (p_0 = initial loan amount)

N = # of payments to pay off $p_N = 0$.

x_k = payment made at time k

r = interest rate per period.

$$p_k = p_{k-1} + r p_{k-1} - x_k$$



Find to make $p_N = 0$

$$x = \frac{(1+r)^N}{(1+r)^N - 1} (r p_0)$$