Recall: $x[n] = e^{j\omega_0 n}$ $n \in \mathbb{Z}$

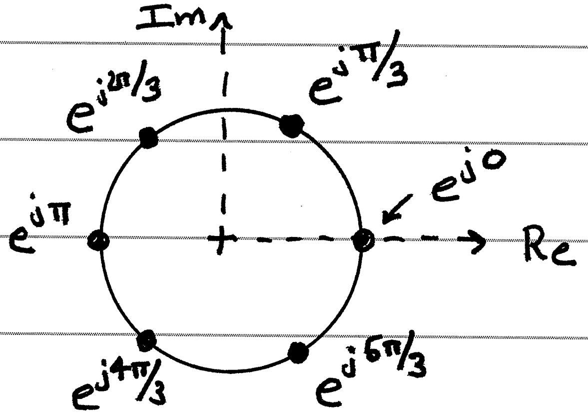
① Periodic in ω_0, the frequency variable, with period 2π:

$$e^{j\omega_0 n} = e^{j(\omega_0 + 2\pi)n} \qquad \forall n \in \mathbb{Z}$$

② Signal $x[n] = e^{j\omega_0 n}$ is periodic in n of period N if and only if ω_0 is of the form

$$\omega_0 = \frac{2\pi}{N} k \qquad k \text{ an integer}$$

↳ since $e^{j\omega_0}$ is periodic in ω_0 we can see that only need consider the N values of ω_0 for $k = 0, 1, 2, \ldots, N-1$



} case $N = 6$
"the 6th roots of unity"

The Harmonically Related Complex Exponentials (HRCE) in DT

Define these

$$\left.\begin{aligned}
\phi_0[n] &= e^{j0\cdot n} && n \in \mathbb{Z}\\
\phi_1[n] &= e^{j2\pi n/N} && \text{"}\\
&\;\vdots\\
\phi_{N-1}[n] &= e^{j2\pi(N-1)n/N} && \text{"}
\end{aligned}\right\}
\begin{gathered}
\phi_k[n] = e^{j2\pi kn/N}\\
n \in \mathbb{Z}\\
k = 0, 1, \dots N-1
\end{gathered}$$

$\Rightarrow$ $\phi_k[n]$ → time index

k ↙ freq. index

Looked at signals as $\phi_k[\cdot]$ time functions, or as vectors ϕ_k

$$\phi_0 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \qquad \phi_1 = \begin{bmatrix} 1 \\ e^{j2\pi/N} \\ \vdots \\ e^{j2\pi(N-1)/N} \end{bmatrix} \qquad \cdots \qquad \phi_{N-1} = \begin{bmatrix} 1 \\ e^{j2\pi(N-1)\cdot 1/N} \\ \vdots \\ e^{j2\pi(N-1)(N-1)/N} \end{bmatrix}$$

Remember the Test Signals

For the vector space of length N discrete-time signals we noted that the N coordinate signals

$$x_0 = [1, 0, \dots\ 0]^T$$

$$x_1 = [0, 1, 0 \dots\ 0]^T$$

$$\vdots$$

$$x_{N-1} = [0, 0 \dots\ 0, 1]^T$$

could be used to uniquely express any length N signal as a linear combination:

$$z[\cdot] = \sum_{k=0}^{N-1} z[k]\, x_k[\cdot]$$

This set of special signals $\{x_k[\cdot]\ k=0,1,\dots N-1\}$ is called a basis of the vector space.

Another Basis

Fact: $\{\phi_k : k=0,1,\dots N-1\}$ is a basis.

$$x[\cdot] = \sum_{k=0}^{N-1} a_k \phi_k[\cdot]$$

or

$$x = \sum_{k=0}^{N-1} \underbrace{a_k}_{\substack{\text{scalar}\\ \mathbb{C}}} \underbrace{\phi_k}_{\substack{\text{vectors}\\ \mathbb{C}^N}}$$

Two conditions define this notion

① every signal must be expressable as a linear combination of basis sigs

② expression is unique.

Also can write as a matrix times a vector:

$$\underbrace{x}_{N\times 1} = \underbrace{\begin{bmatrix}\phi_0 & \phi_1 & \cdots & \phi_{N-1}\end{bmatrix}}_{\substack{N\times N\\ \Phi}} \underbrace{\begin{bmatrix}a_0\\ a_1\\ \vdots\\ a_{N-1}\end{bmatrix}}_{N\times 1}$$

Notation of Φ: Call elements $\varphi_{\underbrace{p}_{\text{row index}}\underbrace{q}_{\text{column index}}}$

$$\varphi_{pq} = e^{j2\pi pq/N}$$

Say $N=3$ $\quad \Phi = \left[e^{j2\pi pq/3}\right]_{p=0\ q=0}^{2}$

$$\begin{array}{c|ccc}
 & q=0 & q=1 & q=2 \\
\hline
p=0 & 1 & 1 & 1 \\
p=1 & 1 & e^{j2\pi/3} & e^{j4\pi/3} \\
p=2 & 1 & e^{j4\pi/3} & \boxed{e^{j8\pi/3}}
\end{array}$$

$e^{j2\pi/3}$

$$e^{j8\pi/3} = e^{j\left(\frac{8\pi}{3}-2\pi\right)} = e^{j\frac{8\pi}{3}} \cdot \underbrace{e^{-j2\pi}}_{1}$$

$$\frac{8\pi}{3} - \frac{6\pi}{3} = \frac{2\pi}{3}$$

Say $N=5$ $$\Phi=\left[e^{j2\pi pq/5}\right]_{p=0\ \ q=0}^{4\ \ \ \ 4}$$

	$q=0$	$q=1$	$q=2$	$q=3$	$q=4$
$p=0$	1	1	1	1	1
$p=1$	1	$e^{j2\pi/5}$	$e^{j4\pi/5}$	$e^{j6\pi/5}$	$e^{j8\pi/5}$
$p=2$	1	$e^{j4\pi/5}$	$e^{j8\pi/5}$	$e^{j12\pi/5}$	$e^{j16\pi/5}$
$p=3$	1	$e^{j6\pi/5}$	$e^{j12\pi/5}$	$e^{j18\pi/5}$	$e^{j24\pi/5}$
$p=4$	1	$e^{j8\pi/5}$	$e^{j16\pi/5}$	$e^{j24\pi/5}$	$e^{j32\pi/5}$

$\{\phi_k : k = 0, 1, \cdots N-1\}$ is a basis for space of N length sigs $\Longleftrightarrow$ The N×N matrix Φ is invertible

$$\Phi^{-1} = \frac{1}{N}\Phi^{H}$$

Hermitian Transpose $\quad \Phi^{H} = (\Phi^{*})^{T}$

Proof for N=3

Find a Φ^{-1} and Show that

$$\Phi\,\Phi^{-1} = I$$

$$\overbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{j2\pi/3} & e^{j4\pi/3} \\ 1 & e^{j4\pi/3} & e^{j8\pi/3} \end{bmatrix}}^{\Phi} \overbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-j2\pi/3} & e^{-j4\pi/3} \\ 1 & e^{-j4\pi/3} & e^{-j8\pi/3} \end{bmatrix}}^{\Phi^H}$$

$$= \begin{bmatrix} 1+1+1=3 & \begin{array}{c} 1+e^{-j2\pi/3}+e^{-j4\pi/3} \\ =0 \end{array} & \begin{array}{c} 1+e^{-j4\pi/3}+e^{-j8\pi/3} \\ =0 \end{array} \\ \begin{array}{c} 1+e^{j2\pi/3}+e^{j4\pi/3} \\ =0 \end{array} & \begin{array}{c} 1+e^{j2\pi/3}e^{-j2\pi/3}+e^{j4\pi/3}e^{-j4\pi/3} \\ =3 \end{array} & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3\cdot I$$

General Proof

Uses $$\sum_{k=0}^{N-1} a^k = \begin{cases} N & \text{if } a=1 \\ \dfrac{1-a^N}{1-a} & \text{if } a \neq 1 \end{cases}$$

applied to a's of form

$$a = e^{j2\pi \ell / N}$$

The Meaning of $\Phi^{-1} = \frac{1}{N}\Phi^{H}$

$$x = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N] \end{bmatrix} = \underbrace{\begin{bmatrix} \phi_0 & \phi_1 & \cdots & \phi_{N-1} \end{bmatrix}}_{\Phi} \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{N-1} \end{bmatrix}$$

$$x = \Phi X \iff \Phi^{-1} x = \Phi^{-1}\Phi X$$
$$= X$$

$$\begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{N-1} \end{bmatrix} = \underbrace{\frac{1}{N}\Phi^{H}}_{\frac{1}{N}\Phi^{*}} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

Freq. Domain

Time Domain

→ This is a Fourier Transform.

$$\frac{1}{N}\Phi^{*} = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \vdots & & & & \\ \vdots & & \boxed{\text{stuff}} & & \end{bmatrix}$$

Think about what happens when use various time sigs

...

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}; \quad x = \phi_3$$

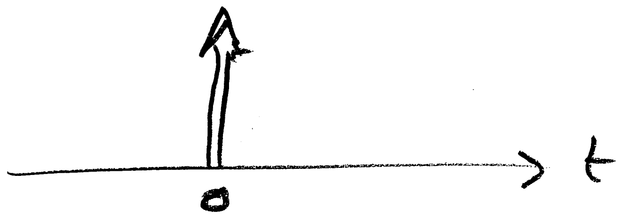
What about CT sigs ?

Dirac Delta $\delta(t) \longrightarrow$ Sometimes people say

$\delta(t) = 0$ for all $t \neq 0$

$\delta(t) =$ $+\infty$ undefined... $t=0$

$\delta[n]$ DT version.



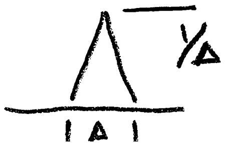
Want $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

$\delta(t)$ is *not* a *normal function*. Called a *functional*

meaning, it only makes sense under an integral ...

$$\int_{-\infty}^{\infty} \delta(t) f(t)\, dt \triangleq f(0)$$

for any continuous $f(t)$.



look as $\Delta \to 0$