(Week 15-d)

Inverse Transforms

Laplace: $$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt \qquad s = \sigma + j\omega$$

$$= \int_{-\infty}^{\infty} x(t)e^{-\sigma t} e^{-j\omega t}\,dt$$

The Laplace transform evaluated at $s = \sigma + j\omega$ is the Fourier Transform of $x(t)e^{-\sigma t}$ evaluated at ω.

Should be able to use the inverse F.T. formula to invert the L.T.

$$\underbrace{x(t)e^{-\sigma t}}_{\tilde{x}_\sigma(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{X(\sigma + j\omega)}_{\tilde{X}_\sigma(j\omega)} e^{j\omega t}\,d\omega$$

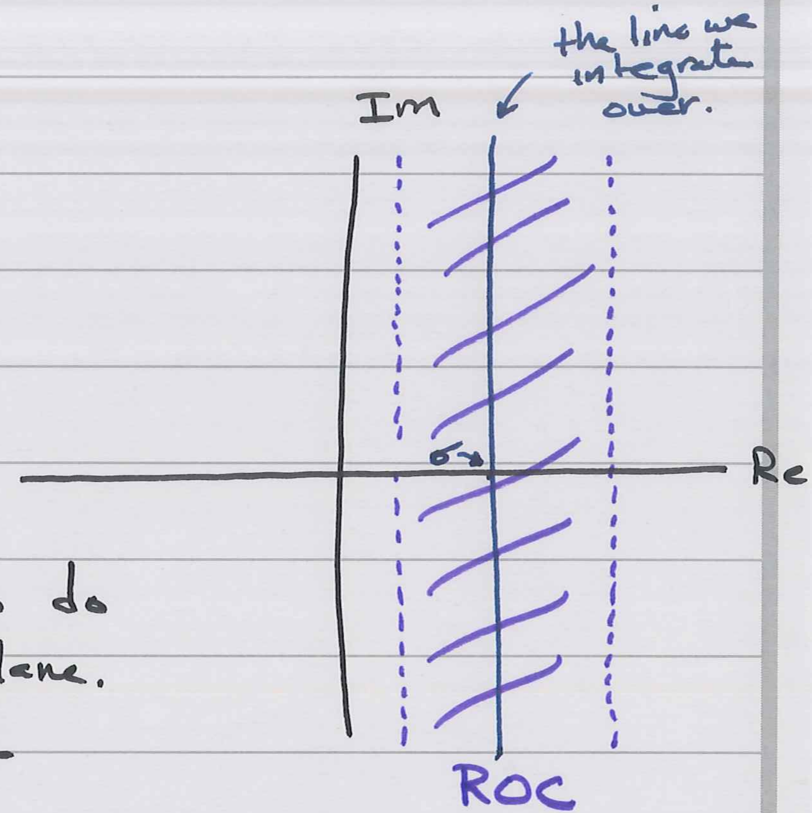
$$\Longrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)\, e^{(\sigma + j\omega)t}\,d\omega.$$

$\Longrightarrow$ Write this as a line integral in the complex plane.

To get line integral formula

$$s = \sigma + j\omega \implies ds = j\,d\omega$$

$$x(t) = \frac{1}{j2\pi}\int_{\sigma - j\omega}^{\sigma + j\omega} X(s)\, e^{st}\, dt$$



This approach to inversion requires us to do contour integration in the complex plane.

The key tools from $\mathbb{C}$ variables are ---

- Jordan's Lemma $\longrightarrow$ converts a line integral into a closed contour integral.
- Cauchy's Theorem $\longrightarrow$ $\displaystyle\oint_C f(s)\,ds = 2\pi j \cdot \left\{ \begin{array}{l} \text{sum of the} \\ \text{residues enclosed} \end{array} \right\}.$

* Outside of 301 scope $\longrightarrow$ Suggest take a course in $\mathbb{C}$ variables.

Lets take a simple example .... no multiple order poles

$$X(s) = \sum_{i=1}^{m} \frac{A_i}{s + a_i}$$

Take the ROC of $X(s)$ and use it to infer the ROCs assoc. with each term in sum

$\mathrm{Re}(s) > -a_i$ $\qquad$ $A_i e^{-a_i t} u(t)$

or

$\mathrm{Re}(s) < -a_i$ $\qquad$ $-A_i e^{-a_i t} u(-t)$

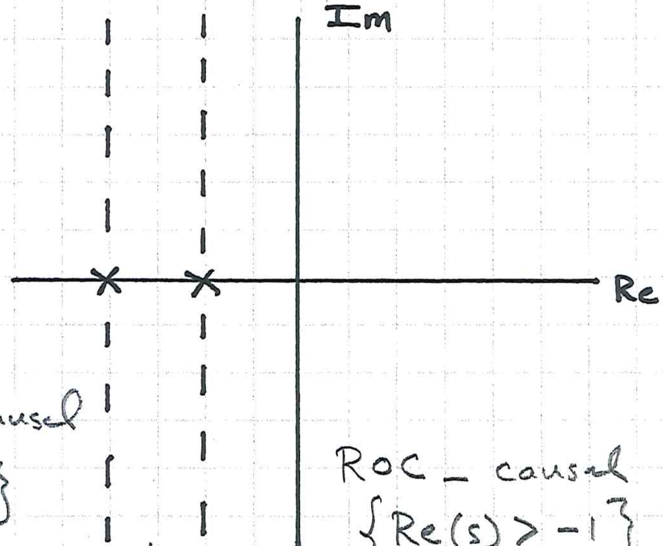
O+W Ex 9.9 – 9.11

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A = \left.\frac{1}{s+2}\right|_{s=-1} = 1$$

$$B = \left.\frac{1}{s+1}\right|_{s=-2} = -1$$

$$\Longrightarrow X(s) = \frac{1}{s+1} - \frac{1}{s+2}$$



Roc_anticausal
$\{Re(s) < -2\}$

Roc_causal
$\{Re(s) > -1\}$

ROC_two sided
$-2 < Re(s) < -1$

① Roc_causal: $x(t) = \{e^{-t} - e^{-2t}\}u(t)$

② Roc_anticausal: $x(t) = \{-e^{-t} + e^{-2t}\}u(-t)$

③ Roc_two sided:
$x(t) = -e^{-t}u(-t) - e^{-2t}u(t)$