(Review of Last Time).

Impulse-Train Sampling

$x(t) =$ an arbitrary CT signal

$p(t) = \sum_n \delta(t-nT)$ = the sampling function

$T =$ sampling interval

$\omega_s = 2\pi/T =$ (radian) sampling frequency

$$x_p(t) = x(t)\,p(t)$$
$$= x(t) \sum_n \delta(t-nT) = \sum_n x(nT)\,\delta(t-nT)$$

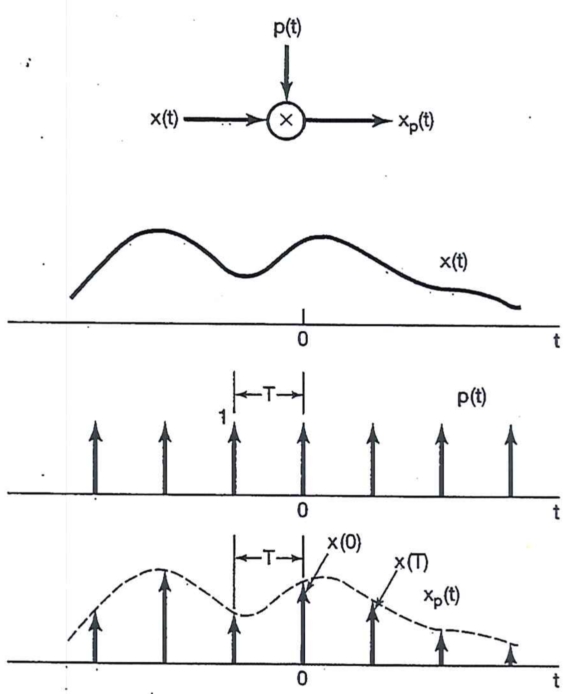


Figure 7.2 Impulse-train sampling.

From the multiplication-in-time/convolution-in-freq. prop. of CTFT we know

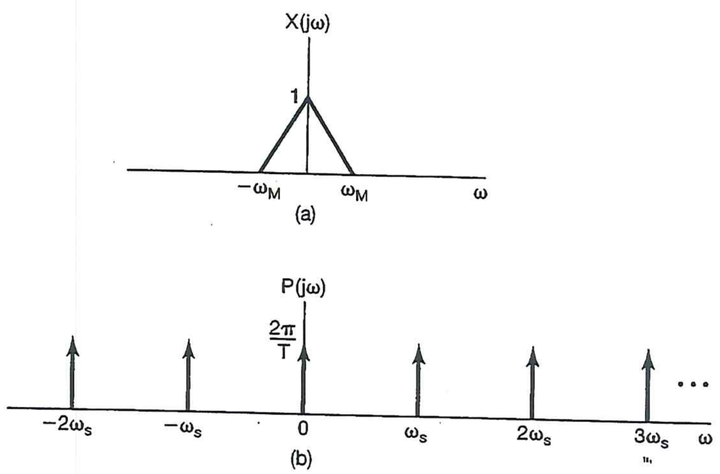
$$X_p(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)\,P(j(\omega-\theta))\,d\theta$$

(Review of Last Time)

Also have

$$\sum_n \delta(t-nT) \longleftrightarrow \frac{2\pi}{T}\sum_k \delta(\omega - k\omega_s) \qquad \omega_s = \frac{2\pi}{T}$$

$$\therefore \quad X_p(j\omega) = \frac{1}{T}\sum_k X(j(\omega - k\omega_s))$$



$\omega_s = 2\pi/T$

Figure 7.3 Effect in the frequency domain of sampling in the time domain: (a) spectrum of original signal; (b) spectrum of sampling function;

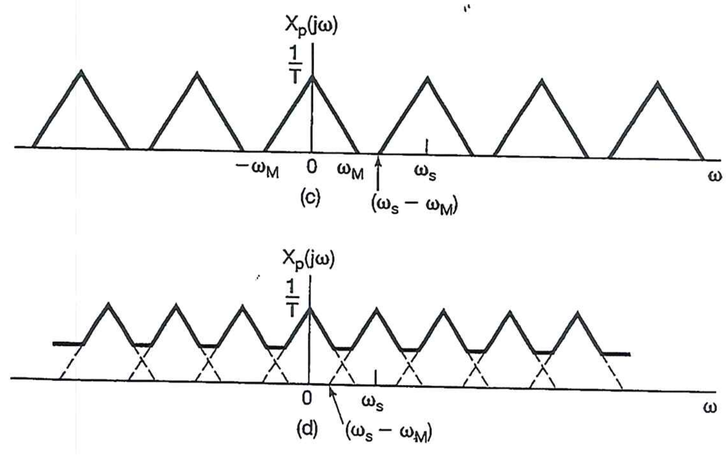


Figure 7.3 *Continued* (c) spectrum of sampled signal with $\omega_s > 2\omega_M$; (d) spectrum of sampled signal with $\omega_s < 2\omega_M$.

<u>Case</u>: $\omega_s - \omega_M > \omega_M \iff \omega_s > 2\omega_M$ no overlap

<u>Case</u>: $\omega_s - \omega_M < \omega_M \iff \omega_s < 2\omega_M$ overlap or aliasing.

(Review of Last Time)

<u>Fact</u> If $\omega_s > 2\omega_M$ (ie Nyquist rate) then $x(t)$ can be recovered from its samples by a lowpass filter.

(This is called the <u>Sampling Theorem</u>. Some names associated with it are H. Nyquist, J. M. Whittaker, C. Shannon)

<u>The proof is "by picture"</u>

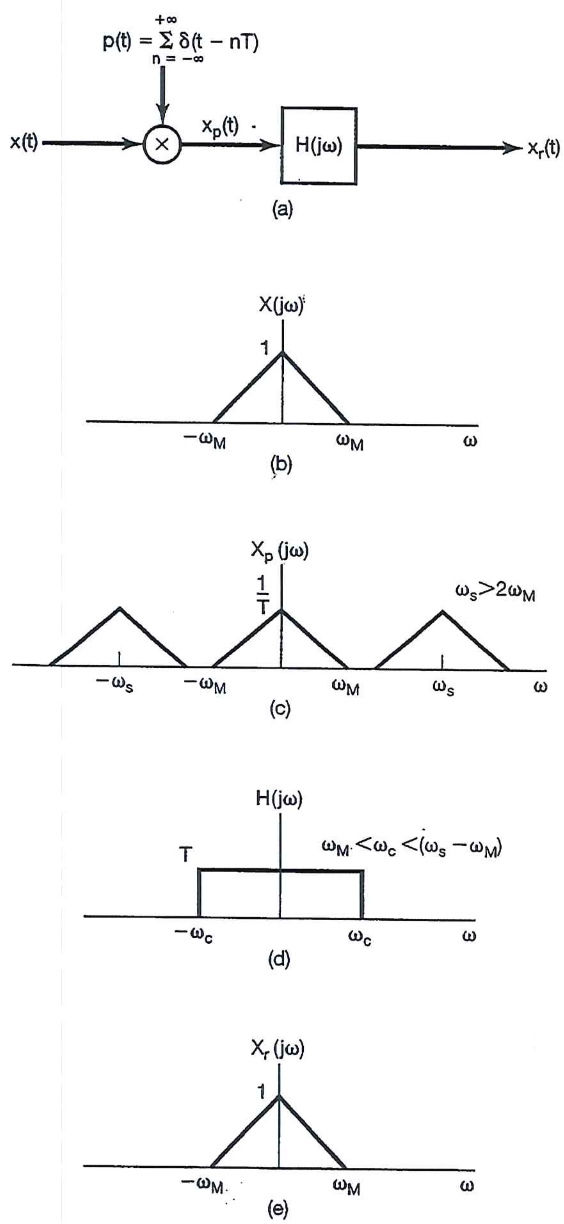


Figure 7.4 Exact recovery of a continuous-time signal from its samples using an ideal lowpass filter: (a) system for sampling and reconstruction; (b) representative spectrum for $x(t)$; (c) corresponding spectrum for $x_p(t)$; (d) ideal lowpass filter to recover $X(j\omega)$ from $X_p(j\omega)$; (e) spectrum of $x_r(t)$.

(Review of Last Time)

The Reconstruction & Interpolation Formula

Consider the reconstruction filter of the previous diagram. Say

$$h_r(t) \longleftrightarrow H_r(j\omega) = \begin{cases} T & |\omega| < \omega_c \\ 0 & \text{else} \end{cases}$$

$$h_r(t) = \frac{\omega_c T \sin(\omega_c t)}{\pi \omega_c t}$$

Clearly

$$x_r(t) = x_p * h_r(t) = \sum_n x(nT)\,\delta(t-nT) * h_r(t)$$

$$= \sum_n x(nT)\, h_r(t-nT)$$

called an interpolation formula because it shows how to fit a continuous curve between sample points.

$$= \sum_n x(nT) \frac{\omega_c T}{\pi} \frac{\sin(\omega_c(t-nT))}{\omega_c(t-nT)}.$$

Notice that if $\omega_c = \omega_m = \frac{1}{2}\omega_s$ then

① $h_r(0) = 1$

② $h_r(nT) = 0$ if $n \neq 0$

(Draw a picture of interlacing).

$\Longrightarrow$ $x_r(nT) = x(nT)$ so it indeed interpolates.

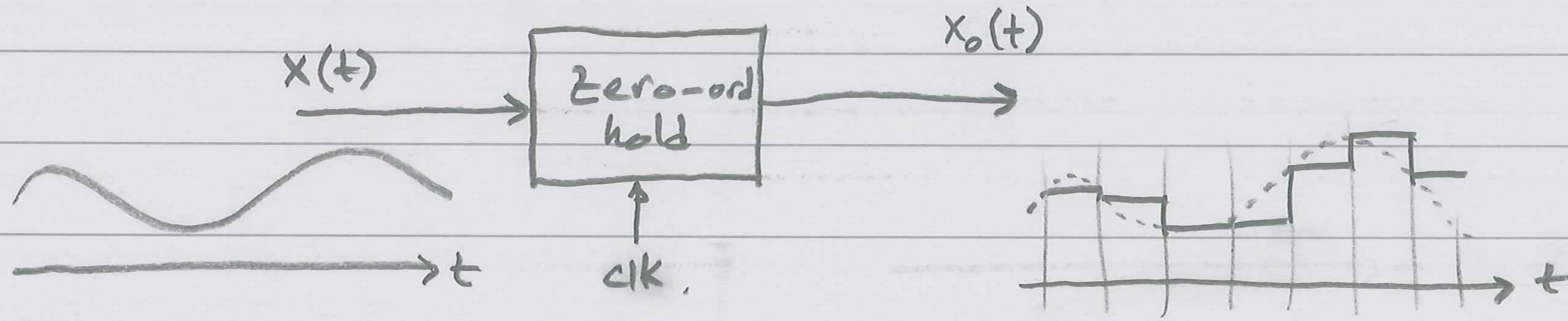
~~What if~~ $\cancel{\omega_m < \omega_c < \omega_s - \omega_m ?}$

$\cancel{h_r(t) = \frac{\sin \omega_c t}{\omega_c t}}$ $\cancel{\omega_c t = \pi k \quad t \neq 0}$

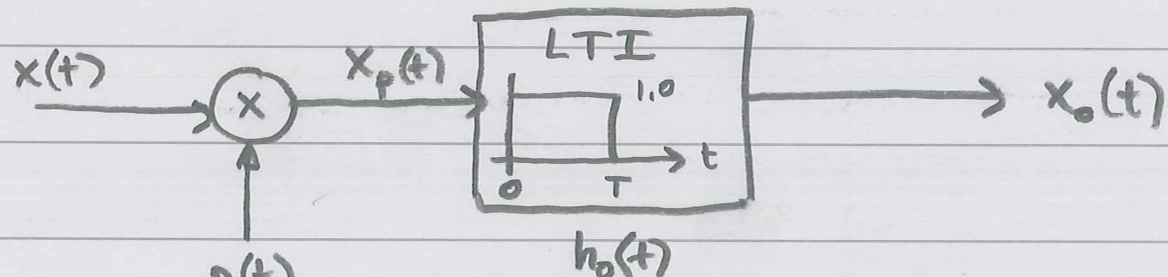
$\cancel{t = \frac{\pi k}{\omega_c}}$

Today: Investigate "More Practical" Schemes for Sampling

Zero-order hold sampling



To make a model (using 301 things)



impulse train sampling

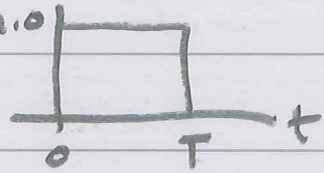
$$x_p(t) = \sum_n x(nT)\,\delta(t-nT)$$

$$\Rightarrow x_0(t) = x_p * h_0(t) = \sum_n x(nT)\,\delta(t-nT) * h_0(t)$$

$$= \sum_n x(nT)\,h_0(t-nT). \longrightarrow (*)$$

To see: ① Effect of ZOH Sampling in freq. domain

② How we might recover $x(t)$ from $x_o(t)$.

$h_o(t) =$ 

$\longleftrightarrow \quad H_o(j\omega) = Te^{-j\omega T/2}\dfrac{2\sin(\omega T/2)}{T\omega}$

From ✱ we know that

$$X_o(j\omega) = H_o(j\omega)X_p(j\omega)$$

$$= H_o(j\omega)\underbrace{\frac{1}{T}\sum_k X(j(\omega - k\omega_s))}_{X_p(j\omega).} \qquad \omega_s = 2\pi/T$$

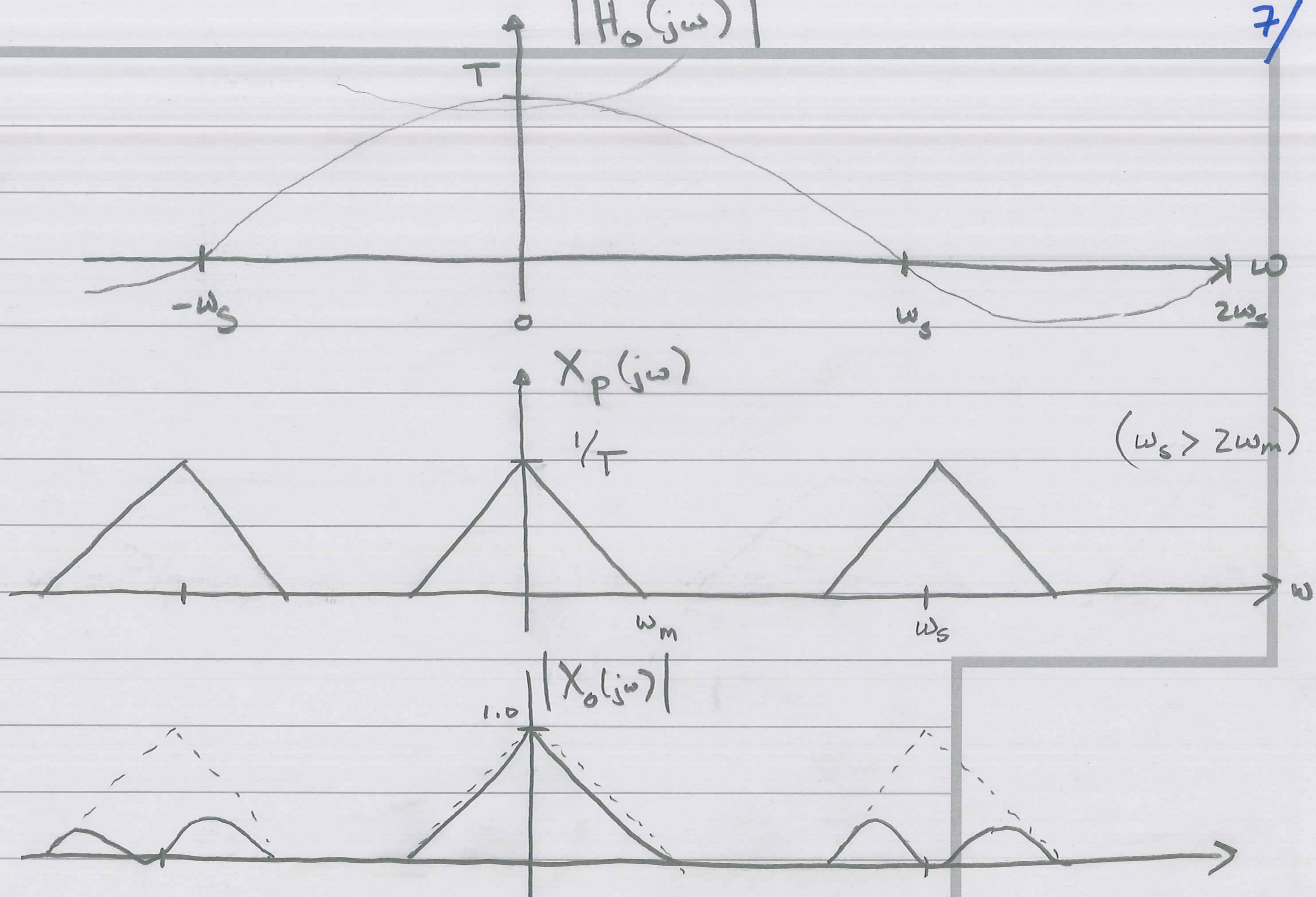
Need to know how this looks.

$$H_o(j\omega) = \underbrace{T\left[\frac{\sin(\omega T/2)}{\omega T/2}\right]}_{??}\cdot e^{-j\omega T/2}$$

$= T$ @ $\omega = 0$

$= 0$ $\frac{\omega T}{2} = \pi k$

$\Longleftrightarrow \omega = \frac{2\pi k}{T} = k\omega_s$

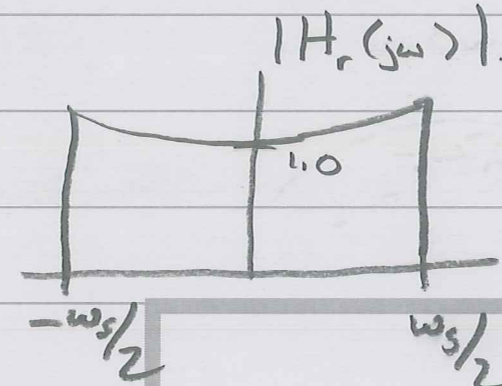

|H_0(jω)|
T
ω
-ω_s
0
ω_s
2ω_s
X_p(jω)
1/T
(ω_s > 2ω_m)
ω
ω_m
ω_s
|X_0(jω)|
1.0

- No zeros in $|\omega| < \omega_m$ provided that we sample @ Nyquist or above.

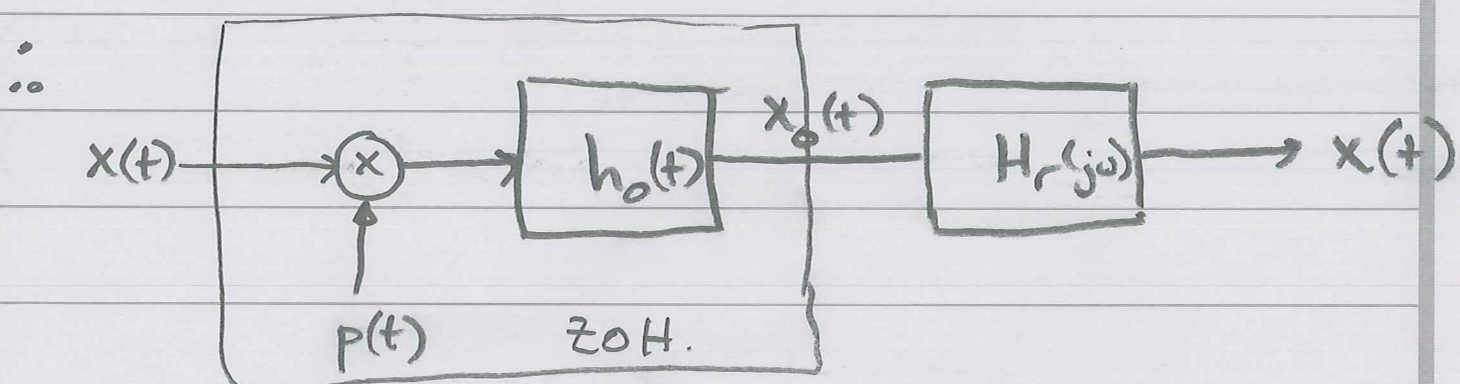
Can use a LP reconstruction filter

$$H_r(j\omega) = \underbrace{\frac{1}{H_o(j\omega)}}_{|\omega| < \omega_s/2} \underbrace{H(j\omega)}_{\text{the ideal LP filter from impulse sampling.}}$$

$$= \frac{\omega e^{+j\omega T/2}}{2 \sin(\omega T/2)} H(j\omega)$$



$\therefore$



Example of Aliasing $x(t) = \cos \omega_0 t$

Fix ω_s sampling rate

Vary ω_0