Impulse Train Sampling

$x(t)$ arbitrary CT signal

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

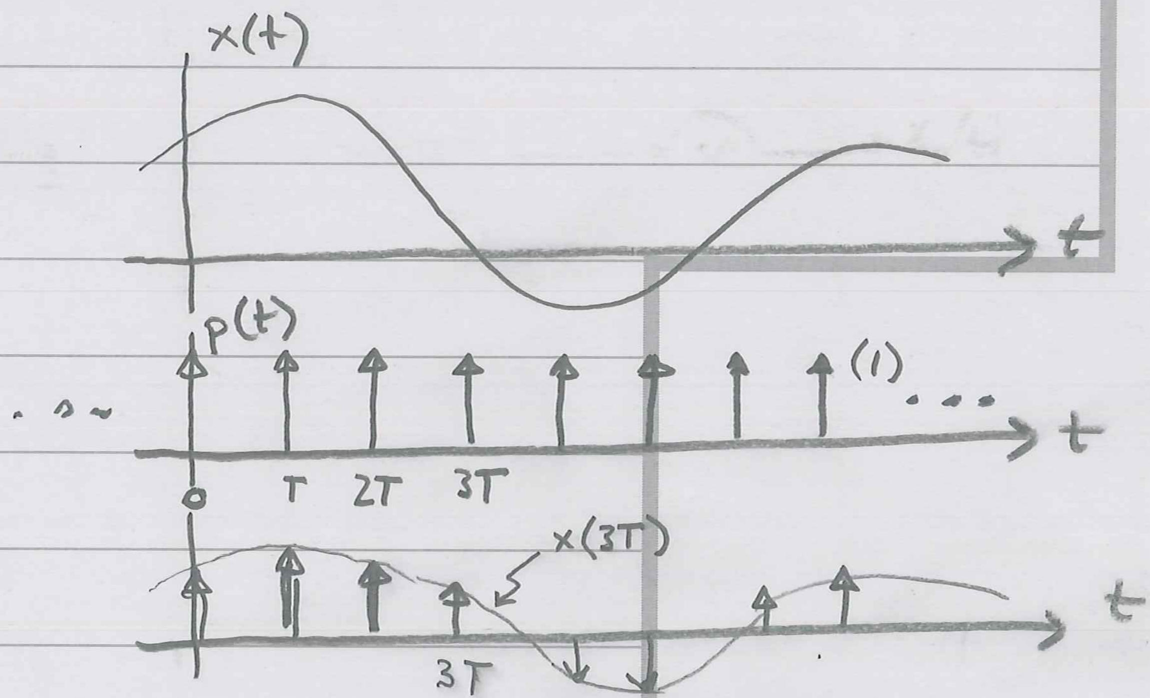
$T =$ sampling interval

$1/T =$ sampling freq.

$\omega_s = \frac{2\pi}{T}$ sampling freq. rad/s.

$$x_p(t) = x(t)\,p(t)$$

$$\begin{tikzcd} x(t) \arrow[r] & \times \arrow[r] & x_p(t) \\ & p(t) \arrow[u] & \end{tikzcd}$$



$x_p(t) =$

$$x(t) \sum_n \delta(t-nT)$$

$$= \sum_n x(t)\,\delta(t-nT)$$

$$= \sum_n x(nT)\,\delta(t-nT)$$

Need to see effect in Freq. Domain:

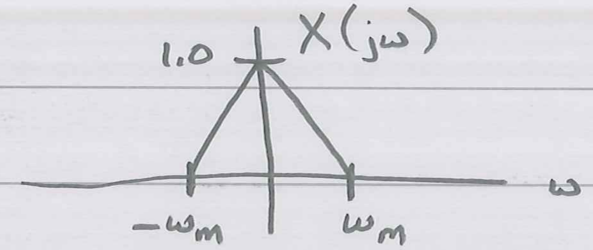
① mult. in time $\Longleftrightarrow$ conv. in freq. $X_p(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)P(j(\omega-\theta))\,d\theta$

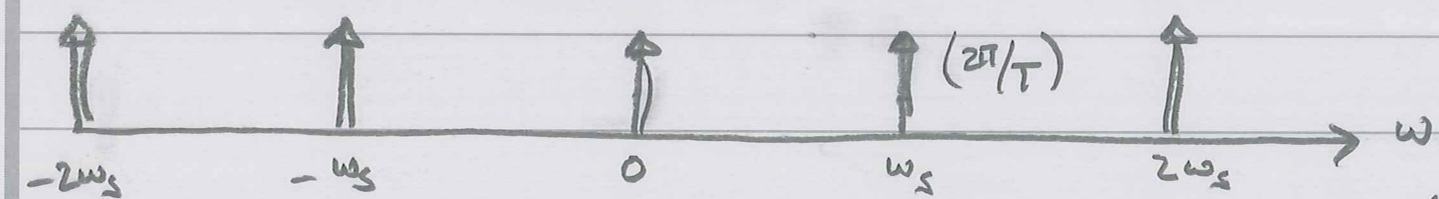
② $\sum_n \delta(t-nT) \longleftrightarrow \frac{2\pi}{T}\sum_k \delta(\omega - k\omega_s)$ $\qquad \omega_s = \frac{2\pi}{T}$

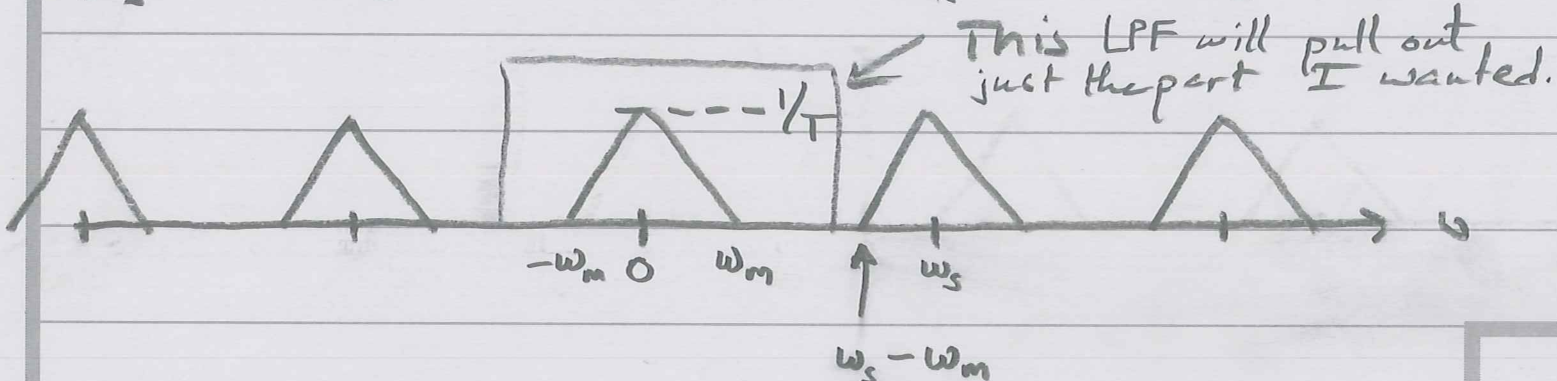
$\quad \overset{\shortparallel}{p(t)} \qquad\qquad \overset{\shortparallel}{P(j\omega)}$.

$\Longrightarrow X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$

Drawing Some Pictures





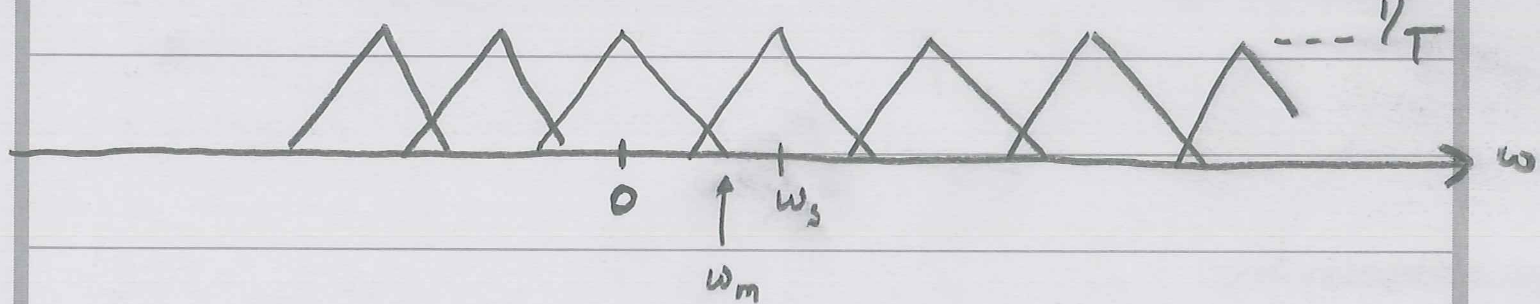


Drawn for case $\omega_s - \omega_m > \omega_m$

$\iff$

$\omega_s > 2\omega_m$.

no ovelap.



overlap case

$\omega_s - \omega_m < \omega_m$

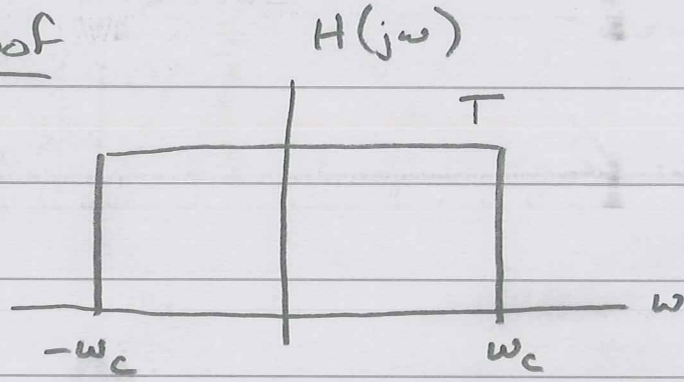
$\omega_s < 2\omega_m$.

aliased case.

Sampling Theorem (Nyquist, Whittaker, Shannon)

Subject to assumptions of prev. picture: If $\omega_s > 2\omega_m$ then $x(t)$ can be reconstructed from its samples by using a low pass filter.

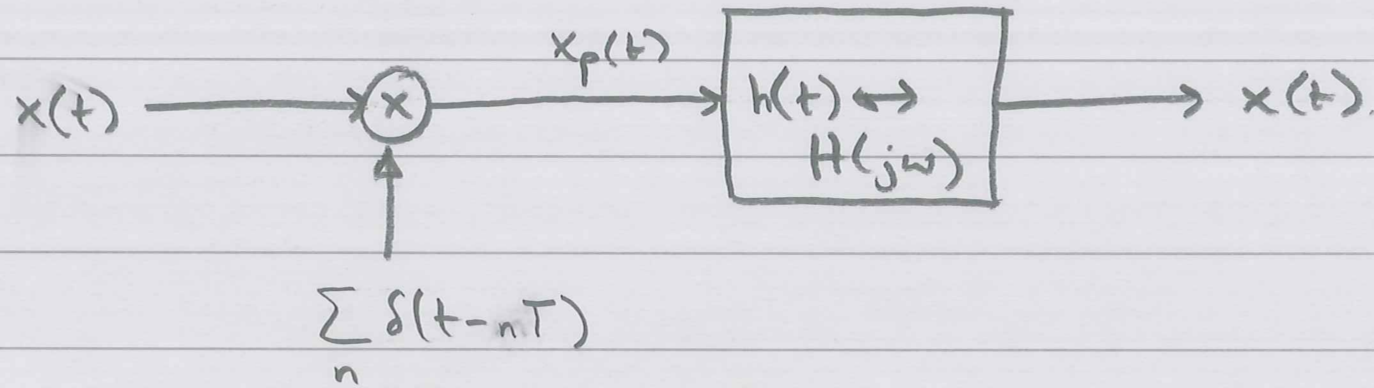
Our proof



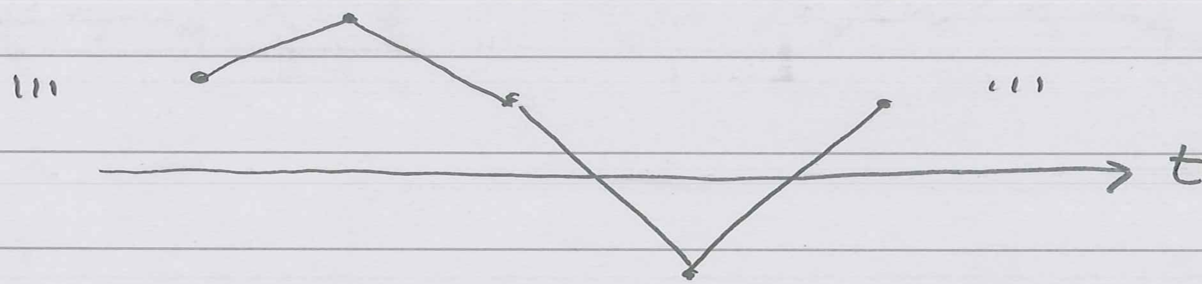
where $\omega_m < \omega_c < \omega_s - \omega_m$

$\Longrightarrow$ Subject to this assumption $\quad X_p(j\omega)H(j\omega) = X(j\omega)$.

Overall



The time domain interpretation is called an interpolater —



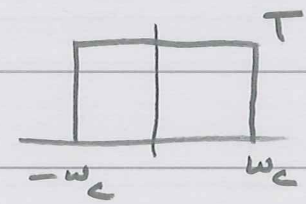
a linear interpolater.

also have polynomial interpolaters, splines,

Our LTI interpolater above is a "bandlimited interpolater"

Reconstruction Filter

$$h_r(t) \longleftrightarrow H_r(j\omega) = \begin{cases} T & |\omega| < \omega_c \\ 0 & \text{else.} \end{cases}$$



$$\parallel$$

$$\omega_c T \, \frac{\sin(\omega_c t)}{\pi \omega_c t}$$

$$\frac{\omega_c T}{\pi} \, \frac{\sin(\omega_c t)}{\omega_c t}$$

$$x_r(t) = x_p * h_r(t) = \sum_n x(nT)\delta(t-nT) * h_r(t).$$

$$= \sum_n x(nT)\, h_r(t-nT) = \text{function}\left(\begin{array}{l}\text{the samples} \\ \text{and an interp} \\ \text{pulse}\end{array}\right).$$

$$= \sum_n x(nT)\; \omega_c T \; \frac{\sin(\omega_c(t-nT))}{\omega_c(t-nT)}.$$

Special case: $\omega_c = \omega_m = \frac{1}{2}\omega_s \implies h_r(0) = \underline{1}$

$h_r(nT) = 0 \quad n \neq 0$