(Week 12 – Wednesday) 1

MT3 Fall 18

Question 6: [23%, Learning Objectives 3, 4, 5, and 6] Consider the following AM transmission system. The input signal is $x(t) = \frac{\sin(t)}{\pi t}$. We first multiply $x(t)$ by $\cos(4t)$. That is,

$$y(t) = x(t) \cdot \cos(4t).$$

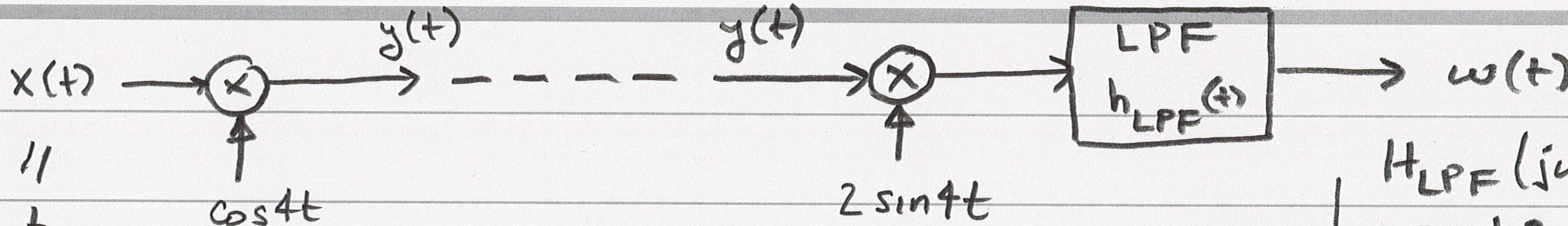
The transmitter then transmits signal $y(t)$ through the antenna.

At the receiver side, we first multiply $y(t)$ by $2\sin(4t)$. That is $z(t) = y(t) \cdot 2\sin(4t)$ and then pass $z(t)$ through a low pass filter with cutoff frequency $W = 7.5$ rad/sec. Denote the final output by $w(t) = z(t) * h_{\text{LPF}}(t)$.

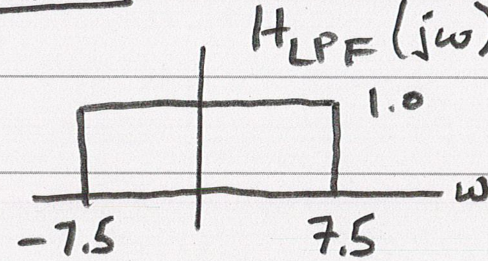
1. [18%] Plot the CTFT $W(j\omega)$ of $w(t)$ for the range of $-10 < \omega < 10$.
2. [5%] Find the expression of $w(t)$.

   Hint 1: This is a more advanced question. Please work on it after you have finished other questions.

Hint 2: If you do not know how to solve this question, you can write down the transmitter and receiver diagrams of AM-DSB when (i) filtering a radio signal to make it of bandwidth 10k Hz; (ii) then sending the filtered signal using carrier frequency 950kHz; and (iii) the receiver needs to demodulate the original signal. You need to carefully mark all the frequency parameters used in your scheme with the correct unit. If your answer is correct, you will receive 13 points.



$x(t) = \frac{\sin t}{\pi t}$



$$z(t) = x(t)\,\overbrace{\cos(4t)\,2\sin(4t)}^{\sin(8t)}$$

$$\Updownarrow$$

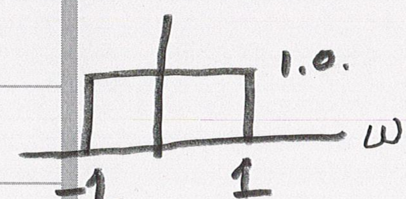
$$Z(j\omega) = \frac{1}{2\pi} X(j\omega) * \mathcal{F}\{\sin 8t\}$$

$$= \frac{1}{2\pi} X(j\omega) * \left\{\frac{\pi}{j}\delta(\omega-8) - \frac{\pi}{j}\delta(\omega+8)\right\}$$

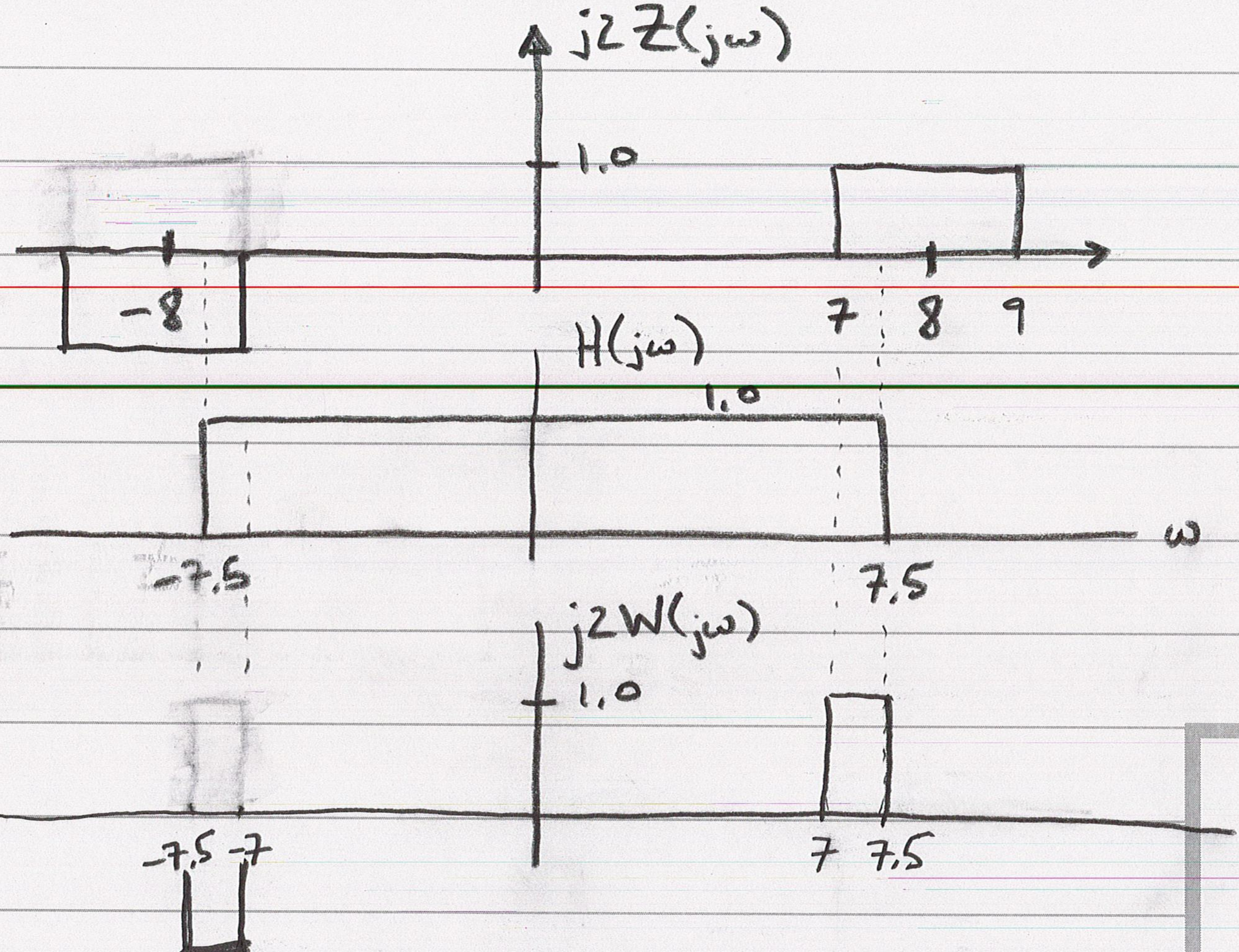
$$= \frac{1}{j2} X(j(\omega-8)) - \frac{1}{j2} X(j(\omega+8)).$$

From table:

$$x(t) = \frac{\sin t}{\pi t} \longleftrightarrow X(j\omega) =$$



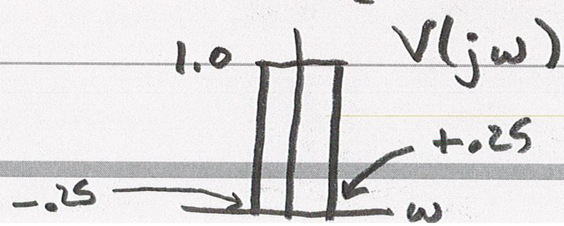
Plot $j2\,Z(j\omega) = X(j(\omega-8)) - X(j(\omega+8))$



$W = ZH$

#2 $W(j\omega) = \dfrac{1}{j2}\left\{ V(j(\omega-7.25)) - V(j(\omega+7.25)) \right\}$

where



$\longleftrightarrow \; v(t) = \dfrac{\sin(0.25t)}{\pi t}$

By comparing with expression for $Z(t)$ see

$$w(t) = v(t)\sin(7.25t)$$

$$= \frac{\sin(0.25t)}{\pi t}\sin(7.25t).$$

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Question 2: [23%, Work-out question, Learning Objectives 2, 3, and 4] Consider a CT-LTI system, for which the impulse response is

$$h(t) = \frac{\sin(5t) - \sin(3t)}{\pi t} \tag{5}$$

Consider a CT signal:

$$x(t) = \sum_{k=0}^{\infty} \sin(k\pi/2t). \tag{6}$$

$\frac{k\pi}{2}t$

1. [10%] Find the Fourier series $(a_k, \frac{2\pi}{N})$ of $x(t)$. Specifically, (i) find the expression of a_k for any arbitrary k, (ii) find the period N, and (iii) plot a_k for the range of $-2 \le k \le 2$.

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2. [13%] Let $y(t)$ denote the output when the input is $x(t)$. Find the Fourier series $(b_k, \frac{2\pi}{N})$ of $x(t)$. Specifically, (i) find the expression of b_k, (ii) find the period N, and (iii) plot b_k for the range of $-3 \le k \le 3$.

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   Hint 1: If you do not know the answer to this sub-question, you can find the Fourier transform $H(j\omega)$ of $h(t)$. You will receive 10 points if your answer is correct.

   Hint 2: The following values may be of use.

$y(t)$

$$\frac{\pi}{2} \approx 1.571 \tag{7}$$

$$\pi \approx 3.142 \tag{8}$$

$$\frac{3\pi}{2} \approx 4.712 \tag{9}$$

$$2\pi \approx 6.283 \tag{10}$$

$$\frac{5\pi}{2} \approx 7.854 \tag{11}$$

$$x(t) = \sum_{k=0}^{\infty} \sin(k\pi t/2) = \sum_{k=0}^{\infty} \frac{e^{jk\pi t/2} - e^{-jk\pi t/2}}{j2}$$

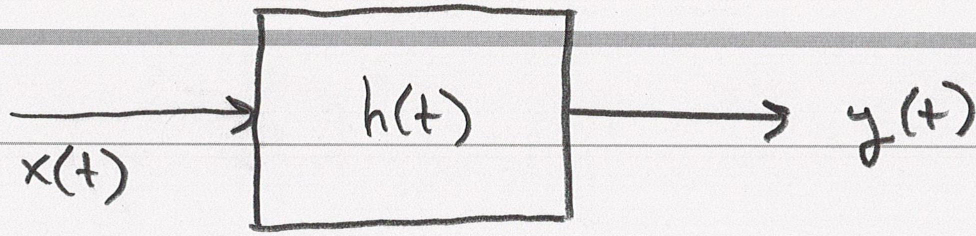
Look @ generic CTFS formula

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \qquad \therefore \frac{\pi}{2} = \frac{2\pi}{T}, \quad T = 4$$

$$= \sum_{k=0}^{\infty} \frac{1}{j2} e^{jk\pi t/2} - \sum_{l=-\infty}^{0} \frac{1}{j2} e^{+jl\pi t/2}$$

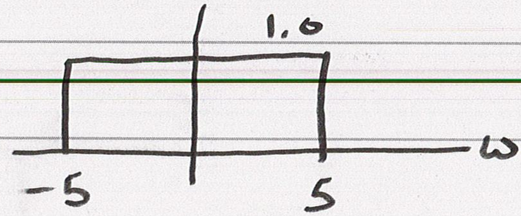
$$\therefore \quad a_k = \begin{cases} \frac{1}{j2} & k = 1, 2, \ldots \\ -\frac{1}{j2} & k = -1, -2, \ldots \\ 0 & k = 0 \end{cases}$$

#2



$$b_k = H(jk\omega_0)\,a_k = H\left(j\frac{k\pi}{2}\right)a_k \qquad \omega_0 = \frac{2\pi}{T} = \frac{\pi}{2}$$

$H(j\omega) =$



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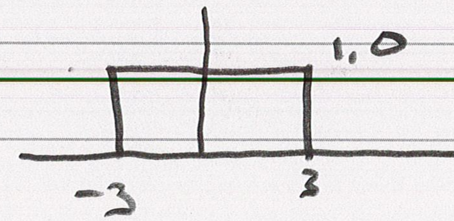

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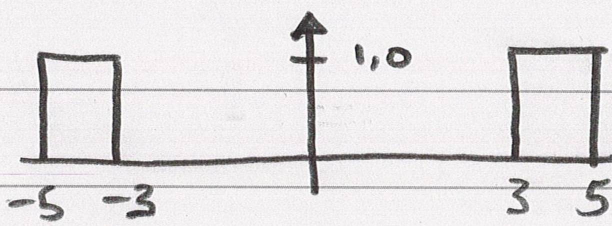

nonzero only if $k = 2, 3$ or $-2, -3$

$$b_k = \begin{cases} \frac{1}{j2} & k = 2, 3 \\ -\frac{1}{j2} & k = -2, -3 \\ 0 & \text{else.} \end{cases}$$

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Question 1: [17%, Work-out question, Learning Objectives 4, 5] Consider a discrete-time periodic signal $x[n]$ with period 20. Denote the corresponding Fourier series a_k, $k = 0$ to 19. Suppose we also know

$$a_k = \begin{cases} 3\cos(j\pi k) & \text{if } 0 \le k \le 9 \\ -1 & \text{if } 11 \le k \le 15 \\ 0 & \text{if } 16 \le k \le 19 \end{cases}. \tag{1}$$

Denote the Fourier series coefficient of $x[n]$ by (a_k, ω_0).

1. [6%] Find the value of $x[0]$.
2. [6%] Find the value of $\sum_{n=-3}^{16} |x[n]|^2$.
3. [5%] Find the value of $\sum_{n=0}^{19} (-1)^n x[n]$.

Correction: 3cos(j\pi k) should be 3cos (\pi k)

and 11<= k <= 15 should be 10<= k <= 15.

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \qquad N = 20$$

$$\updownarrow$$

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

#1 $$x[0] = \sum_{k=0}^{19} a_k = 3 \underbrace{\sum_{k=0}^{9} \overbrace{\cos(\pi k)}^{(-1)^k}}_{0} + \underbrace{\sum_{k=10}^{15} (-1)}_{-6} \Rightarrow x[0] = -6$$

#2 $$\sum_{n=0}^{19} |x[n]|^2 = 20 \sum_{k=0}^{19} |a_k|^2 = 20 \left\{ 9 \cdot 10 + 6 \right\} = 1920$$

#3 $y[n] \triangleq (-1)^n x[n]$ If b_k are the FS coeffs of $y[n]$ then

$$\sum_{n=0}^{19} y[n] = 20 \cdot b_0 = 20 \cdot a_{-10} = 20\, a_{10} = -20.$$

(Direct Soln to part 3)

#3 $\sum_{n=0}^{19} (-1)^n x[n]$ $\qquad -1 = e^{j\pi}$

$$\Longrightarrow (-1)^n x[n] = (-1)^n \sum_{k=0}^{9} 3(-1)^k e^{jk\left(\frac{2\pi}{20}\right)n} + (-1)^n \sum_{k=10}^{15} (-1) e^{j\cdots}$$

$$= \sum_{k=0}^{9} 3(-1)^k \underbrace{e^{jk\left(\frac{2\pi}{20}\right)n} e^{j\pi n}} + \sum_{k=10}^{15} (-1) e^{jk\left(\frac{2\pi}{20}\right)n} e^{j\pi n}$$

$$e^{j\left[k\left(\frac{2\pi}{20}\right) + \frac{20\pi}{20}\right]n}$$

$$e^{j\left[(k+10)\left(\frac{2\pi}{20}\right)\right]n}$$

$\therefore$ Just shifts coeffs. ~~No change~~

$$(-1)^n x[n] \triangleq y[n] \longleftrightarrow b_k$$

$$\therefore b_k = a_{k-10}$$

$$\Longrightarrow b_0 = a_{-10} = a_{10} = -1$$

$$\sum_{n=0}^{19} (-1)^n x[n] = N \cdot b_0 = 20(-1) = -20$$

Another Way to Solve #3

Freq. shift property $\underbrace{e^{jM(2\pi/N)n}}\, x[n] \longleftrightarrow a_{k-M}$

In our case this equals $(-1)^n = e^{j\pi n}$
and that happens if

$$\frac{M \cdot 2\pi}{N} = \pi \quad \cdots \quad N = 20 \text{ given}$$

$$\Longrightarrow M = 10.$$

$\therefore$ If $y[n] = (-1)^n x[n]$ $\Longrightarrow b_k = a_{k-10}$

$$\updownarrow$$

$$b_k$$

$$\sum_{n=0}^{19} y[n] = 20 \cdot b_0 = 20 \cdot a_{-10} = 20 \cdot a_{10} = -20.$$

MT3 Fall 15

(Did not get to this in class).

Question 1: [18%, Work-out question, Learning Objectives 1, 2, 3] Consider a discrete-time signal

$$x[n] = \begin{cases} 0.5e^{jn} & \text{if } 1 \leq n \leq 10 \\ 0.5 & \text{if } 11 \leq n \leq 20\ . \\ \text{is periodic with period 20} & \end{cases} \tag{1}$$

Denote the Fourier series coefficient of $x[n]$ by (a_k, ω_0).

1. [6%] Find the value of a_0.
2. [6%] Find the value of $\sum_{k=0}^{19} a_k$.
3. [6%] Find the value of $\sum_{k=0}^{19} |a_k|^2$.

Hint: You may need to use the following formulas: If $|r| \neq 1$, then we have

$$\sum_{k=1}^{K} ar^{k-1} = \frac{a(1-r^K)}{1-r}. \tag{2}$$

If $|r| < 1$, then we have

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}. \tag{3}$$

If $|r| \geq 1$, then we have

$$\sum_{k=1}^{\infty} ar^{k-1} \text{ does not exist.} \tag{4}$$

$$x[n] = \sum_{k=0}^{19} a_k e^{jk(2\pi/N)n} \qquad N = 20$$

$$\updownarrow$$

$$a_k = \frac{1}{N} \sum_{n=0}^{19} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$\boxed{1}$

$$x[20] = x[0] = 0.5$$

$$a_0 = \frac{1}{20} \sum_n x[n]$$

$$= \frac{1}{20}\left\{ \sum_{n=1}^{10} 0.5(e^j)^n + \sum_{n=11}^{20} (0.5) \right\}$$

$$= \frac{1}{20}\left\{ 0.5(e^j) \sum_{n=1}^{10} (e^j)^{n-1} + 10 \cdot (0.5) \right\}$$

$$= \frac{1}{20}\left\{ 0.5(e^j) \frac{1-e^{j10}}{1-e^j} + 5 \right\}$$

$\boxed{2}$

$$\sum_{k=0}^{19} a_k = x[0] = 0.5$$

$\boxed{3}$

$$\sum_{k=0}^{19} |a_k|^2 = \frac{1}{20} \sum_{n=0}^{19} |x[n]|^2$$

$$= \frac{1}{20} \left\{ \sum_{n=1}^{10} (0.5)^2 \left| e^{jn} \right|^2 + \sum_{n=11}^{20} (0.5)^2 \right\}$$

$$= \frac{20 \cdot (0.5)^2}{20} = 0.25$$