(Week 12 - Monday)

Discrete Time Fourier Transform $x[n]$ aperiodic

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{+j\omega n}\,d\omega = x[n] \longleftrightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Note ① If $\tilde{x}[n] \longleftrightarrow \tilde{X}_k$ say periodic of per $= N$

further extract an aperiodic signal $x[n]$ by

$$x[n] = \begin{cases} \tilde{x}[n] & n = 0, 1, 2 \ldots N-1 \\ 0 & \text{else.} \end{cases}$$

Then turns out

$$\tilde{X}_k = \frac{1}{N} X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}} \qquad k = 0, 1, 2 \ldots N-1$$

Note ② $X(e^{j\omega})$ periodic per. 2π

$\omega \approx 0, \pm 2\pi, \pm 4\pi \ldots$ approx DC.

$\omega \approx \pm\pi, \pm 3\pi \ldots$ are the highest freqs.

(Text Fig 5.3)

Note ③ Math. speaking we've already covered the DTFT .... its just a Fourier Series.

$$\sum_{k=1}^{\infty} \alpha^k = \sum_{k=0}^{\infty} \alpha^k - 1$$

MT3 Fall 2017

Question 5: [12%, Work-out question, Learning Objectives 4, 5, and 6] Consider a discrete time signal

$$x[n] = 2^{-|n|} e^{j3n} \tag{5}$$

Find the corresponding Fourier transform $X(e^{j\omega})$.

Hint: You may need the following formula: If $|r| \neq 1$, then

$$\sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha}$$

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}. \tag{6}$$

$$X(e^{j\omega}) = \sum_{n} 2^{-|n|} e^{j3n} e^{-j\omega n} = \sum_{n<0} \cdots + \sum_{n=0} + \cdots + \sum_{n>0} \cdots$$

$$= \sum_{n=-\infty}^{-1} 2^{+n} e^{j3n} e^{-j\omega n} + 1 + \sum_{n=1}^{\infty} 2^{-n} e^{j3n} e^{-j\omega n}$$

$k=-n$

$$= \sum_{k=1}^{\infty} \left(2^{-1} e^{-j3} e^{j\omega}\right)^k + 1 + \sum_{n=1}^{\infty} \left(2^{-1} e^{j3} e^{-j\omega}\right)^n$$

$$= \frac{1}{1-2^{-1}e^{-j3}e^{j\omega}} - 1 + \cancel{1} + \frac{1}{1-2^{-1}e^{j3}e^{-j\omega}} \cancel{-1}$$

MT 3 Nov '19

Question 4: [20%, Work-out question, Learning Objectives 2, 3, 4 and 5] Consider the following DT signal $x[n] = e^{j3(n-100)} \cdot 2^{-n+100} U[n-100]$.

1. [8%] Find the expression of $X(e^{j\omega})$.
2. [6%] Find the value of $\int_{\pi}^{3\pi} X(e^{j\omega})d\omega$.
3. [6%] Find the value of $\int_{0}^{10\pi} |X(e^{j\omega})|^2 d\omega$.

Hint: The following formula may be useful: If $|r| < 1$, then

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}. \tag{7}$$

$$X(e^{jw}) = \sum_{n} e^{j3(n-100)} 2^{-n+100} u[n-100] e^{-jwn}$$

$$= \sum_{n=100}^{\infty} e^{j3(n-100)} 2^{-(n-100)} e^{-jwn}$$

$k = n-100$

$k+100 = n$

$e^{jw(k+100)}$

$$= \sum_{k=0}^{\infty} \underbrace{e^{j3k} 2^{-k} e^{-jwk}}_{\left(e^{j3} 2^{-1} e^{-jw}\right)^k} e^{-jw100}$$

$$= \frac{1}{1 - e^{j3} 2^{-1} e^{-jw}} \cdot e^{-jw100}$$

$\boxed{2}$ $$\int_{\pi}^{3\pi} X(e^{j\omega})\, d\omega = \int_{-\pi}^{\pi} X(e^{j\omega})\, d\omega = 2\pi\, x[0] = 0$$

$\boxed{3}$ $$\int_{0}^{10\pi} |X(e^{j\omega})|^2 d\omega = 5 \int_{0}^{2\pi} |X(e^{j\omega})|^2 d\omega$$

$$\underset{\text{Parseval}}{=} 5 \cdot 2\pi \sum_n |x[n]|^2$$

$$|x[n]| = 2^{-n}\, 2^{100}\, u[n-100]$$

$$|x[n]|^2 = 2^{-2n}\, 2^{200}\, u[n-100]$$

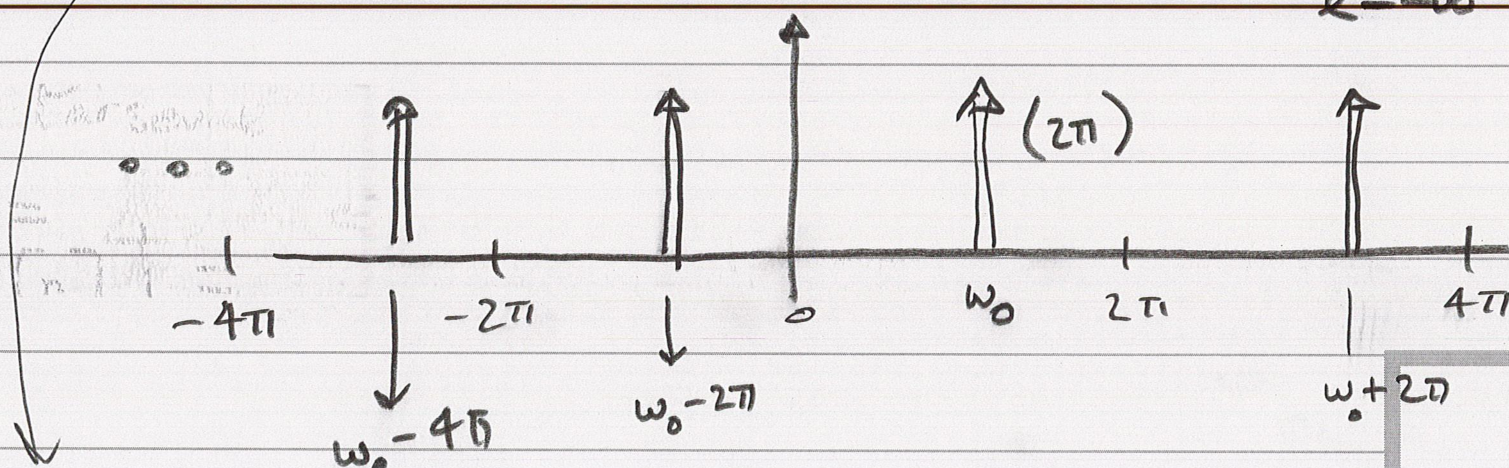
$$\sum_{n=0}^{\infty} |x[n]|^2 = 2^{200} \sum_{n=100}^{\infty} 2^{-2n} = 4/3$$

$$\int_{0}^{10\pi} |X(e^{j\omega})|^2 = \frac{4}{3} \cdot 10\pi$$

DTFT for Periodic Signals

For any real # ω_0

$$x[n] = e^{j\omega_0 n} \longleftrightarrow X(e^{j\omega}) = 2\pi \sum_{\ell=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi\ell)$$



Direct approach fails

$$\sum_{n=-\infty}^{\infty} e^{j\omega_0 n} e^{-j\omega n} = ?$$

$\rightarrow$ It's a delta $\rightarrow$ use inverse transform & guess.

MT3 Nov 2019

Question 5: [18%, Work-out question, Learning Objectives 4, 5, and 6] Consider a discrete time signal

$$x[n] = \sin(1.25\pi n) + 2e^{j2.5\pi n} \tag{8}$$

Plot the corresponding DTFT $X(e^{j\omega})$ for the range of $-2\pi < \omega < 2\pi$.

Hint: There is no need to write down the expression of $X(e^{j\omega})$. A plot is sufficient.

$$x[n] = \underbrace{\sin(1.25\pi n)} + 2e^{j2.5\pi n}$$

$$\frac{e^{j1.25\pi n} - e^{-j1.25\pi n}}{j2} + 2e^{j2.5\pi n}$$

$\omega_0 = 1.25\pi$

$\omega_0 = -1.25\pi$

$\omega_0 = 2.5\pi$

We already have DTFT$\left\{e^{j\omega_0 n}\right\}$