(Week 11 Monday)

Fourier Transform Properties (cont'd.)

$$\mathcal{F}\{x(t)\} = X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\mathcal{F}^{-1}\{X(j\omega)\} = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$x(t) \longleftrightarrow X(j\omega)$$

Differentiation in Time $\quad \frac{dx}{dt} \longleftrightarrow j\omega X(j\omega)$

To show, start with IFT

$$\frac{d}{dt}x(t) = \frac{d}{dt}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{+j\omega t}\, d\omega\right\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\underbrace{\frac{d}{dt}e^{j\omega t}}_{j\omega e^{j\omega t}}\, d\omega \implies$$

This proves it by uniqueness of FTs.

$$x(t) \longleftrightarrow X(j\omega)$$

Integration in Time $\quad \int_{-\infty}^{t} x(\tau)\, d\tau \longleftrightarrow \frac{1}{j\omega}X(j\omega) + \pi X(0)\,\delta(\omega)$

To prove, defer until we talk about FT of a unit step.

Time/Freq. Scaling

$$\left.\begin{array}{l} x(t) \longleftrightarrow X(j\omega) \\ x(at) \longleftrightarrow \frac{1}{|a|}X\left(j\frac{\omega}{a}\right) \end{array}\right\}$$

interp:
$a > 1$ compress in time $\iff$ expand in freq.
$0 < a < 1$ ... the opposite

$$x(t) \longleftrightarrow X(j\omega)$$

Duality Property

$$\underbrace{X(j\omega)\Big|_{\omega \mapsto -t}}_{X(-jt)} \longleftrightarrow \underbrace{2\pi x(t)\Big|_{t \mapsto \omega}}_{2\pi x(\omega)}$$

$$x(t) \longleftrightarrow X(j\omega) \iff \underbrace{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega}$$

redefine variables until this inverse FT looks like a Forward FT.

Let $t \mapsto \lambda$ $$2\pi\, x(\lambda) = \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega\lambda}\, d\omega$$

Let $\omega \mapsto -s$ $$2\pi\, x(\lambda) = \int_{-\infty}^{\infty} X(-js)\, e^{-js\lambda}\, ds$$

Let $\lambda \mapsto \omega$ $$2\pi\, x(\omega) = \int_{-\infty}^{\infty} X(-js)\, e^{-js\omega}\, ds$$

Let $s \mapsto t$ $$2\pi\, x(\omega) = \int_{-\infty}^{\infty} X(-jt)\, e^{-j\omega t}\, dt$$

Examples

① Book: $e^{-|t|} \longleftrightarrow \frac{2}{1+\omega^2} \Longleftrightarrow \frac{2}{1+t^2} \longleftrightarrow 2\pi e^{-|\omega|}$

② $e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega} \Longleftrightarrow \frac{1}{a-jt} \longleftrightarrow 2\pi e^{-a\omega}u(\omega)$

$\mathrm{Re}(a) > 0$

"Back to the Basics" Derivation

$$\frac{1}{a+j\omega} = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t}\, dt$$

$(t \mapsto \lambda)$

$$\frac{1}{a+j\omega} = \int_{-\infty}^{\infty} e^{-a\lambda} u(\lambda) e^{-j\omega\lambda}\, d\lambda$$

$(\omega \mapsto -t)$

$$\frac{1}{a-jt} = \int_{-\infty}^{\infty} e^{-a\lambda} u(\lambda) e^{+j\lambda t}\, d\lambda$$

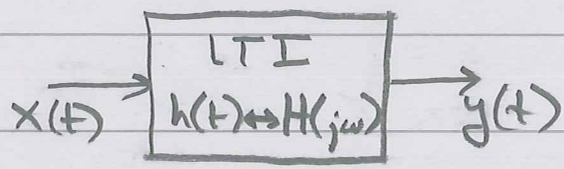
$(\lambda \mapsto \omega)$

$$\frac{1}{a-jt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\, e^{-a\omega} u(\omega) e^{j\omega t}\, d\omega$$

Other Important $\quad x(t) \leftrightarrow X(j\omega)$

Parseval: $$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

Convolution Property:
$$x(t) \leftrightarrow X(j\omega)$$
$$h(t) \leftrightarrow H(j\omega)$$
$$y(t) \leftrightarrow Y(j\omega)$$



$$y(t) = x * h(t)$$
$$\updownarrow$$
$$Y(j\omega) = X(j\omega)\, H(j\omega)$$