Summary of Week 1

- Definition of Signals and Examples
  + discrete-time
  + continuous-time
- Definition of a System
  + linear vs. non-linear
  + time-varying vs. time-invarient
- Idea of using test signals to characterize linear systems
- Other classifications of signals
  + real-valued vs. complex-valued
  + energy vs. power

Main lecture: broad examples of signals and systems

Here: Take a mathematical/abstract view

Focus: discrete-time

Discrete-time signal

$$x : \mathbb{D} \longrightarrow \mathbb{R} \text{ or } \mathbb{C}$$

$$\mathbb{D} \subseteq \mathbb{Z} = \{\ldots -2, -1, 0, +1, +2 \ldots\}$$

Distinction betw. Signal / function

x, $x[\cdot]$

vs

Values

$x[n]$ or $x[0]$ or $x[61]$

Discrete-time system

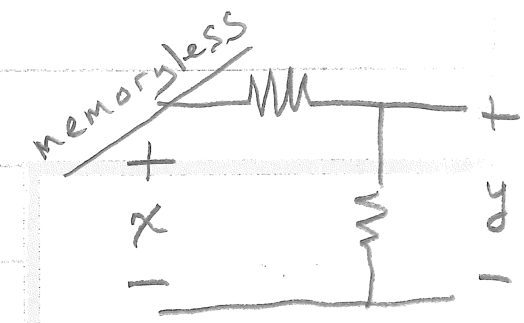
$$\begin{Bmatrix} \text{DT} \\ \text{signals} \end{Bmatrix} \xrightarrow{S} \begin{Bmatrix} \text{DT} \\ \text{signals} \end{Bmatrix}$$

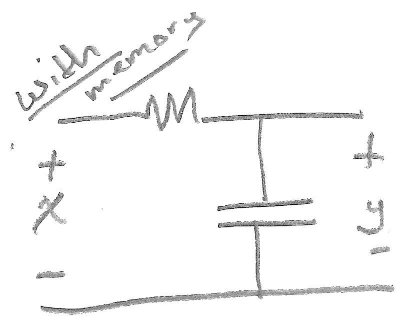
$$x \longmapsto S(x) = y$$

$$x[\cdot] \longmapsto S(x[\cdot]) = y[\cdot]$$

or

$$\left.\begin{aligned} x[n] = S(x[n]) = y[n] \end{aligned}\right\} \text{not ideal.}$$





Linear System

Set of signals at input and output must have a vector space structure:

α scalar
$x[\cdot]$ sig. $\Longrightarrow$ $\alpha x[\cdot]$ is a sig.

$x[\cdot], y[\cdot] \Longrightarrow x[\cdot] + y[\cdot]$ is a sig.

S is linear if:

$$S(\alpha x[\cdot]) = \alpha \cdot S(x[\cdot])$$

$$S(x[\cdot] + y[\cdot]) = S(x[\cdot]) + S(y[\cdot])$$

Test Signals: DT → LT Example

$$\mathcal{S} : \mathbb{I} \longrightarrow \mathbb{O}$$

$\mathbb{I} = \left\{ x[\cdot] : \begin{matrix}\text{defined for} \\ n = 0, 1, \dots N-1\end{matrix} \right\}$

$\mathbb{O} = \left\{ y[\cdot] : \begin{matrix}\text{defined for} \\ n = 0, 1, \dots M-1\end{matrix} \right\}$

Consider special input sigs:

$$\left.\begin{aligned} x_0[\cdot] &= [1, 0, 0 \dots 0] \\ x_1[\cdot] &= [0, 1, 0 \dots 0] \\ &\vdots \\ x_{N-1}[\cdot] &= [0, 0 \dots 1] \end{aligned}\right\}$$

test signals.
or
coordinate signals.

Let $\underbrace{h_k[\cdot]}_{\text{outputs}} = \mathcal{S}\left(x_k[\cdot]\right) \qquad k = 0, 1, \dots N-1$

Linear System Fact

If $S : \mathbb{I} \rightarrow \mathbb{O}$ is a linear system, then its response to an arbitrary input is a linear combination of the N responses to the N test coordinate signals.

Proof Any input sig $x[\cdot]$ of length N is a unique lin. comb. of the prev. test sigs

(a basis of the input sig. vector space)

$$x[\cdot] = [x[0], x[1] \ldots x[N-1]]$$

$$= x[0] \cdot \underbrace{[1, 0, 0 \ldots 0]}_{x_0[\cdot]} + x[1] \cdot \underbrace{[0, 1, 0 \ldots 0]}_{x_1[\cdot]} + \cdots +$$

$$x[N-1] \cdot \underbrace{[0, 0 \ldots 1]}_{x_{N-1}[\cdot]}$$

$$y[\cdot] = S(x[\cdot]) = S\left(\sum_{k=0}^{N-1} x[k]\, x_k[\cdot]\right) = \sum_{k=0}^{N-1} S(x[k]\, x_k[\cdot])$$

$$= \sum_{k=0}^{N-1} x[k]\, S(x_k[\cdot]) = \sum_{k=0}^{N-1} x[k]\, h_k[\cdot]$$

Evaluate at $n = 0, 1, 2 \ldots M-1$

$$y[n] = \sum_{k=0}^{N-1} x[k]\, h_k[n]$$

superposition sum.

Matrix Notation

$$y[n] = \sum_{k=0}^{N-1} x[k]\, h_k[n] \qquad n = 0, 1, \ldots M-1$$

$$\begin{pmatrix} y[0] \\ y[1] \\ \vdots \\ y[M-1] \end{pmatrix} = \underbrace{\begin{pmatrix} h_0[0] & h_1[0] & \ldots & h_{N-1}[0] \\ h_0[1] & h_1[1] & & h_{N-1}[1] \\ & & & \\ h_0[M-1] & & & h_{N-1}[M-1] \end{pmatrix}}_{\mathbb{H}} \begin{pmatrix} x[0] \\ \vdots \\ x[N-1] \end{pmatrix}$$

$\mathbb{H}$ is an $M \times N$ matrix.