The Falling Slinky Problem

Jonathan Poggie

School of Aeronautics and Astronautics, Purdue University

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If you hold a Slinky by one end, and let go, the top of the slinky falls, but the bottom stays in place: `https://www.youtube.com/watch?v=rCw5JXD18y4`. The same thing goes for a chain, a spring, and even a solid bar. Why? I know others have solved this problem [1, 2], but it's fun to try to figure it out for yourself.

Let's consider a one-dimensional model problem of a falling bar; the physics are very close to the Slinky problem, but a little simpler.[1] We'll take the x-direction as down, with the top of the bar at $x = 0$, and the bottom at $x = L$. The vertical component of the momentum equation (Newton's second law) in elasticity is:

$$\rho \frac{\partial^2 u}{\partial t^2} = E \frac{\partial^2 u}{\partial x^2} + \rho g \tag{1}$$

where x is position along the bar, t is time, u is the displacement of the bar, and g is the acceleration of gravity. Here E is an elastic modulus, ρ is the density, and $c = \sqrt{E/\rho}$ is the wave speed in the material.

The initial condition at $t = 0$ is the bar hanging in equilibrium. The elastic force due to the stretch of the bar balances the pull of gravity. In this initial state, the momentum equation (1) reduces to:

$$0 = E \frac{d^2 u_0}{dx^2} + \rho g \tag{2}$$

The boundary conditions are that the top is held fixed, $u_0(0) = 0$, and the bottom is free, with zero stress and zero strain, $(du_0/dx)_{x=L} = 0$. The initial displacement is thus:

$$u_0(x) = \frac{\rho g L^2}{E} \left(\frac{x}{L} - \frac{1}{2} \frac{x^2}{L^2} \right) \tag{3}$$

Once you let go of the bar, the stress at both endpoints is zero since there is nothing to constrain the ends. Thus, at both $x = 0$ and $x = L$, we have $\partial u / \partial x = 0$ for $t > 0$. The message that you have let go of the end of the bar travels as a wave of speed $c = \sqrt{E/\rho}$ from the top of the bar toward the bottom.

[1] A Slinky, for example, cannot support compression.

Looking back at Eq. (1), we see that it is inhomogeneous. It admits a particular solution $u = \frac{1}{2}gt^2$, a freefall solution. Thus, we'll try a solution of the form:

$$u(x,t) = \begin{cases} f(x-ct)+h(x+ct)+\frac{1}{2}gt^2 & x < ct \\ u_0(x) & x > ct \end{cases} \tag{4}$$

This is a freefall component plus a propagating wave solution for the upper part of the bar, and the initial condition for the lower, undisturbed part of the bar.

Enforcing continuity of $u(x,t)$ at $x = ct$, and using $\partial u/\partial x = 0$ at $x = 0$, we find the solution for the displacement to be:

$$\frac{u(x,t)}{\rho g L^2/E} = \begin{cases} \frac{ct}{L} - \frac{1}{2}\frac{x^2}{L^2} & x < ct \\ \frac{x}{L} - \frac{1}{2}\frac{x^2}{L^2} & x > ct \end{cases} \tag{5}$$

The stress is $\sigma = E\,(\partial u/\partial x)$, so the corresponding stress distribution is:

$$\frac{\sigma(x,t)}{\rho g L} = \begin{cases} -x/L & x < ct \\ 1-x/L & x > ct \end{cases} \tag{6}$$

Here the convention is that a positive stress corresponds to tension in the bar. Notice that the stress changes sign at $x = ct$; there is a kind of shock wave there. The corresponding speed is $\partial u/\partial t$, which has the form:

$$\frac{c}{gL}\frac{\partial u}{\partial t} = \begin{cases} 1 & x < ct \\ 0 & x > ct \end{cases} \tag{7}$$

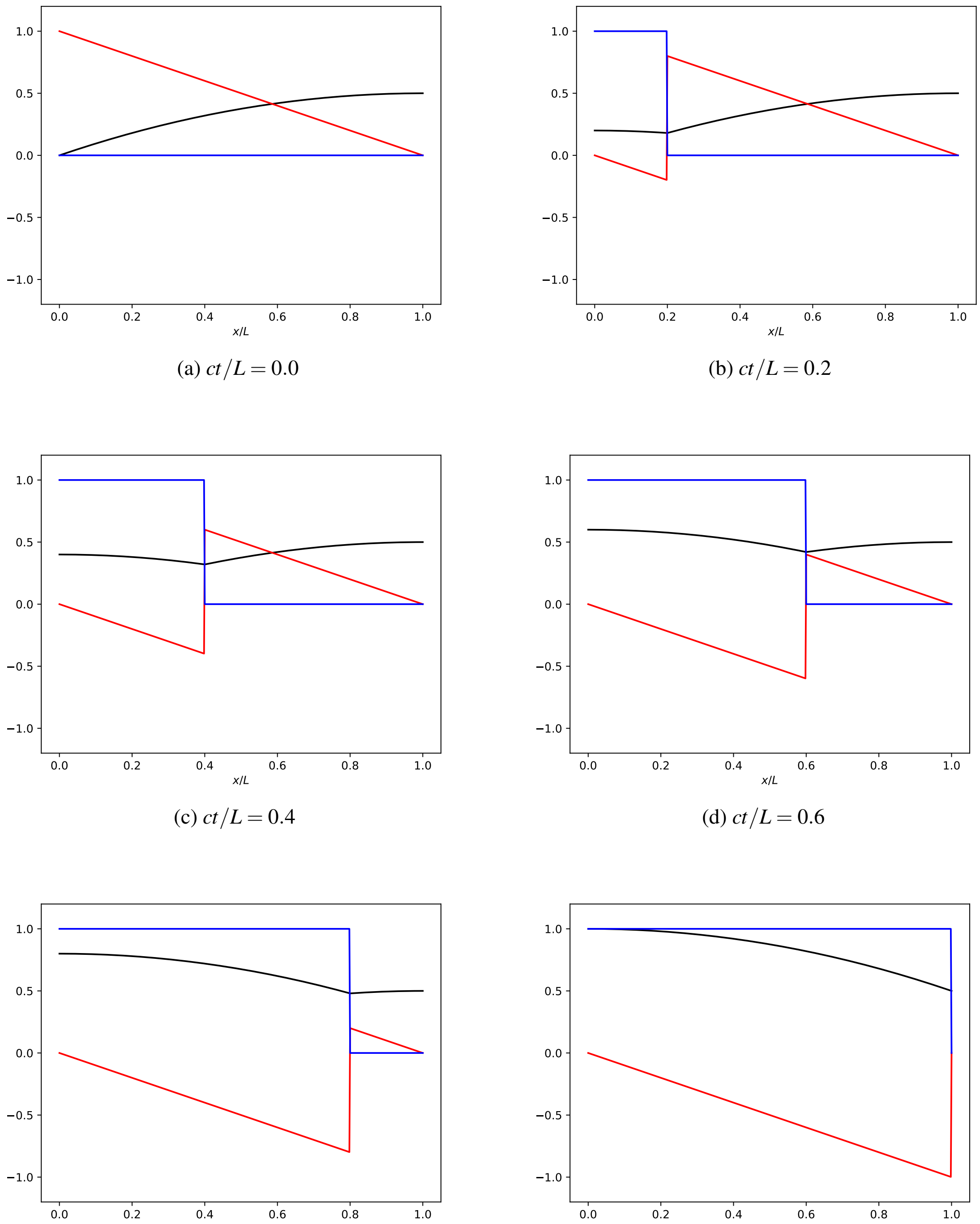
which explains the video. The bar has a constant downward velocity above the discontinuity at $x = ct$, and zero velocity below that station.

The results are illustrated in Fig. 1 as profiles of displacement, stress, and velocity for different times. Figure 1a shows the initial condition. At this time, there is tension throughout the bar to maintain static equilibrium against gravity, and the velocity is zero. For $t > 0$ in Figs. 1b–1e, a kind of shock wave propagates through the domain. Ahead of the wave ($x > ct$) is the initial condition, behind the wave ($x < ct$), the displacement changes, the stress becomes compressive, and a downward velocity appears. Finally, the shock hits the end of the bar, and would reflect if the solution were continued (Fig. 1f).

References

[1] J. M. Aguirregabiria, A. Hernández, and M. Rivas. Falling elastic bars and springs. *American Journal of Physics*, 75(7):583–587, 2007.

[2] R. C. Cross and M. S. Wheatland. Modeling a falling slinky. *American Journal of Physics*, 80 (12):1051–1060, 2012.



(a) $ct/L = 0.0$

(b) $ct/L = 0.2$

(c) $ct/L = 0.4$

(d) $ct/L = 0.6$

(e) $ct/L = 0.8$

(f) $ct/L = 1.0$

Figure 1: Falling bar problem at different times. Black: displacement $Eu/\rho g L^2$, red: stress $\sigma/\rho g L$, blue: velocity $c/gL(\partial u/\partial t)$.