Lecture 16: Switched Linear Quadratic Regulation
Optimal Control of D-T Nonswitched Systems

A discrete-time controlled dynamical system

\[ x_{k+1} = f(x_k, u_k), \quad k = 0, 1, \ldots \]

**Problem:** Given a time horizon \( k \in \mathbb{N} \cup \{\infty\} \), find the optimal input sequence \( u = (u_0, \ldots, u_{N-1}) \) that minimizes

\[
J(u) = \sum_{k=0}^{N-1} \ell(x_k, u_k) + \phi(x_N)
\]

- Running cost \( \ell(x_k, u_k) \geq 0 \) and terminal cost \( \phi(x_N) \geq 0 \)
- State constraint \( x \in \mathcal{X} \) and control constraint \( u \in \mathcal{U} \)
D-T Linear Quadratic Regulation

A discrete-time LTI system with given initial condition \( x_0 \):

\[
\begin{align*}
x_{k+1} &= Ax_k + Bu_k \\
y_k &= Cx_k + Du_k
\end{align*}
\]

**Problem:** find optimal input sequence \( u = (u_0, \ldots, u_{N-1}) \) that minimizes

\[
J(u) = \sum_{k=0}^{N-1} \left( x_k^T Q x_k + u_k^T R u_k \right) + x_N^T Q_f x_N
\]

- **State weight matrix** \( Q = Q^T \succeq 0 \)
- **Control weight matrix** \( R = R^T \succ 0 \) (no free control)
- **Terminal state weight matrix** \( Q_f = Q_f^T \succeq 0 \)
Switched Linear Quadratic Regulation

**Problem:** For the switched linear system

\[
\begin{align*}
    x_{k+1} &= A_{\sigma_k} x_k + B_{\sigma_k} u_k \\
    y_k &= C_{\sigma_k} x_k + D_{\sigma_k} u_k
\end{align*}
\]

Find the optimal input sequence \((u_0, \ldots, u_{N-1})\) and mode sequence \((\sigma_0, \ldots, \sigma_{N-1})\) that minimize the cost function

\[
\sum_{k=0}^{N-1} \left( x_k^T Q_{\sigma_k} x_k + u_k^T R_{\sigma_k} u_k \right) + x_N^T Q_f x_N
\]

- Both dynamics and costs are different in different modes
- Special case: no continuous input \((B_i = 0, D_i = 0\) for all \(i)\)
- Challenge: number of mode sequences increases exponentially with \(N\)
Example: Energy Efficient Building Control

- **Goal:**
  1. Maintain zone temperatures near setpoints (loss-of-productivity cost)
  2. Less energy consumption (energy cost)

- Control: cooling/heating equipments (some on/off only)
Solution Roadmap

- Solve LQR problem using dynamic programming method
- Extend the method to solve SLQR problem
- Complexity reduction techniques

LQR as a constrained optimization problem:

Minimize \[
\sum_{k=0}^{N-1} \left( x_k^T Q x_k + u_k^T R u_k \right) + x_N^T Q_f x_N
\]

subject to \[ x_{k+1} = A x_k + B u_k, \quad k = 0, \ldots, N - 1 \]
\[ x_0 \text{ fixed} \]
Direct Solution

LQR as a least square problem:

Minimize $\mathbf{x}^T \begin{bmatrix} Q & Q & \cdots & Q_f \end{bmatrix} \mathbf{x} + \mathbf{u}^T \begin{bmatrix} R & R & \cdots & R \\ Q & R & \cdots & R \\ \vdots & \vdots & \ddots & \vdots \\ Q_f & \cdots & \cdots & R \end{bmatrix} \mathbf{u}$

such that

$\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_N \end{bmatrix} = \begin{bmatrix} \mathbf{B} & 0 & \cdots \\ \mathbf{A} \mathbf{B} & \mathbf{B} & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}^{N-1} \mathbf{B} & \mathbf{A}^{N-2} \mathbf{B} & \cdots & \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{N-1} \end{bmatrix} + \begin{bmatrix} \mathbf{A} \\ \mathbf{A}^2 \\ \vdots \\ \mathbf{A}^N \end{bmatrix} \mathbf{x}_0$

Optimal control sequence is $\mathbf{u}^* = - (\mathbf{R} + \mathbf{G}^T \mathbf{Q} \mathbf{G})^{-1} \mathbf{G}^T \mathbf{Q} \mathbf{H} \mathbf{x}_0$

- Not numerically viable for large $N$ (e.g., $N = \infty$)
Dynamical Programming Approach

**Idea:** Solve a sequence of optimal control problems over time horizons \( \{t, \ldots, N\} \), for \( t = N, N - 1, \ldots, 0 \)

- **Value function** at time \( t \) is the optimal cost over horizon \( \{t, \ldots, N\} \):
  \[
  V_t(x) = \min_{u_t, u_{t+1}, \ldots, u_{N-1}} \sum_{k=t}^{N-1} (x_k^T Q x_k + u_k^T R u_k) + x_N^T Q_f x_N
  \]
  with the initial condition \( x_t = x \)

- **Value function iteration:** \( V_{t-1}(\cdot) \) can be computed based on \( V_t(\cdot) \)

- Optimal cost of original problem is \( V_0(x_0) \)

- Optimal input sequence can be recovered from value functions
Motivating Example

- Start from point A
- End at point B
- Each step only move right
- Cost labeled on each edge

**Problem:** find the path from A to B with the least cost

- For $\ell$-by-$\ell$ grid, total number of legal paths is $\frac{(2\ell)!}{(\ell!)^2}$
Value Function

Value function $V(z)$ at node $z$ is the least cost to reach $B$ from $z$

**Value function iteration:** $V(z)$ satisfies

$$V(z) = \min\{w_u + V(z_u'), w_d + V(z_d')\}$$

**Principle of Optimality:** If $x_0^* = z \rightarrow x_1^* \rightarrow x_2^* \rightarrow \cdots \rightarrow B$ is a least-cost path from $z$ to $B$, then any truncation of it is also a least-cost path e.g., $x_1^* \rightarrow x_2^* \rightarrow \cdots \rightarrow B$ is a least-cost path from $x_1^*$ to $B$

Reduced computational complexity: for $\ell$-by-$\ell$ grid, only need to compute $\ell^2$ value functions, each with fixed complexity
Iteration Results

- Stage 1: compute the value functions from right to left
- Stage 2: recover the least-cost path from left to right
Value Functions of LQR Problem

**Value function** at time $t \in \{0, 1, \ldots, N\}$ and state $x \in \mathbb{R}^n$ is

$$V_t(x) = \min_{u_t, u_{t+1}, \ldots, u_{N-1}} \left\{ \sum_{k=t}^{N-1} \left( x_k^T Q x_k + u_k^T R u_k \right) + x_N^T Q_f x_N \middle| x_t = x \right\}$$

- $V_t(x)$ is the optimal cost from time $t$ to $N$ starting from $x_t = x$
- $V_0(x_0)$ is the optimal cost of the original LQR problem

**Bellman equation**: (value function iteration)

$$V_t(x) = \min_{u_t=v} \left[ x^T Q x + v^T R v + V_{t+1}(Ax + Bv) \right]$$

- **Principle of Optimality**: cost-to-go from next state $x_{t+1}$ is also optimal
Value Function Iteration

- Value function at time $N$ is quadratic: $V_N(x) = x^T Q_f x$

- Suppose $V_{t+1}(x) = x^T P_{t+1} x$ is quadratic, then
  
  $$V_t(x) = \min_v \left[ x^T Q x + v^T R v + V_{t+1}(A x + B v) \right]$$

  $$= \min_v \begin{bmatrix} x \\ v \end{bmatrix}^T \begin{bmatrix} Q + A^T P_{t+1} A & A^T P_{t+1} B \\ B^T P_{t+1} A & R + B^T P_{t+1} B \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

  $$= x^T P_t x$$

  where $P_t$ is the Schur complement of $R + B^T P_{t+1} B$ in the block matrix:

  $$P_t := \rho(P_{t+1}) = \underbrace{Q + A^T P_{t+1} A - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A}_{\text{Riccati mapping}}$$

- Minimum achieved by the optimal control
  
  $$u_t^* = - (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A x = -K_t x$$

  $$\underbrace{K_t}_{\text{Kalman gain}}$$
Useful Properties of Riccati Mapping

Riccati mapping $\rho: \mathbb{S}_+ \rightarrow \mathbb{S}_+$ maps PSD matrices to PSD matrices:

$$P_t = Q + A^T P_{t+1} A - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A$$

- derived from $x^T P_t x = \min_v \left[ \begin{bmatrix} x \\ v \end{bmatrix}^T \left[ \begin{bmatrix} Q + A^T P_{t+1} A & A^T P_{t+1} B \\ B^T P_{t+1} & R + B^T P_{t+1} B \end{bmatrix} \right] \begin{bmatrix} x \\ v \end{bmatrix} \right]$

- $\rho$ is PSD-monotone:

$$P \preceq P' \text{ implies } \rho(P) \preceq \rho(P')$$

- $\rho$ is PSD-concave, i.e., for $P, P' \in \mathbb{S}_+$ and $\theta \in [0, 1]$

$$\rho \left( \theta P + (1 - \theta) P' \right) \succeq \theta \cdot \rho(P) + (1 - \theta) \cdot \rho(P')$$

LQR Solution Algorithm

Stage 1: Compute the value functions backward in time:

Set $P_N = Q_f$

for $t = N - 1, \ldots, 1, 0$ do

$$P_t = Q + A^T P_{t+1} A - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A$$

end for

Return $x_0^T P_0 x_0$ as the optimal cost

Stage 2: Compute the optimal controls and state solution forward in time:

Set $x_0^* = x_0$

for $t = 0, 1, \ldots, N - 1$ do

$$u_t^* = -(R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A x_t^*$$

$$x_{t+1}^* = A x_t^* + B u_t^*$$

end for

Return $u_t^*$ and $x_t^*$ as the optimal control and state sequences
Inverted Pendulum: Model

States $x = [z \quad \theta \quad \dot{z} \quad \dot{\theta}]^T$:
- $\theta$: angle of pendulum
- $\dot{\theta}$: angular velocity of pendulum
- $z$: horizontal position of cart
- $\dot{z}$: velocity of cart

Linearized dynamics near $x_e = [0 \quad 0 \quad 0 \quad 0]^T$ (sampled at $T = 0.5$ s):

$$x_{k+1} = \begin{bmatrix}
1 & 0.1054 & 0.06128 & 0.01688 \\
0 & 5.753 & -1.859 & 1.154 \\
0 & 0.5287 & -0.1617 & 0.1037 \\
0 & 27.31 & -9.328 & 5.665
\end{bmatrix} x_k + \begin{bmatrix}
0.05763 \\
0.2442 \\
0.1526 \\
1.225
\end{bmatrix} u_k$$
**Inverted Pendulum: Stabilization**

\[ x(0) \rightarrow x_e \]

\[ x_0 = [-1 \ 0 \ 0 \ 0]^T \]

**Goal:** Find \( u_0, \ldots, u_{N-1} \) to minimize

\[
J = \alpha \sum_{k=1}^{N} \|x_k\|^2 + \beta \sum_{k=0}^{N-1} |u_k|^2
\]

- LQR formulation: \( Q = Q_f = \alpha I \), \( R = \beta \)
- Stabilization with energy consideration
Inverted Pendulum: Solutions

Optimal solution with horizon $10$  ($\alpha = 10, \beta = 1$)

Optimal solution with horizon $10$  ($\alpha = 10^4, \beta = 1$)
Infinite Horizon Case \((N = \infty)\)

- If \((A, B)\) is stabilizable, then Riccati iteration will converge to a solution \(P_{ss}\) of the **Algebraic Riccati Equation (ARE)**:

\[
P_{ss} = Q + A^T P_{ss} A - A^T P_{ss} B (R + B^T P_{ss} B)^{-1} B^T P_{ss} A
\]

- If further \((Q, A)\) is detectable, then \(P_{ss}\) is unique, and under the steady-state optimal control gain

\[
K_{ss} = (R + B^T P_{ss} B)^{-1} B^T P_{ss} A,
\]

the closed-loop system \(A_{cl} = A - BK_{ss}\) is stable

  - Hence stabilization can be studied via LQR

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Switched LQR Problem

Find control sequence $u_0, \ldots, u_{N-1}$ and mode sequence $\sigma_0, \ldots, \sigma_{N-1}$ to

minimize $\left\lfloor \sum_{k=0}^{N-1} \left( x_k^T Q_{\sigma_k} x_k + u_k^T R_{\sigma_k} u_k \right) + x_N^T Q_f x_N \right\rfloor$

subject to $x_{k+1} = A_{\sigma_k} x_k + B_{\sigma_k} u_k$, $k = 0, \ldots, N - 1$, and fixed $x_0$

Value function at each $t = 0, 1, \ldots, N$ and $x$ is the optimal cost over horizon $\{t, \ldots, N\}$ assuming $x_t = x$

$$V_t(x) = \min_{\sigma_t, \ldots, \sigma_{N-1}} \sum_{u_t, \ldots, u_{N-1}} \sum_{k=t}^{N-1} \left( x_k^T Q_{\sigma_k} x_k^T + u_k^T R_{\sigma_k} u_k \right) + x_N^T Q_f x_N$$

- $V_0(x_0)$ is the optimal cost of the original SLQR problem
- $V_t(x)$ does not depend on mode (why?)
SLQR Bellman Equation

Value functions can be obtained from the iteration

\[ V_t(x) = \min_{\sigma_t = \sigma, u_t = v} \left[ x^T Q_\sigma x + v^T R_\sigma v + V_{t+1}(A_\sigma x + B_\sigma v) \right] \]

- Optimal control and mode are the ones achieving minimum above
  - Optimal state feedback switching policy \( \sigma^*_t(x) \)
  - Optimal state feedback control policy \( u^*_t(x) \)

- Value functions are in general not quadratic

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Value Function at $t = N - 1$

\begin{align*}
V_{N-1}(x) &= \min_{\sigma} \min_{\nu} \left[ x^T Q_{\sigma} x + \nu^T R_{\sigma} \nu + V_N(A_{\sigma} x + B_{\sigma} \nu) \right] \\
&= \min_{\sigma} \left[ x^T \rho_{\sigma}(Q_f) x \right]
\end{align*}

Here, $\rho_{\sigma}$ is the Riccati mapping of subsystem $(A_{\sigma}, B_{\sigma})$ with weights $Q_{\sigma}, R_{\sigma}$.
Optimal Control at $t = N - 1$

$V_{N-1}(x)$ is pointwise minimum of a number of quadratic functions

$$V_{N-1}(x) = \min_{P \in \mathcal{P}_{N-1}} x^T P x$$

where $\mathcal{P}_{N-1} = \{\rho_1(Q_f), \ldots, \rho_M(Q_f)\} := \rho_M(Q_f)$

- State space partitioned into sectors (cones)
- Each sector has an optimal mode and a Kalman gain
- May have multiple sectors with different Kalman gains for a mode
General $t$ Case

Suppose $V_{t+1}(x) = \min_{P \in \mathcal{P}_{t+1}} x^T P x$ for some set $\mathcal{P}_{t+1} \subset \mathbb{S}^n_+$

At time $t$, the value function $V_t(x)$ is given by

$$V_t(x) = \min_{P \in \mathcal{P}_t} x^T P x$$

where $\mathcal{P}_t$ is obtained from $\mathcal{P}_{t+1}$ by **switched Riccati mapping**:

$$\mathcal{P}_t = \rho \mathcal{M}(\mathcal{P}_{t+1}) := \{\rho_i(P) \mid P \in \mathcal{P}_{t+1}, i \in \mathcal{M}\} \subset \mathbb{S}^n_+$$
SLQR Solution Algorithm

Stage 1: Compute the value functions backward in time:

Set \( \mathcal{P}_N = \{Q_f\} \)

for \( t = N - 1, \ldots, 1, 0 \) do

\[
\mathcal{P}_t = \rho_{\mathcal{M}}(\mathcal{P}_{t+1})
\]

end for

Return \( \min_{\mathcal{P} \in \mathcal{P}_0} x_0^T P x_0 \) as the optimal cost

Stage 2: Compute optimal mode/control/state sequences forward in time:

Set \( x_0^* = x_0 \)

for \( t = 0, 1, \ldots, N - 1 \) do

\[
(\sigma_t^*, u_t^*) = \arg \min_{\sigma, v} \left[ (x_t^*)^T Q_\sigma x_t^* + v^T R_\sigma v + V_{t+1}(A_{\sigma} x_t^* + B_{\sigma} v) \right]
\]

\[
x_{t+1}^* = A_{\sigma_t^*} x_t^* + B_{\sigma_t^*} u_t^*
\]

end for

Return \( \sigma_t^*, u_t^* \) and \( x_t^* \) as the optimal mode/control/state sequences
Complexity Reduction

**Issue:** Number of matrices in $\mathcal{P}_t$ grows exponentially

In the set $\mathcal{P}_t$ defining the value function $V_t(x) = \min_{P \in \mathcal{P}_t} x^T P x$

- Matrix $P \in \mathcal{P}_t$ is called **effective** if for at least one $x \neq 0$

  $$x^T P x < x^T P' x, \quad \forall P' \in \mathcal{P}_t \setminus \{P\}$$

- Otherwise $P$ is called **redundant** and can be discarded without loss (due to monotonicity of Riccati mapping)

**Sufficient condition** for $P \in \mathcal{P}_t$ to be redundant:

$$P \succeq \text{a convex combination of matrices in } \mathcal{P}_t \setminus \{P\}$$

- Checked via an LMI feasibility problem
Decision Tree Pruning

\[
\begin{align*}
\rho_1(\rho_1(Q_f)) & \quad \sigma = 1 \\
\rho_2(\rho_1(Q_f)) & \quad \sigma = 2 \\
\rho_1(\rho_2(Q_f)) & \quad \sigma = 1 \\
\rho_2(\rho_2(Q_f)) & \quad \sigma = 2 \\
Q_f & \quad \sigma = 1 \\
\end{align*}
\]
Example

Switched LQR problem specified by

\[
A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.5 & 1 \\ 0 & 1.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

\[Q_\sigma = I, \quad R_\sigma = 1, \quad N = 20\]
Further Reduction by Relaxation

Matrix $P$ in $\mathcal{P}_t$ is called $\varepsilon$-redundant if $P + \varepsilon I$ is redundant

- $\varepsilon > 0$ is a small relaxation parameter
- A sufficient condition is given by

$$P + \varepsilon I \succeq \text{a convex combination of matrices in } \mathcal{P}_t \setminus \{P\}$$

- Oftentimes a small $\varepsilon$ could result in significant reduction

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Example

\[ A_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 1 \\ 0 & 0.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \]

\[ Q_1 = Q_2 = I, \quad R_1 = R_2 = 1, \quad Q_f = I, \quad N = 100 \]
Example (cont.)

Optimal switching policy (Gray region: mode 1 optimal; Black region: mode 2 optimal)
Another Example

\[
A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]
\[
A_2 = \begin{bmatrix} 1.5 & 1 \\ 0 & 1.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]
\[
Q_\sigma = I, \quad R_\sigma = 1, \quad N = 20
\]

- Without any reduction, complexity grows exponentially
- With reduction, complexity saturates at 360 matrices
- With relaxation (\(\varepsilon = 10^{-3}\)), complexity saturates at 14 matrices
SLQR Problem with Switching Cost

Cost function to be minimized:

\[
\sum_{k=0}^{N-1} \left( x_k^T Q_{\sigma_k} x_k + u_k^T R_{\sigma_k} u_k + w(\sigma_k, \sigma_{k+1})(x_k) \right) + x_N^T Q_f x_N
\]

Value function \( V_t(\sigma, x) \) is the optimal cost-to-go starting from \( x_t = x \) with previous mode being \( \sigma_{t-1} = \sigma \)

Bellman equation:

\[
V_t(\sigma, x) = \min_{\sigma_t = \sigma', u_t = v} \left[ x^T Q_{\sigma'} x + v^T R_{\sigma'} v + w(\sigma, \sigma')(x) + V_{t+1}(\sigma', A_{\sigma'} x + B_{\sigma'} v) \right]
\]

- Optimal switching policy \( \sigma_t^*(\sigma, x) \) and optimal control \( u_t^*(\sigma, x) \)
- Both depend on previous mode \( \sigma \) and current state \( x \)
- Same technique if \( w(\sigma, \sigma')(\cdot) \) is quadratic, linear, or constant,
Continuous-Time LQR Problem

For the continuous-time LTI system

\[
\dot{x} = Ax + Bu, \quad x(0) = x_0
\]

Find the control input \( u(t) \) over the time horizon \([0, t_f]\) to

\[
\text{minimize } \int_0^{t_f} \left( x^T Q x + u^T R u \right) dt + x(t_f)^T Q_f x(t_f)
\]

- \( Q = Q^T \succeq 0, \quad R = R^T \succ 0, \quad Q_f = Q_f^T \succeq 0 \)

**Value function** \( V_t(x) \) is the optimal cost-to-go at time \( t \) from state \( x \):

\[
V_t(x) = \min_{u(s), s \in [t, t_f]} \int_t^{t_f} \left[ x(s)^T Q x(s)^T + u(s)^T R u(s) \right] ds + x(t_f)^T Q_f x(t_f)
\]

- Value functions are still quadratic: \( V_t(x) = x^T P(t)x, \ t \in [0, t_f] \)
Continuous-Time Bellman Equation

As $\delta \to 0$, we obtain the **Riccati (matrix) differential equation**:

$$-\dot{P}(t) = Q + P(t)A + A^T P(t) - P(t)BR^{-1}B^T P(t), \quad P(t_f) = Q_f$$

The optimal control is a linear state feedback controller:

$$u^*_t = -R^{-1}B^T P(t)x$$
Continuous-Time SLQR Problem

For the continuous-time switched linear system

\[
\dot{x} = A_\sigma x + B_\sigma u, \quad x(0) = x_0
\]

Find the mode \( \sigma(t) \in \mathcal{M} \) and input \( u(t) \) over the time horizon \([0, t_f]\) to minimize

\[
\int_0^{t_f} \left( x^T Q_\sigma x + u^T R_\sigma u \right) dt + x(t_f)^T Q_f x(t_f)
\]

- \( Q_\sigma = Q_\sigma^T \succeq 0, \quad R_\sigma = R_\sigma^T \succ 0, \quad Q_f = Q_f^T \succeq 0 \)

Value function \( V_t(x) \) is the optimal cost-to-go at time \( t \) from state \( x \):

\[
V_t(x) = \min_{u(s), \sigma(s)} \int_t^{t_f} \left[ x(s)^T Q_{\sigma(s)} x(s) + u(s)^T R_{\sigma(s)} u(s) \right] ds + x(t_f)^T Q_f x(t_f)
\]
Value Functions of C.-T. SLQR Problem

The value function $V_t(x)$ is still of the form

$$V_t(x) = \inf_{P \in \mathcal{P}(t)} x^T P x$$

- $|\mathcal{P}(t)| = \infty$ as infinite switchings possible in a short time interval
- $\mathcal{P}(t)$ can be computed from the Riccati differential inclusion

$$-\dot{P}(t) \in \{Q_\lambda + P(t)A_\lambda + A_\lambda^T P(t) - P(t)B_\lambda R_\lambda^{-1} B_\lambda^T P(t) : \lambda \in S\}$$

where $A_\lambda, B_\lambda, Q_\lambda, R_\lambda$ is any convex combination of $A_i, B_i, Q_i, R_i$ for $i \in \mathcal{M}$

In general, $\mathcal{P}(t)$ is very difficult to compute analytically and numerically
- Impose dwell-time or maximum switching times constraints
- Discretize the C.-T. SLS into D.-T. SLS