Lecture 15: Stabilization II
Switching Stabilization of Discrete-Time SLS

D-T switched linear system $x(k + 1) = A_{\sigma(k)} x(k), \ k = 0, 1, \ldots$

- Mode $\sigma(k) \in \Sigma = \{1, \ldots, m\}$
- Assume each $A_i$ is not (Schur) stable

SLS is called **switching stabilizable** if for any $x(0)$, there exists a switching sequence $\sigma(\cdot)$ under which $x(k) \to 0$ as $k \to \infty$

- **exponentially switching stabilizable** if for some $\rho \in [0, 1)$, $c \geq 0$,
  \[
  \|x(k)\| \leq c \rho^k \|x(0)\|, \ \forall k, \forall x(0). \tag{1}
  \]

- Two stability notions are equivalent
- **Switching stabilizing rate** $\rho_*$ is the smallest $\rho$ such that (1) holds.
Stabilization via Open-Loop Switching Policies

Example: \( A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.4 & 0 \\ 0 & 3 \end{bmatrix}. \)

- \( A_1A_2 = A_2A_1 \) is stable
- Periodic switching policy \( \sigma(\cdot) = \{1, 2, 1, 2, \ldots\} \) stabilizes the system regardless of \( x(0) \)

SLS is called **open-loop switching stabilizable** if there exists a switching sequence \( \sigma(\cdot) \) such that

\[
\lim_{k \to \infty} A_{\sigma(k-1)}A_{\sigma(k-2)} \cdots A_{\sigma(0)} = 0
\]
Joint Spectral Subradius

Joint spectral subradius of a set of matrices \( \mathcal{A} = \{A_1, \ldots, A_m\} \) is

\[
\tilde{\rho}^*_*(\mathcal{A}) := \lim_{k \to \infty} \min_{A_{i_1}, \ldots, A_{i_k} \in \mathcal{A}} \|A_{i_1} \cdots A_{i_k}\|^{1/k}
\]

- Independent of the choice of induced matrix norm \( \|\cdot\| \)
- Replacing min with max, we get the joint spectral radius \( \rho^*(\mathcal{A}) \)
- Equivalently, can use the spectral radius \( \rho(\cdot) \) instead of \( \|\cdot\| \)
- Stabilizing rate using open-loop switching polices
Example I

\[
A_1 = \begin{bmatrix} 100 & 0 \\ 0 & 0.99 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}
\]

- \( \theta > 0 \) very small, or \( \theta = \alpha \pi \) for some irrational \( \alpha > 0 \)
- Not stabilizable by open-loop switching policies: for any \( \sigma(\cdot) \),

\[
\det \left( A_{\sigma(k-1)}A_{\sigma(k-2)} \cdots A_{\sigma(0)} \right) \geq 1
\]

\[
\Rightarrow \quad \rho \left( A_{\sigma(k-1)}A_{\sigma(k-2)} \cdots A_{\sigma(0)} \right) \geq 1
\]

Hence \( \tilde{\rho}_* \geq 1 \)

- SLS is switching stabilizable, i.e., \( \rho_* < 1 \)

Conclusion: \( \tilde{\rho}_* \geq \rho_* \) with strict inequality in general

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Example II

SLS with 
\[ A_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \]

- \( A_2A_1^{k-1} = A_2 \), which implies \( \tilde{\rho}_* \leq 1 \)

- From \( x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \), \( x_2(k) \) is nondecreasing regardless of \( \sigma \); hence \( \rho_* \geq 1 \)

- Conclusion: \( \rho_* = 1 \)

- \( \rho^* \) cannot be exactly achieved by any closed-loop switching policy

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Remarks

Numerical computation of $\tilde{\rho}_*$ and $\rho_*$ is very difficult

- Known to be NP-hard problems
- In a sense more difficult than computing JSR $\rho^*$
- No easy lower/upper bounds using induced norms
**Joint Control & Switching Stabilization**

**Stabilization Problem:** For switched system $\dot{x} = f_\sigma(x, u)$, design a switching law $\sigma(x)$ and continuous controllers $u_\sigma(x)$ such that the closed-loop system is asymptotically stable at e.g. $x_e = 0$

Example: temperature regulation of multi-zone buildings
Switched Controlled Linear Systems

Switched controlled linear system

\[ \dot{x} = A_\sigma x + B_\sigma u, \]

Mode-dependent linear state feedback control

- A set of linear state feedback controllers with gains \( K_1, \ldots, K_m \):
  \[ u_\sigma(x) = K_\sigma x, \quad \sigma \in \Sigma = \{1, \ldots, m\} \]

- Closed-loop system is a switched linear system
  \[ \dot{x} = (A_\sigma + B_\sigma K_\sigma) x \]

- A state-dependent switching policy \( \sigma(x) \)

Need to consider more general (i.e., nonlinear) controllers?
Control Lyapunov Function

Controlled Lyapunov Function: A PD function \( V(x) \) such that

1. \( \dot{V}(x; f_\sigma(x, u)) \) exists for all \( x, \sigma \in \Sigma, u \in \mathcal{U} \);
2. \( \inf_{\sigma \in \Sigma, u \in \mathcal{U}} \dot{V}(x; f_\sigma(x, u)) < 0 \) for all \( x \neq 0 \)

Stabilizing switching and control policy:

\[
(\sigma^*(x), u^*(x)) \in \left\{ (\sigma, u) \mid \dot{V}(x; f_\sigma(x, u)) < 0 \right\}
\]

Quadratically stabilizable if quadratic CLF \( V(x) = x^T P x \) exists

- All directional derivatives exist
- Stronger than stabilizability
Quadratic Stabilization of SCLS

Switched controlled linear system $\dot{x} = A_\sigma x + B_\sigma u$

- $V(x) = x^T P x$ is a CLF if
  $$\min_{i \in \Sigma} \inf_u (2Px) \cdot (A_i x + B_i u) < 0, \quad \forall x \neq 0$$
  $$\Leftrightarrow \min_{i \in \Sigma} x^T (PA_i + A_i^T P)x < 0 \text{ whenever } 0 \neq x \in \text{Range}(P^{-1} B^\perp)$$
  $$\Leftrightarrow \min_{i \in \Sigma} z^T (B^\perp)^T (A_i Q + QA_i^T) B^\perp z < 0, \quad \forall z \neq 0 \quad (Q := P^{-1})$$
  $$\Leftrightarrow (B^\perp)^T \left[ \left( \sum \alpha_i A_i \right) Q + Q \left( \sum \alpha_i A_i^T \right) \right] B^\perp < 0 \text{ for some } \alpha \in \Delta_{\Sigma}$$

**Sufficient condition:** $\exists \alpha \in \Delta_{\Sigma}$ (i.e., $\alpha_i \geq 0, \forall i \in \Sigma, \sum_i \alpha_i = 1$) such that the LTI system $(\sum_i \alpha_i A_i, [B_1 \cdots B_m])$ is stabilizable.
Stabilization of Discrete-Time SCLS

D-T SCLS \( x(k + 1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k), \quad k = 0, 1, \ldots \)

**Idea:** Consider CLF \( V_P(x) = \min_{P \in \mathcal{P}} x^T P x \), where \( \mathcal{P} = \{P_1, \ldots, P_\ell\} \)

\( V_P(\cdot) \) is a CLF if and only if

\[
\min_{i \in \Sigma} \min_{P \in \mathcal{P}} \inf_u \left[ V_P(A_i x + B_i u) + u^T R u \right] < V_P(x), \quad \forall x \neq 0 \quad (R \succ 0)
\]

\[\Leftrightarrow\]

\[
\min_{i \in \Sigma} \min_{P \in \mathcal{P}} \min_{\rho_i(P)} \inf_u \left[ x^T \begin{bmatrix} A_i^T P A_i & A_i^T P B_i \\ B_i^T P A_i & R + B_i^T P B_i \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right] < V_P(x), \quad \forall x \neq 0
\]

\[
\Leftrightarrow \quad \min_{i \in \Sigma} \min_{P \in \mathcal{P}} \min_{Q \in \rho(P)} x^T (A_i^T P A_i - A_i P B_i (R + B_i^T P B_i)^{-1} B_i^T P A_i) x < V_P(x), \quad \forall x \neq 0
\]

\( \rho_i(P) \): **Riccati mapping** of subsystem \( i \)

\[
\Leftrightarrow \quad \min_{Q \in \rho(P)} x^T Q x < \min_{P \in \mathcal{P}} x^T P x, \quad \forall x \neq 0
\]

where \( \rho(P) \) is the matrix set \( \{\rho_i(P) | i \in \Sigma, P \in \mathcal{P}\} \)
Sufficient Stabilization Condition

\( V_P(\cdot) \) is a CLF if and only if

\[
\text{For any } P \in \mathcal{P}, \min_{Q \in \rho(P)} x^T Q x < x^T P x, \quad \forall x \neq 0
\]

A sufficient condition is

\[
\text{For any } P \in \mathcal{P}, \quad \sum_{Q \in \rho(P)} \beta_P^Q \cdot Q \prec P \text{ for some } \beta_P^Q \geq 0, \quad \sum_{Q \in \rho(P)} \beta_P^Q = 1
\]

- With \( \mathcal{P} \) given, \( \rho(\mathcal{P}) \) are known (with cardinality \( |\mathcal{P}| \cdot |\Sigma| \))
- Solved as a group of SDP feasibility problems in \( \beta \)'s
- Replace \( P \) on the RHS with \( \gamma P \) to estimate the stabilizing rate
- Update \( \mathcal{P} \) with \( \rho(\mathcal{P}) \) can improve the quality of CLF

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The SCLS is (exponentially) stabilizable if and only if there exists a CLF of the form \( V_P(x) = \min_{P \in \mathcal{P}} x^T P x \) for some matrix set \( \mathcal{P} \).

- \( V_P(x) \) can be obtained by solving an optimal control problem
- Size of \( \mathcal{P} \) could be very large
- Possible to reduce complexity (more on this later)

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Example

\[ A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1.5 & 1 \\ 0 & 1.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{(neither stabilizable)} \]

Typical decision regions (\( \mathcal{P} = \{P_1, P_2\} \))

Stabilized state solutions

Piecewise linear feedback controller
(different gains on different sectors)
Remarks

Some further topics:

- Joint control & switching stabilizability using general CLFs
  - Composite (maximum, minimum) quadratic functions
  - Piecewise quadratic functions
  - Sum of squares (SOS)
- Stabilizability of switched polynomial systems via SOS
- Optimal control approach to stabilization
- Stabilization using receding horizon control or Model Predictive Control (MPC)