Lecture 15: Switching Stabilization
Stabilization Problem

**Problem:** Given the (nonlinear) system \( \dot{x} = f(x, u) \), design a state feedback controller \( u = u(x, t) \in \mathcal{U} \) so that the closed-loop system

\[
\dot{x} = f(x, u(x))
\]

is asymptotically/exponentially stable at the equilibrium \( x_e \)

**Example: Inverted Pendulum**

System dynamics:

\[
\begin{cases}
(M + m)\ddot{x} = m\ell\dddot{\theta} \cos \theta + m\ell(\dot{\theta})^2 = F \\
\ell\dddot{\theta} - g \sin \theta = \ddot{x} \cos \theta
\end{cases}
\]

Design input \( F \) to stabilize the system at

\[
\theta = \dot{\theta} = x = \dot{x} = 0
\]

(from Wiki)
Controlled Lyapunov Function

Controlled Lyapunov function (CLF): a $C^1$ positive definite (PD) function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that (assume $x_e = 0$)

$$\inf_{u \in \mathcal{U}} \left[ \frac{\partial V}{\partial x}(x) \cdot f(x, u) \right] < 0, \quad \forall x \neq 0$$

• $V$ decreases along at least one control direction

Theorem (CLF)

If a CLF exists, then the system $\dot{x} = f(x, u)$ can be stabilized by a state feedback controller $u^*(x)$ such that $u^*(0) = 0$ and

$$u^*(x) \in \left\{ u \in \mathcal{U} \left| \frac{\partial V}{\partial x}(x) \cdot f(x, u) < 0 \right. \right\}, \quad \forall x \neq 0$$

• Choose $u^*(x)$ for regularity (e.g. Lipschitz continuity)
• Quadratically stabilizable if a quadratic CLF $V(x) = x^T P x$ exists
Exponential Controlled Lyapunov Function

**Exponential CLF:** a $C^1$ positive definition function $V$ such that

$$\inf_{u \in U} \left[ \frac{\partial V}{\partial x}(x) \cdot f(x, u) \right] < -\alpha V(x), \quad \forall x \neq 0, \quad (\text{for some } \alpha > 0)$$

- Alternative conditions for ECLF: $\exists \beta_1, \beta_2, \beta_3 > 0$ such that
  1. $\beta_1 \|x\|^2 \leq V(x) \leq \beta_2 \|x\|^2$
  2. $\inf_{u \in U} \left[ \frac{\partial V}{\partial x}(x) \cdot f(x, u) \right] < -\beta_3 \|x\|^2$

**Theorem (ECLF)**

If an ECLF exists, then the system $\dot{x} = f(x, u)$ can be exponentially stabilized by a state feedback controller

$$u^*(x) = \arg \min_{u \in U} \left[ \frac{\partial V}{\partial x}(x) \cdot f(x, u) \right], \quad \forall x \neq 0$$
Example: LTI Systems

LTI system \( \dot{x} = Ax + Bu \) has a quadratic CLF \( V(x) = x^T Px, \ P \succ 0 \), if

\[
\inf_u (2Px) \cdot (Ax + Bu) < 0, \ \forall x \neq 0
\]

\[
\iff x^T (PA + A^T P)x + \inf_u (2Px) \cdot (Bu) < 0, \ \forall x \neq 0
\]

\[
\iff x^T (PA + A^T P)x < 0 \text{ whenever } x \in \text{Range}(P^{-1}B^\perp), \ x \neq 0
\]

\[
\iff (B^\perp)^T (AP^{-1} + P^{-1}A)B^\perp < 0
\]

\[
\iff AQ + QA + BY + Y^TB^T < 0 \text{ for some } Y \quad (Q = P^{-1})
\]

where we use the Finsler’s Lemma (HW 1, Problem 9).

**Conclusion:** LTI system is quadratically stabilizable if and only if it is stabilizable by a linear state feedback controller \( u = Kx = YQ^{-1}x \)
CLF for Discrete-Time Systems

D-T system $x(k + 1) = f(x(k), u(k)), k \in \mathbb{N}$

CLF is a PD function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that

$$\inf_{u \in \mathcal{U}} [V(f(x, u)) - V(x)] < 0, \quad \forall x \neq 0$$

If a CLF exists, the system is stabilized to $x_e = 0$ by the controller

$$u^*(x) \in \{ u \in \mathcal{U} | V(f(x, u)) - V(x) < 0 \}$$

- Example: $u^*(x) = \arg \min_{u \in \mathcal{U}} V(f(x, u))$ (if exists)
- Extension to exponential CLF
Switching Stabilization of Hybrid Systems

Switching Stabilization Problem: For switched system

\[ \dot{x}(t) = f_{\sigma(t)}(x(t)), \quad t \geq 0 \]

with mode \( \sigma(t) \in \Sigma := \{1, \ldots, m\} \), design a state-feedback switching law \( \sigma : \mathbb{R}^n \rightarrow \Sigma \) such that the closed-loop system \( \dot{x} = f_{\sigma(x)}(x) \) is stable at the equilibrium \( x_e = 0 \)

- Switching signal \( \sigma(t) \) is the only control input
- Problem is trivial if any of the subsystem is stable
- Switching policy partitions \( \mathbb{R}^n \) into \( m \) decision regions
- More general (e.g. pointwise, time-varying) switching stabilization
Simple Inverted Pendulum

Pendulum on horizontal track (no friction)

Control input is pivotal point $O$ acceleration:

$$a \in \{-a_0, 0, a_0\}$$

Dynamics

$$J \ddot{\theta} = -mg\ell \sin(\theta) + m\ell \cos(\theta)$$

Goal: stabilize pendulum at upright position $B$

- Hybrid system with state $x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$ and three modes

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Inverted Pendulum: Mode 1 \((a = 0)\)

\[ J = mg\ell = 1 \]

**Dynamics**

\[
\ddot{\theta} = -mg\ell \sin(\theta)
\]

**Solution along level curves of**

\[ E_1 = \frac{1}{2} J\dot{\theta}^2 + mg\ell(1 - \cos(\theta)) \]

Kinetic Energy \(T = \frac{1}{2} J\dot{\theta}^2\)

Potential Energy \(V = mg\ell(1 - \cos(\theta))\)

Total Energy \(E = T + V\)
Inverted Pendulum: Mode 2 \( (a = a_0) \)

\[ J = m g \ell = a_0 = 1 \]

Dynamics

\[ \ddot{\theta} = -m g \ell \sin(\theta) + m a_0 \ell \cos(\theta) \]

Solution along level curves of (fictional) total energy \( (\theta_0 = \tan^{-1} \frac{a_0}{g}) \):

\[ E_2 = \frac{1}{2} J \dot{\theta}^2 + m (g^2 + a_0^2)^{\frac{1}{2}} \ell (1 - \cos(\theta - \theta_0)) \]

For mode 3 \( (a = -a_0) \), replace \( \theta_0 \) with \( -\theta_0 \) to obtain \( E_3 \)
Inverted Pendulum: Swing Up

Switching-stabilizing strategy

- Switch when the pendulum is stationary ($\dot{\theta} = 0$)
- Switch to a mode that pumps energy into system (along natural motion)
- Last mode is mode 0
- Mode 0 can be replaced with stabilizing controller of linearized system
General Switching Stabilization Method

Idea: Look for PD CLF $V(x)$ such that

$$\min_{i \in \Sigma} \left[ \frac{\partial V}{\partial x}(x) \cdot f_i(x) \right] < 0, \quad \forall x \neq 0$$

- May need to consider piecewise CLF $C^1$, e.g.

$$V(x) = \max_j x^T Q_j x, \quad V(x) = \min_j x^T Q_j x$$

- Maintaining regularity (continuity) of closed-loop system vector field is tricky; often relax to piecewise continuous

- Need to consider sliding mode solution (nonissue for D-T systems)
Example

Switched linear system $\dot{x} = A_{\sigma}x$ with two unstable subsystems

$$A_1 = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & -5 \\ -5 & -1 \end{bmatrix}$$

Since $\tilde{A} = \frac{5}{8}A_1 + \frac{3}{8}A_2 = -I_2$ is stable, there exists $P > 0$ (e.g. $P = I_2$) s.t.

$$P\tilde{A} + \tilde{A}^T P < 0$$

$$\Rightarrow \frac{5}{8}(PA_1 + A_1^T P) + \frac{3}{8}(PA_2 + A_2^T P) < 0$$

$$\Rightarrow \min \left\{ x^T (PA_1 + A_1^T P)x, \ x^T (PA_2 + A_2^T P)x \right\} < 0, \ \forall x \neq 0$$

Stabilizing switching strategy

$$\sigma^*(x) = \arg \min_{i=1,2} \left\{ x^T (PA_i + A_i^T P)x \right\} = \begin{cases} 1 & \text{if } x_1x_2 \leq 0 \\ 2 & \text{if } x_1x_2 > 0 \end{cases}$$
More General Examples

Switched system $\dot{x} = f_\sigma(x)$, $\sigma \in \{1, \ldots, m\}$

Suppose $V(\cdot)$ is a Lyapunov function of the non-switched system

$$
\dot{x} = \sum_{i=1}^{m} \lambda_i(x) f_i(x)
$$

where for any $x$, $\lambda_i(x) \geq 0$ and $\sum_i \lambda_i(x) = 1$. Then

- $V(\cdot)$ is a CLF of the switched system:
  $$
  \min_{i \in \Sigma} \left[ \frac{\partial V}{\partial x}(x) \cdot f_i(x) \right] < 0, \quad \forall x \neq 0
  $$

- A stabilizing switching strategy is given by
  $$
  \sigma^*(x) = \arg \min_{i \in \Sigma} \left[ \frac{\partial V}{\partial x}(x) \cdot f_i(x) \right]
  $$
Controlled Lyapunov Function of Switched Systems

**CLF** of switched system \( \dot{x} = f_\sigma(x) \) is a PD function \( V(x) \):

1. **directional derivative** \( \dot{V}(x; f_i(x)) \) exists for all \( x \in \mathbb{R}^n \), \( i \in \Sigma \);
2. \( \min_{i \in \Sigma} \dot{V}(x; f_i(x)) < 0 \) for all \( x \neq 0 \)

- \( V(\cdot) \) is continuous but not necessarily \( C^1 \)
- Directional derivative \( \dot{V}(x; v) := \lim_{\delta \downarrow 0} \frac{V(x+\delta v) - V(x)}{\delta} \)
- **Stabilizing switching law** given by
  \[
  \sigma^*(x) = \arg \min_{i \in \Sigma} \dot{V}(x; f_i(x))
  \]
- \( \sigma^*(\cdot) \) results in a partition \( \mathbb{R}^n = \bigcup_{i \in \Sigma} \Omega_i \), where \( \Omega_i = \{x \mid \sigma^*(x) = i\} \)
- **Additional requirements** for sliding mode solutions
Sliding Mode Solutions

Switching law $\sigma^*(\cdot)$ may result in sliding mode solutions

Example: SLS $\dot{x} = A_\sigma x$ for $\sigma \in \{1, 2\}$

CLF $V(x) = \max\{x^TP_1x, x^TP_2x\}$

CLF $V(x) = \min\{x^TP_1x, x^TP_2x\}$

$\dot{V}(x; A_1x) < 0$ or $\dot{V}(x; A_2x) < 0$

not necessary for

$\dot{V}(x; \alpha_1A_1x + \alpha_2A_2x) < 0$

$\dot{V}(x; A_1x) < 0$ and $\dot{V}(x; A_2x) < 0$

not sufficient for

$\dot{V}(x; \alpha_1A_1x + \alpha_2A_2x) < 0$
Quadratic Stabilizability

Switched System is **quadratically stabilizable** if a quadratic CLF exists, i.e., $V(x) = x^T P x$ for some $P \succ 0$ such that

$$\min_{i \in \Sigma} \left[ x^T P f_i(x) \right] < 0, \quad \forall x \neq 0$$

- Stabilizing switching law $\sigma^*(x) = \arg \min_{i \in \Sigma} \left[ x^T P f_i(x) \right]

- No need to worry about sliding-mode solution since $V(x)$ is smooth

- SLS $\dot{x} = A_{\sigma} x$ is quadratic stabilizable if $\exists P \succ 0$ such that

$$\min_{i \in \Sigma} \left[ x^T (PA_i + A_i^T P) x \right] < 0, \quad \forall x \neq 0$$

- **Sufficient condition**: a convex combination of $A_i$ is stable

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Need for General CLFs

SLS $\dot{x} = A_\sigma x$, with $A_1 = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & -0.5 \\ 1 & 0 \end{bmatrix}$

- Switching stabilizable but there exists on quadratic CLF

**Idea:** Consider composite quadratic CLF: $V_P(x) = \min_{j \in J} x^T P_j x$
- Level sets are unions of ellipsoids

Composite Quadratic CLFs

\[ V_P(x) = \min_{j \in J} x^T P_j x \]

- Number of matrices in \( P = \{ P_j \}_{j \in J} \) could be large than \( |\Sigma| \)
- \( V_P(\cdot) \) not smooth only when minimum is achieved by multiple \( j \)

\( V_P \) is a CLF for the SLS \( \dot{x} = A_{\sigma} x \) if

\[ \min_{i \in \Sigma} \dot{V}_P(x; f_i(x)) < 0, \quad \forall x \neq 0 \]

**Theorem**

*If \( V_P \) is a CLF, then the SLS is stabilized by the switching law*

\[ \sigma^*(x) = \arg \min_{i \in \Sigma} \dot{V}_P(x; f_i(x)) \]
**LMI Stabilizability Test**

**Fact:** $V_P$ is a CLF if the following problem is feasible

\[
\text{find } P_j \succ 0, \alpha_{ij} \in \mathbb{R}, \beta_{jk} \in \mathbb{R}, i \in \Sigma, j, k \in J
\]

such that

\[
\left( \sum_{i \in \Sigma} \alpha_{ij} A_i \right)^T P_j + P_j \left( \sum_{i \in \Sigma} \alpha_{ij} A_i \right) - \sum_{k \in J} \beta_{jk} (P_j - P_k) \prec 0
\]

\[
\alpha_{ij} \geq 0, \sum_{i \in \Sigma} \alpha_{ij} = 1, \beta_{jk} \geq 0, \forall i \in \Sigma, \forall j, k \in J
\]

- Proof uses $S$ procedure and considering sliding mode*
- Bilinear Matrix Inequality (BMI) problem via e.g. path-following method†
- Piecewise quadratic CLF for discrete-time SLS‡

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‡ Switching stabilization and $\ell_2$ gain performance controller synthesis for discrete-time switched linear systems, H. Lin and J. Antsaklis, CDC, 2006.