Lecture 11: Stability I
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• Stability of continuous time systems

• Lyapunov stability of continuous time systems

• LaSalle invariance principle of continuous time systems

• Stability of linear systems

• Stability of hybrid systems
Stability of the Continuous Systems

\[ \dot{x} = f(x), \quad x(0) = x_0 \]

where \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is globally Lipschitz continuous

**Definition** (*Equilibrium point*): \( x_e \in \mathbb{R}^n \) for which \( f(x_e) = 0 \)

**Definition** (*Stability*): The equilibrium point \( x_e \in \mathbb{R}^n \) is stable (in the sense of Lyapunov) if

\[ \forall \epsilon > 0, \exists \delta > 0 : \|x(t_0) - x_e\| < \delta \Rightarrow \|x(t) - x_e\| \leq \epsilon, \forall t \geq t_0 \geq 0 \]

If the system starts close to the equilibrium, it remains close to it since then.
Example: Pendulum

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 \]

- $x_{eq} = (0,0)$ stable
- $x_{eq} = (\pi,0)$ unstable
Example: Van der Pole oscillator

\[ \dot{x}_1 = x_2 \]

\[ \dot{x}_2 = -x_1 + 0.5(1 - x_1^2)x_2 \]

\( x^* \text{ Lyapunov stable} \)
Example: Van der Pole oscillator

E.g., Van der Pol oscillator

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - 0.5(1 - x_1^2)x_2
\end{align*}
\]
Stability of Continuous Systems

\[ \dot{x} = f(x), \ x(0) = x_0 \quad (1) \]

where \( f : \mathbb{R}^n \to \mathbb{R}^n \) is globally Lipschitz continuous

More definitions for stability: exponentially stable, globally (asymptotically, exponentially) stable, locally (asymptotically, exponentially) stable, unstable, ....

Consider a continuously differentiable \((C^1)\) function \( V : \mathbb{R}^n \to \mathbb{R} \)

The rate of change of \( V \) along solutions of (1) is computed as:

\[ \dot{V}(t) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) = \frac{\partial V}{\partial x} f(x) \]

*Lie derivative* of \( V \) with respect to \( f \)
Lyapunov Stability Theorem

Aleksandr Mikhailovich Lyapunov

(6 June 1857 - 3 Nov 1918, Russia)
Lyapunov Stability Theorem

**Theorem (Lyapunov Stability Theorem)** Let \( x_e = 0 \) be an equilibrium point of (1) and \( D \subset \mathcal{R}^n \) a set containing \( x_e = 0 \). If \( V : D \to \mathcal{R} \) is a \( C^1 \) function such that

\[
\begin{align*}
V(0) &= 0 \\
V(x) &> 0, \quad \forall x \in D \setminus \{0\} \\
\dot{V}(x) &\leq 0, \quad \forall x \in D
\end{align*}
\]

Then \( x_e \) is stable. Further more, if \( x_e = 0 \) is stable and

\[
\dot{V}(x) < 0, \quad \forall x \in D
\]

Then \( x_e \) is *asymptotically* stable.
Lyapunov Stability Theorem

Sketch of proof

\[ B_\epsilon = \{ x \in \mathbb{R}^n : \|x\| < \epsilon \} \]

\[ \Omega_{C2} = \{ x \in \mathbb{R}^n : V(x) < C_2 \} \subset B_\epsilon \]

\[ B_\delta = \{ x \in \mathbb{R}^n : \|x\| < \delta \} \in \Omega_{C2} \]

\[ \forall \epsilon > 0, \exists \delta > 0, \quad \|x_0\| < \delta \rightarrow \|x(t)\| < \epsilon, \forall t \geq t \]
Example: Pendulum

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 \\
k &\equiv \text{friction coefficient} \\
x_1 &\equiv \theta \\
x_2 &\equiv \dot{\theta}
\end{align*}
\]

\[
V(x) := \frac{g}{l} (1 - \cos x_1) + \frac{x_2^2}{2} \geq 0
\]

positive definite because \(V(x) = 0\) only for \(x_1 = 2k\pi \ k \in \mathbb{Z}\) & \(x_2 = 0\)

- Finding Lyapunov functions in general HARD.
LaSalle’s Invariance Principle

\[ \dot{x}(t) = f(x), \ x \in \mathcal{R}^n \]

\( M \in \mathcal{R}^n \) is an **invariant set** if \( x(t_0) \in M \Rightarrow x(t) \in M, \ \forall t \geq t_0 \)

**Theorem (LaSalle’s Invariance Principle)** Suppose there is a continuously differentiable, positive definite, and radially unbounded function \( V : \mathcal{R}^n \rightarrow \mathcal{R} \) such that

\[ \dot{V}(x) = \frac{\partial V}{\partial x}(x - x_e) f(x) \leq W(x) \leq 0, \ \forall x \in \mathcal{R}^n \]

Then \( x_e \) is a Lyapunov stable equilibrium. \( x(t) \) converges to the largest invariant set \( M \) contained in

\[ E := \{ x \in \mathcal{R}^n : W(x) = 0 \} \]

- If in a domain about the equilibrium, we can find a Lyapunov function whose Lie derivative is negative semidefinite, and we can show that no trajectory can stay identically at points where \( \dot{V}(x) = 0 \) except at \( x_e \), then the equilibrium is asymptotically stable.
LaSalle’s Invariance Principle

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 \\
k &\equiv \text{friction coefficient} \\
x_1 &\equiv \theta \\
x_2 &\equiv \dot{\theta}
\end{align*}
\]

\[
V(x) := \frac{g}{l} (1 - \cos x_1) + \frac{x_2^2}{2} \geq 0
\]

For \( x_{eq} = (0,0) \)
\[
\frac{\partial V}{\partial x} (x - x_{eq}) f(x) = -\frac{k}{m} x_2^2 \leq 0 \quad \forall x \in \mathbb{R}^n
\]

\[
E := \{ (x_1, x_2) : x_1 \in \mathbb{R}, x_2 = 0 \} = \{ x : \dot{V}(x) = 0 \}
\]

Inner \( E \), the ODE becomes
\[
\begin{align*}
\dot{x}_1 &= x_2 = 0 \\
0 &= \dot{x}_2 = -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 = -\frac{g}{l} \sin x_1 \\
x_1 &= 0, \quad (\pi < x_1 < \pi)
\end{align*}
\]

\( x(t) \) converges to \( M = \{ x_e = (0,0) \} : \text{maximum invariant set} \)
LaSalle’s Invariance Principle

Remarks

• When $W(x) = 0$ only for $x = x_e$, then $E = \{x_e\}$.
  Since $M \subset E$, $M = \{x_e\}$ and therefore $x(t) \to x_e$

=> Asymptotic stability

• Even when $E$ is larger than $\{x_e\}$, usually $M = \{x_e\}$ and can conclude asymptotic stability

• The equilibrium point and the limit cycle are an invariant set
Linear Systems

\[ \dot{x} = Ax \text{ where } x \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n} \]

\[ \Rightarrow x(t) = e^{At}x(0), \quad t \geq 0 \]

Lyapunov function: \( V(x) = x^T P x, \quad P > 0 \)

**Theorem:** The equilibrium \( x_e \equiv 0 \) is asymptotically stable *if and only if* for every positive definite symmetric matrix \( Q \) the Lyapunov equation

\[ A^T P + PA = -Q \]

has a unique solution \( P \) which is a *positive definite symmetric* matrix.

- \( P \) is positive definite = all eigenvalues are positive
  \[ \iff x^T P x > 0 \quad \forall x \neq 0 \]

- \( P \) is positive semi-definite = all eigenvalues are positive or zero
  \[ \iff x^T P x \geq 0 \quad \forall x \neq 0 \]
Linear Systems

Proof:

(if) \( \dot{V} = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P)x = -x^T Q x < 0 \)

(only if) Consider \( P = \int_0^\infty e^{A^T t} Q e^{A t} \, dt \)

\( PA + A^T P = \int_0^\infty e^{A^T t} Q e^{A t} A \, dt + \int_0^\infty A^T e^{A^T t} Q e^{A t} \, dt \)

\( = \int_0^\infty \frac{d}{dt} e^{A^T t} Q e^{A t} \, dt = -Q \)
Linear Systems

P is unique:

Proof) Assume there is another solution \( \hat{P} \neq P \)

\[
0 = e^{A^T t}(Q - Q)e^{At} \\
= e^{A^T t}[(P - \hat{P})A + A^T (P - \hat{P})]e^{At} \\
= \frac{d}{dt} e^{A^T t}(P - \hat{P})e^{At}
\]

\[
\Rightarrow e^{A^T t}(P - \hat{P})e^{At} \quad \text{is constant for all } t \geq 0
\]

\[
e^{A^T 0}(P - \hat{P})e^{A0} = \lim_{t \to \infty} e^{A^T t}(P - \hat{P})e^{At}
\]

\[
P - \hat{P} = 0
\]
Example: Linear System

\[ \dot{x} = Ax \]
\[ A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \]
\[ \lambda(A) = -1 \pm i \]

Lyapunov equation: \[ A^T P + PA = -Q \]

MATLAB
\text{LYAP} \text{ solves continuous-time Lyapunov equations.}
P = \text{LYAP}(A,Q) \text{ solves the special form of the Lyapunov matrix equation:}
\[ A^*P + PA' = -Q \]

\[ Q = I \quad P = 0.5I \]

Asymptotically STABLE!
Stability of Hybrid Systems

Autonomous hybrid automaton

\[ H = (Q, X, Init, f, Dom, R) \]

**Definition** (*Equilibrium*) The continuous state \( x_e = 0 \in \mathbb{R}^n \) is an equilibrium point of \( H \) if:

1. \( f(q, 0) = 0, \ \forall q \in Q \)
2. \( R(q, 0) \subseteq Q \times 0 \)

- Discrete transitions are allowed out of \((q, 0)\), while the system makes a jump to \((q', 0)\).
- If \((q_0, 0) \in Init\) and \((\tau, (q, x))\) is the execution of \( H \) starting from \((q_0, 0)\), then \( x(t) = 0 \) for all \( t \in \tau \)
Stability of Hybrid Systems

\[ H = (Q, X, \text{Init}, f, \text{Dom}, R) \]

**Definition (Stable Equilibrium)** The equilibrium \( x_e = 0 \in \mathbb{R}^n \) is **stable** if for all \((\tau, (q, x))\) starting at \((q_0, 0)\):

\[ \forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } ||x_0|| < \delta \Rightarrow ||x(t)|| < \epsilon, \forall t \in \tau \]

**Definition (Asymptotically Stable Equilibrium)** The equilibrium \( x_e = 0 \in \mathbb{R}^n \) is **asymptotically stable** if it is stable and \( \delta \) can be chosen so that for all \((\tau, (q, x))\) starting at \((q_0, x_e)\)

\[ ||x_0|| < \delta \Rightarrow \lim_{t \to \tau \infty} ||x(t)|| = 0 \]
Example: Switching Linear System

• Would a hybrid system for which the continuous system in each discrete is stable be stable?

Not Always!

Example: Consider the hybrid automaton $H$ with:

- $Q = \{q_1, q_2\}, X = \mathbb{R}^2$
- $\text{Init} = Q \times \{x \in X : ||x|| > 0\}$
- $f(q_1, x) = A_1 x$ and $f(q_2, x) = A_2 x$, with:

$$A_1 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 100 \\ -10 & -1 \end{bmatrix}$$

- $\text{Dom} = \{q_1, \{x \in \mathbb{R}^2 : x_1 x_2 \leq 0\}\} \cup \{q_2, \{x \in \mathbb{R}^2 : x_1 x_2 \geq 0\}\}$
- $R(q_1, \{x \in \mathbb{R}^2 : x_1 x_2 \geq 0\}) = (q_2, x)$ and $R(q_2, \{x \in \mathbb{R}^2 : x_1 x_2 \leq 0\}) = (q_1, x)$
Example: Consider the hybrid automaton $H$ with:

- $Q = \{q_1, q_2\}$, $X = \mathbb{R}^2$
- $\text{Init} = Q \times \{x \in X : ||x|| > 0\}$
- $f(q_1, x) = A_1 x$ and $f(q_2, x) = A_2 x$, with:
  
  \[
  A_1 = \begin{bmatrix}
  -1 & 10 \\
  -100 & -1
  \end{bmatrix},
  A_2 = \begin{bmatrix}
  -1 & 100 \\
  -10 & -1
  \end{bmatrix}
  \]

- $\text{Dom} = \{q_1, \{x \in \mathbb{R}^2 : x_1 x_2 \leq 0\}\} \cup \{q_2, \{x \in \mathbb{R}^2 : x_1 x_2 \geq 0\}\}$
- $R(q_1, \{x \in \mathbb{R}^2 : x_1 x_2 \geq 0\}) = (q_2, x)$ and $R(q_2, \{x \in \mathbb{R}^2 : x_1 x_2 \leq 0\}) = (q_1, x)$

\[
f(q_1, 0) = f(q_2, 0) = 0 \land R(q_1, 0) = (q_2, 0), \ R(q_2, 0) = (q_1, 0)
\]

$X_e = (0, 0)$ is an equilibrium of $H$

Eigenvalues of the both systems: $-1 \pm j\sqrt{1000}$

Both systems are asymptotically stable
Example: Switching Linear System

- Each continuous system is stable.
Example: Switching Linear System

$q_1$: quadrants 2 and 4
$q_2$: quadrants 1 and 3
unstable

$q_1$: quadrants 1 and 3
$q_2$: quadrants 2 and 4
stable

In general, the stability of a hybrid system cannot be analyzed by studying individual continuous systems.