Lecture 6: Computational Tools II
Motivation

- LMI $F(z) = F_0 + z_1 F_1 + \cdots + z_k F_k \succeq 0$, $F_i \in \mathbb{S}^n$, recast as

$$x^T F(z)x = \sum_{i,j} f_{ij}(z)x_i x_j \geq 0, \quad \forall x_1, \ldots, x_n \in \mathbb{R}$$

- Nonegativity of a quadratic multivariate polynomial
- Coefficients are affine functions of constrained variable $z$
- Feasible set of $z$ is intersection of half spaces (one for each $x$)

- Generalization to higher order multivariate polynomials:

$$\sum_{\alpha \in I} f_{\alpha}(z)x_1^{\alpha_1} \cdots x_n^{\alpha_n} \geq 0, \quad \forall x_1, \ldots, x_n \in \mathbb{R}$$

  - Assume coefficients $f_{\alpha}(z)$ are affine in $z$
  - This defines a convex feasible set of $z$

- In many cases the constraints are trivial, e.g., $f(z)x_1^2 x_2 \geq 0$
P.S.D. and SOS Polynomials

\( P^d_n \): set of all polynomials in \( n \) variables of degree up to \( d \)

\[
p(x) = \sum_{\alpha \in \mathcal{I}} c_\alpha x^{\alpha}, \quad \text{where } \alpha = (\alpha_1, \ldots, \alpha_n), \ x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}
\]

- Each \( x^{\alpha} \) is a monomial with degree \(|\alpha| = \alpha_1 + \cdots + \alpha_n\)

\( P^d_{n,+} \): set of positive semidefinite \( p(x) \), i.e., \( p(x) \geq 0, \ \forall x \in \mathbb{R}^n \)

- Example: \( p_1(x) = 2x_1^2x_2^4 + 0.2x_2^2 \), \( p_2(x) = 2x_1^4 + 5x_2^4 - x_1^2x_2^2 + 2x_1x_2 \)
- \( P^d_{n,+} \) is a convex cone in \( P^d_n \)
- Determining if \( p(x) \) is in \( P^d_{n,+} \) is NP-hard

\( \Sigma^d_n \): set of \( p(x) \) that are sum of squares (SOS) of polynomials

- Example: \( p_1(x), p_2(x), p_3(x) = (x_1 - x_1^3x_2)^2 + (x_2 - 1)^2 \)
- \( \Sigma^d_n \) is a convex (sub)cone in \( P^d_{n,+} \)
- Checking if \( p(x) \) is in \( \Sigma^d_n \) is easy
P.S.D. Polynomials ≠ SOS Polynomials

- In general, $\Sigma_d^n \subsetneq P_{n,+}^d$, especially for large $n$ and $d$
  - Example of p.s.d. but not SOS polynomials:
    \[
    p(x) = x_1^2x_2^4 + x_1^4x_2^2 + 1 - 3x_1^2x_2^2
    \] (T. Motzkin, 1967)

- $\Sigma_d^n = P_{n,+}^d$ only for (Hilbert, 1888):
  - Univariate polynomials ($n = 1$)
  - Quadratic polynomials ($d = 2$)
  - Bivariate quartic polynomials ($d = 4$ and $n = 2$)

- Hilbert Seventeenth Problem: every p.s.d. real polynomial is the SOS of rational functions (i.e., fraction of polynomials)
  - Answered affirmatively by Emil Artin in 1927
Gram Matrix Representation of SOS

A SOS $p(x)$ has the **Gram matrix representation**

$$p(x) = \begin{bmatrix} x^\alpha \\ \vdots \\ x^\zeta \end{bmatrix}^T Q \begin{bmatrix} x^\alpha \\ \vdots \\ x^\zeta \end{bmatrix}$$

- $x^\alpha, \ldots, x^\zeta$ are monomials, $Q \succeq 0$

Example:

$$4x_1^4 + 10x_2^4 - 2x_1^2x_2^2 + 4x_1^3x_2 = (2x_1^2 - 3x_2^2 + x_1x_2)^2 + (x_2^2 + 3x_1x_2)^2$$

$$= \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}^T \begin{bmatrix} 4 & -6 & 2 \\ -6 & 10 & 0 \\ 2 & 0 & 10 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}$$

$$Q = L^T L$$ where $L = \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$
SOS Constraints are LMI

“\( p(x) = \sum_{\alpha \in I} c_\alpha x^\alpha \) is SOS” as a constraint on \( c_\alpha \) is equivalent to

\[
v(x)^T Q v(x) = p(x), \quad Q \succeq 0
\]

- \( v(x) \) is a vector of sufficiently rich monomials in \( x \)
- New constrained variable \( Q \) has entries that are affine in \( c_\alpha \)’s

Example:

\[
p(x) = c_1 x_1^4 + c_2 x_2^4 + c_3 x_1^2 x_2^2 + c_4 x_1^3 x_2 = \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1 x_2 \end{bmatrix}^T \begin{bmatrix} c_1 & q_{12} & c_4/2 \\ q_{21} & c_2 & 0 \\ c_4/2 & 0 & q_{33} \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1 x_2 \end{bmatrix}
\]

Constraint “\( p(x) \) is SOS” is equivalent to the LMI constraint

\[
q_{12} = q_{21}, \quad q_{12} + q_{21} + q_{33} = c_3, \quad Q \succeq 0
\]
# SOS Optimization

## SOS feasibility problem:

\[
\text{find } p(x) \\
\text{such that } A(x)p(x) + b(x) \text{ are SOS} \\
A_e(x)p(x) + b_e(x) = 0
\]

- \(A(x), A_e(x), b(x), b_e(x)\): given matrices (vectors) of polynomials
- \(p(x)\): vector of polynomials to be solved

## SOS optimization problem:

\[
\begin{align*}
\text{minimize } & \quad w^T c \\
\text{subject to } & \quad A(x)p(x) + b(x) \text{ are SOS} \\
& \quad A_e(x)p(x) + b_e(x) = 0
\end{align*}
\]
SOS Software

- Toolbox specifically for SOS: **SOSTOOLS**
- Generic solvers: SeduMi, SDTP3, CSDP, SDPNAL, SDPA
- **YALMIP** has built-in (basic) SOS solver

SOS has been applied for

- Stability analysis, region of attraction estimation
- Reachability analysis
- Controller design
- Probability density function

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Stability Example

Given nonlinear system
\[
\begin{align*}
\dot{x}_1 &= -x_2 + \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3 \\
\dot{x}_2 &= 3x_1 - x_2
\end{align*}
\]

**Problem:** find a global Lyapunov function \( V(x) \) of the system:

\[
V(x) \geq 0, \quad \frac{d}{dt} V(x(t)) \leq 0, \quad \forall x \in \mathbb{R}^2
\]

Solve via the SOS feasibility problem:

\[
\text{find } V(x) = \sum_{0 \leq i+j \leq 4} c_{ij} x_1^i x_2^j \text{ such that } V(x) \text{ is SOS}
\]

\[
- \nabla V(x) \cdot \begin{bmatrix} -x_2 + \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3 \\ 3x_1 - x_2 \end{bmatrix} \text{ is SOS}
\]

Parrilo (ACC’06)
Stability Example (cont.)

Plot of solved $V(x)$ (level sets in red dashed) and state trajectories (blue):

\[
V = 4.5819x^2 - 1.5786xy + 1.7834y^2 - 0.12739x^3 + 2.5189x^2y - 0.34069xy^2 + 0.61188y^3 + 0.47537x^4 - 0.052424x^3y + 0.44289x^2y^2 + 0.0000018868xy^3 + 0.090723y^4
\]
Generalized S-Procedure

Given polynomials $p(x)$, $g_i(x)$ (not necessarily SOS), the condition

$$p(x) \geq 0 \text{ whenever } g_i(x) \geq 0 \text{ for all } i$$

has the following sufficient (but not necessary) conditions

$$p(x) - \sum_i \lambda_i g_i(x) \text{ is SOS for some } \lambda_i \geq 0$$

or more generally

$$p(x) - \sum_i f_i(x) g_i(x) \text{ is SOS for some SOS } f_i(x)$$
Local Stability

Nonlinear system \[ \begin{align*}
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_1, x_2)
\end{align*} \]

with \( f_1 \) and \( f_2 \) polynomials

Problem: Find a local Lyapunov function around equilibrium 0

\[ V(x) \geq 0, \quad \frac{d}{dt} V(x(t)) \leq 0, \quad \forall x \in \mathbb{R}^2 \text{ with } \|x\| \leq r \]

Want \( V(x) \geq 0 \) and \( -\dot{V}(x) \geq 0 \) on the set \( \{r^2 - x_1^2 - x_2^2 \geq 0\} \):

\[
\begin{align*}
\text{find} & \quad V(x) \\
\text{s.t.} & \quad V(x) - f_1(x) \cdot (r^2 - x_1^2 - x_2^2) \text{ is SOS for some SOS } f_1(x) \\
& \quad -\frac{d}{dt} V(x(t)) - f_2(x) \cdot (r^2 - x_1^2 - x_2^2) \text{ is SOS for some SOS } f_2(x)
\end{align*}
\]