Lecture 2: Discrete and Continuous Dynamics
Dynamics of Hybrid Systems

The state of hybrid systems:
- Continuous state \( x \in X \) where \( X \subset \mathbb{R}^n \)
- Discrete state \( q \in Q \) where \( Q = \{ q_1, q_2, \ldots \} \)

Dynamics of hybrid systems:
- Continuous dynamics of \( x \): differential equations
- Discrete dynamics of \( q \): automata (transition systems)
Automata

An automaton is the tuple $\mathcal{A} = (Q, \Sigma, \to, Q_0, Q_f)$

1. $Q$: set of states
2. $\Sigma$: set of input symbols (alphabet)
3. $\to \subset Q \times \Sigma \times Q$: set of transitions
   - Ex: $(q_1, a, q_2) \in \to$ is the transition $q_1 \xrightarrow{a} q_2$
4. $Q_0$: set of initial states
5. $Q_f$: set of terminating states

Examples:
Language Accepted by Automaton

An **input word** is a finite string \( a_1a_2 \cdots a_n \) of symbols from \( \Sigma \)
- Could be the empty string \( \varepsilon \)
- \( \Sigma^* \): set of all input words generated by the alphabet \( \Sigma \)

The input word \( a_1a_2 \cdots a_n \) is accepted by the automaton \( \mathcal{A} \) if there exists a sequence (or **run**) of states \( q_0, q_1, \ldots, q_n \) such that
- \( q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \cdots q_{n-1} \xrightarrow{a_n} q_n \)
- \( q_0 \in Q_0 \)
- \( q_n \in Q_f \)

The **language accepted by the automaton** \( \mathcal{A} \) is the set of all input words accepted by \( \mathcal{A} \), and denoted by \( L(\mathcal{A}) \) (subset of \( \Sigma^* \))

- Given a language, is there a finite automaton that accepts it?
- Smallest automaton that accepts a given language
- Algorithms to check if two automata accept the same language
Properties of Automata

Automaton is **blocking** if there is a state with no transitions out

Automaton is **deterministic** if there is at most one run for any given initial state and input word
Continuous Dynamical Systems

ODE:
\[
\begin{align*}
\dot{x} &= f(x, u, t), \quad x(t_0) = x_0 \\
y &= g(x, u, t)
\end{align*}
\]

- $x \in \mathbb{R}^n$: state
- $u \in \mathbb{R}^m$: input
- $y \in \mathbb{R}^p$: output
- $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^n$ vector field

- Time-invariant (TI) systems: $\dot{x} = f(x, u), \; y = g(x, u)$
- Linear TI systems: $\dot{x} = Ax + Bu, \; y = Cx + Du$
- Autonomous TI systems: $\dot{x} = f(x), \; y = g(x)$

\[
\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x
\]
Existence of Classical Solutions

A classical solution of ODE $\dot{x} = f(x, t), \ x(t_0) = x_0$, is given by

- $x : I \to \mathbb{R}^n$ is $C^1$ on some time interval $I = [t_0, t_0 + \tau]$  
- $\dot{x}(t) = f(x(t), t)$ for all $t \in I$  
- $x(t_0) = x_0$

Theorem (Peano Existence Theorem)

If $f$ is continuous near $(x_0, t_0)$, then a classical solution exists for some $I$

- Continuity of $f$ is needed, e.g., $\dot{x} = -\text{sgn}(x) = \begin{cases} -1 & \text{if } x \geq 0 \\ 1 & \text{if } x < 0 \end{cases}$
- Solutions may not be unique, e.g., $\dot{x} = \sqrt{|x|}, \ x(0) = 0$
- Solution is called maximal if $I$ is largest possible
Uniqueness of Classical Solutions

Theorem (Picard’s Theorem)

If $f(x, t)$ is uniformly Lipschitz continuous in $x$ and continuous in $t$ near $(x_0, t_0)$, then the ODE $\dot{x} = f(x, t)$, $x(t_0) = x_0$, has a unique solution on some time interval $I$

- Uniform Lipschitz continuity in $x$ means that there is a constant $M$ such that

  $$|f(x_1, t) - f(x_2, t)| \leq M \cdot |x_1 - x_2|, \quad \forall x_1, x_2, \forall t$$

- If Lipschitz continuity holds globally, then $I = [t_0, \infty)$

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Weak Solutions

For the ODE $\dot{x} = f(x, t)$, $x(t_0) = x_0$, a solution in the sense of Caratheodory is given by $x(t)$, $t \in \mathcal{I} = [t_0, t_0 + \tau]$, such that

1. $x$ is absolutely continuous on $\mathcal{I}$
2. $\dot{x} = f(x, t)$ for almost all $t \in \mathcal{I}$
3. $x(t_0) = x_0$

Example: $\dot{x} = f(x, t) = \begin{cases} 0 & \text{if } t \in [0, 1] \\ 1 & \text{if } t > 1 \end{cases}$, $x(0) = 0$

A weak (but not classical) solution: $x(t) = \begin{cases} 0 & \text{if } t \in [0, 1] \\ t - 1 & \text{if } t > 1 \end{cases}$
Existence of Weak Solutions

**Theorem (Caratheodory’s Existence Theorem)**

A weak solution exists on some $\mathcal{I}$ if the following hold near $(x_0, t_0)$

1. $f(x, t)$ is continuous in $x$ for each fixed $t$
2. $f(x, t)$ is measurable in $t$ for each fixed $x$
3. $|f(x, t)| \leq m(t)$ for some integrable function $m(t)$

- Continuity in $t$ is no longer needed
- Can be extended to global version and with control input $u$

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Examples

Example 1: \[ \dot{x} = \begin{cases} 
0 & \text{if } t \in [0, 1], \\
1 & \text{if } t > 1
\end{cases} , \quad x(0) = 0 \]

Example 2: \[ \dot{x} = \begin{cases} 
1 & \text{if } x > 0 \\
0.5 & \text{if } x = 0, \\
-1 & \text{if } x < 0
\end{cases} , \quad x(0) = 0 \]

What about \[ \dot{x} = \begin{cases} 
-1 & \text{if } x \geq 0 \\
1 & \text{if } x < 0
\end{cases} , \quad x(0) = 0? \]
Filippov Solution

- Partition of state space $\Omega = \Omega_1 \cup \Omega_2$
- Dynamics $\dot{x} = \begin{cases} f_1(x) & x \in \Omega_1 \\ f_2(x) & x \in \Omega_2 \end{cases}$
- Discontinuity on boundary $S$: $f_1(x) \neq f_2(x)$

Filippov solution $\hat{x}(t)$: absolutely continuous and for almost all $t$,

$$\frac{d\hat{x}(t)}{dt} \in \mathcal{F}(\hat{x}) := \begin{cases} \{f_1(\hat{x})\} & \text{in interior of } \Omega_1 \\ \{f_2(\hat{x})\} & \text{in interior of } \Omega_2 \\ co\{f_1(\hat{x}), f_2(\hat{x})\} & \text{on boundary } S \end{cases}$$

- Obtained from hysteretic regulation of size $\Delta$ with $\Delta \to 0$

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