OPTIMAL COLLISION AVOIDANCE AND FORMATION SWITCHING ON RIEMANNIAN MANIFOLDS*

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Abstract. The problems of optimal collision avoidance and optimal formation switching for multiple agents moving on a Riemannian manifold are studied. Optimal solutions are those that minimize a cost function defined by the integral of some Lagrangian function. Various optimality conditions are obtained based on the assumption that the Riemannian manifold admits a group of isometries with respect to which the Lagrangian function is invariant. These conditions include the conservation of momentum maps along optimal solutions as well as bounds on the conserved quantities derived through second variational analysis and topological considerations. The results are illustrated by examples. In particular, an example is worked out in details to show the limitation of the obtained optimality conditions. Some generalizations of the addressed problems are also presented.

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1. Introduction. We study optimal collision avoidance (OCA) and optimal formation switching (OFS) for multiple agents moving on a Riemannian manifold. OCA and OFS are closely related problems where a group of agents move from a set of initial positions to a set of destination positions within a certain time interval, while maintaining at all time instants a pairwise distance of at least $r > 0$. The trajectories followed by the agents are optimal if the weighted sum of their energies is minimized, the weights representing agent priorities. We try to characterize these optimal trajectories.

The motivating application for this research is aircraft conflict resolution [13, 19], which in the level flight case is exactly the OCA problem in $\mathbb{R}^2$ with r equal to 5 nautical miles in en-route airspace and 3 nautical miles close to airports. The OCA problem also finds application in the field of multi-user wireless communication, where each user has to be assigned a (possibly time-varying) subspace in the signal space, and sufficient separation among these subspaces should be maintained for satisfactory signal-to-noise ratio. In this case the appropriate Riemannian manifold is a Grassmann manifold [32]. In the OFS problem, trajectories are subject to additional separation constraints. Relevant applications of the OFS problem in robotics include, for example, multiple mobile robots cooperating to carry a common object, or a multi-link reconfigurable robot performing configuration switchings [5]. Here we consider only holonomic constraints, as opposed to the numerous papers dealing with nonholonomic constraints (see e.g. [2, 4, 17, 18, 30, 31]). Other relevant work includes, for example, [9, 20], where the problems of stable and optimal coordinate control of vehicle formations are studied.

Our analysis of the OCA and OFS problems rests on the assumption that the Riemannian manifold on which the agents are moving admits a group of symmetries. Under this symmetry assumption, we derive various necessary conditions for a solution to be optimal, which generalize the results obtained in [14] for the OCA problem in $\mathbb{R}^2$. Generally speaking, what makes the OCA and OFS problems difficult to solve is the presence of the separation requirements, which introduces nonsmooth boundary constraints. Without these constraints, the classical Noether theorem ([1, 23]) would apply to reduce the degrees of freedom of the problems by establishing the conservation of some quantities called momentum maps.

The main contribution of this paper consists in

- the extension of the Noether theorem to OCA and OFS problems with nonsmooth boundaries;
- the introduction of bounds on the conserved quantities that apply uniformly to solutions to all OCA and OFS problems. Some of these bounds can be further improved by exploiting the structure of the specific problem under consideration;
- the generalization of the obtained results to OCA and OFS problems for bodies with arbitrary shapes

*Part of the results in this paper have appeared in [15, 16].

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and to the optimal control of a class of hybrid systems.

For simplicity, we assume in this paper that solutions to the OCA and OFS problems belong to the class of continuous and piecewise smooth trajectories, which is a reasonable assumption in most of the practical situations mentioned above. However, even when the underlying manifold is simple, it is a nontrivial task to prove that a solution exists in this class for arbitrary starting and destination positions of the agents. Therefore, all the results obtained should be understood to hold under the provision that a solution does exist in the class of continuous and piecewise smooth trajectories for the considered starting and destination positions.

The paper is organized as follows. In Section 2, we formulate the OCA and OFS problems for multiple agents moving on a Riemannian manifold, and introduce the symmetry assumption on this manifold used throughout the paper. In Section 3, we derive various necessary conditions that apply uniformly to solutions to all OCA and OFS problems. In particular, using some preliminary results in Section 3.1, we show in Section 3.2 that a version of the classical Noether theorem, namely, the preservation of momentum maps, still apply in our problems that are nonsmooth in nature. Bounds on the momentum maps are derived in Sections 3.2 and 3.3 through a second variational analysis and a topological analysis, respectively. In Section 4, an example is presented to show that the derived necessary conditions are not sufficient to characterize the optimal solutions, especially when the number of agents is large. Section 5 contains two natural generalizations of our results. Finally, in Section 6, some conclusions and possible future directions of research are outlined.

Throughout the paper, the results are illustrated using several recurrent examples: the Euclidean space $\mathbb{R}^n$, the sphere $\mathbf{S}^n$, a group G with a bi-invariant metric, the Grassmann and the Stiefel manifolds.

2. Problem Formulation. In this section, we formulate precisely the OCA and OFS problems addressed in the paper. First of all, we need to introduce some notations and recall a few concepts in differential geometry.

Let M be a C^∞ Riemannian manifold. For each $q \in M$, we denote by $\langle \cdot, \cdot \rangle_q$ and $\|\cdot\|_q$ (or simply $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$) the Riemannian metric and the corresponding norm on the tangent space T_qM, respectively. Fix $t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$, and consider a curve $\gamma : [t_0, t_1] \to M$. The arc length of γ is defined as $l_\gamma = \int_{t_0}^{t_1} \|\dot\gamma(t)\|\, dt$. Note that, unless otherwise stated, we shall always assume that curves in M are continuous and piecewise C^∞. For this class of curves the arc length is well defined. The distance $d_M(q_0, q_1)$ between two arbitrary points q_0 and q_1 in M is by definition the infimum of the arc length of all curves connecting q_0 and q_1: $d_M(q_0, q_1) = \inf\{l_\gamma : \ \gamma : [t_0, t_1] \to M, \gamma(t_0) = q_0, \ \gamma(t_1) = q_1\}$. A geodesic in M is a locally distance-minimizing curve. More precisely, $\gamma : [t_0, t_1] \to M$ is a geodesic if and only if for any $t \in (t_0, t_1)$, there exists an $\epsilon > 0$ small enough such that the arc length of γ restricted on $[t-\epsilon, t+\epsilon]$ is equal to $d_M(\gamma(t-\epsilon), \gamma(t+\epsilon))$. In this paper, we assumed that M is connected and complete, and that all geodesics in M are parameterized proportionally to arc length.

Let $L : TM \to \mathbb{R}$ be a *Lagrangian function*, i.e., a smooth function defined on the tangent bundle $TM = \{T_qM : \ q \in M\}$ of M that is nonnegative and convex on each fiber T_qM, $q \in M$. For each curve $\gamma : [t_0, t_1] \to M$, we define the *cost* of γ as

$$\hat{J}(\gamma) = \int_{t_0}^{t_1} L[\dot\gamma(t)]\, dt. \tag{2.1}$$

The curves joining two fixed points in M with minimal cost are extremals of the functional $\hat{J}$, and in any canonical local coordinates of TM, say, $(x_1, \ldots, x_n, \dot x_1, \ldots, \dot x_n)$, $n = \dim(M)$, they are characterized by the Euler-Lagrange equations [1]:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot x_i} = \frac{\partial L}{\partial x_i}, \quad i = 1, \ldots, n.$$

As an example one can take $L = \frac{1}{2}\|\cdot\|^2$, i.e., $L(v) = \frac{1}{2}\|v\|_q^2$, $\forall v \in T_qM, q \in M$. In this case the Euler-Lagrange equations describe the geodesics in M.

Consider an (ordered) k-tuple of points of M, $\langle q_i \rangle_{i=1}^k = (q_1, \ldots, q_k)$, where k is a positive integer. We say that $\langle q_i \rangle_{i=1}^k$ satisfies the *r-separation condition* for some positive r if $d_M(q_i, q_j) \geq r$ for all $i \neq j$. Let $\langle a_i \rangle_{i=1}^k$

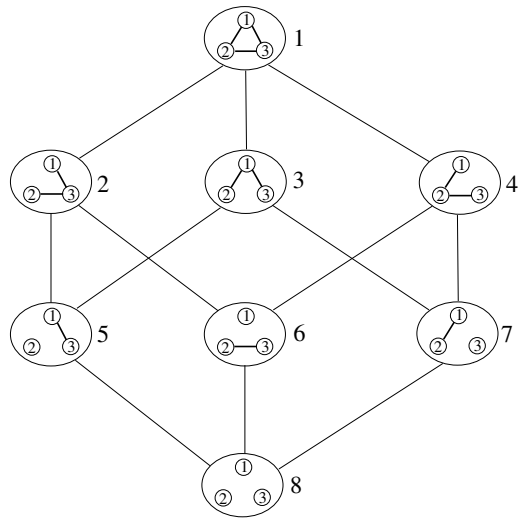

FIG. 2.1. *Hasse diagram of $\mathcal{F}$ when $M = \mathbb{R}^2$ and $k = 3$.*

and $\langle b_i \rangle_{i=1}^k$ be two k-tuples of points of M, each of which satisfies the r-separation condition. $\langle a_i \rangle_{i=1}^k$ is called the *starting position* and $\langle b_i \rangle_{i=1}^k$ the *destination position*.

Let $\gamma = \langle \gamma_i \rangle_{i=1}^k$ be a k-tuple of curves in M defined on $[t_0, t_1]$ such that $\gamma_i(t_0) = a_i, \gamma_i(t_1) = b_i$, for $i = 1, \ldots, k$. One can interpret γ as the joint trajectory of k agents moving on M that start from $\langle a_i \rangle_{i=1}^k$ at time t_0 and end at $\langle b_i \rangle_{i=1}^k$ at time t_1. γ is said to be *collision-free* if the k-tuple $\langle \gamma_i(t) \rangle_{i=1}^k$ satisfies the r-separation condition for each $t \in [t_0, t_1]$. Equivalently, if the agents are Riemannian disks of radius $r/2$ in M whose centers follow γ, then γ is collision-free if and only if no two agents overlap during $[t_0, t_1]$. Naturally, r must be small enough so that it is possible to pack k disks of radius $r/2$ in M.

Using these notations, we can now formulate the first problem we are going to study.

Problem 1 (Optimal Collision Avoidance (OCA)) *Among all collision-free $\gamma = \langle \gamma_i \rangle_{i=1}^k$ that start from $\langle a_i \rangle_{i=1}^k$ at time t_0 and end at $\langle b_i \rangle_{i=1}^k$ at time t_1, find the ones that minimize the cost*

$$J(\gamma) = \sum_{i=1}^{k} \lambda_i \hat{J}(\gamma_i), \tag{2.2}$$

where $\langle \lambda_i \rangle_{i=1}^k$ is a k-tuple of positive real numbers and $\hat{J}(\cdot)$ is defined in (2.1).

The k-tuple $\langle \lambda_i \rangle_{i=1}^k$ of weighting coefficients in the overall cost $J(\gamma)$ represents the priorities of the k agents, with a larger λ_i corresponding to a higher priority for agent i.

The OCA problem can be formulated in an alternative way by viewing each k-tuple of points of M as a single point in $M^{(k)} = M \times \cdots \times M$. According to this interpretation, γ becomes a curve in $M^{(k)}$ starting from $(a_1, \ldots, a_k)$ at time t_0 and ending at $(b_1, \ldots, b_k)$ at time t_1, while avoiding the obstacle

$$W = \cup_{i \neq j} \{(q_1, \ldots, q_k) \in M^{(k)} : d_M(q_i, q_j) < r\}. \tag{2.3}$$

As a result, solutions to the OCA problem are cost-minimizing curves in $M^{(k)} \setminus W$ connecting two fixed points. In particular, if $L = \frac{1}{2}\|\cdot\|^2$, then these solutions are geodesics in $M^{(k)} \setminus W$ with a proper choice of metric, a viewpoint taken in [12] to study the conflict resolution problem in $\mathbb{R}^2$.

To define the OFS problem we need to introduce some further notions. Given a k-tuple $\langle q_i \rangle_{i=1}^k$ of points of M satisfying the r-separation condition, we define the *formation pattern* of $\langle q_i \rangle_{i=1}^k$ as a graph $(\mathcal{V}, \mathcal{E})$ whose set of vertices $\mathcal{V}$ is given by $\mathcal{V} = \{1, \ldots, k\}$ and whose set of edges $\mathcal{E}$ contains the edge (i, j) between vertex i and vertex j if and only if $d_M(q_i, q_j) = r$. Let $\gamma = \langle \gamma_i \rangle_{i=1}^k$ be a collision-free k-tuple of curves in M defined

on $[t_0, t_1]$. Then, for each $t \in [t_0, t_1]$, the formation pattern of γ at time t is defined to be the formation pattern of the k-tuple of points $\langle \gamma_i(t) \rangle_{i=1}^k$.

Remark 1 *For given M, r and k, not all graphs with k vertices can represent the formation pattern of some k-tuple of points of M satisfying the r-separation condition. For example, if $M = \mathbb{R}^2$ and $k = 4$, the complete graph with four vertices is not the formation pattern of any $\langle q_i \rangle_{i=1}^4$ satisfying the r-separation condition, regardless of $r > 0$. In fact, each formation pattern $(\mathcal{V}, \mathcal{E})$ corresponds to a nonempty subset of $M^{(k)} \setminus W$, namely, those $(q_1, \ldots, q_k) \in M^{(k)} \setminus W$ satisfying $d_M(q_i, q_j) = r$ if $(i, j) \in \mathcal{E}$ and $d_M(q_i, q_j) > r$ otherwise. In particular, if $\mathcal{E}$ contains no edges, then $(\mathcal{V}, \mathcal{E})$ corresponds to the interior of $M^{(k)} \setminus W$.*

Denote by $\mathcal{F}$ the set of all formation patterns. A partial order $\prec$ is defined on $\mathcal{F}$ such that two formation patterns $(\mathcal{V}_1, \mathcal{E}_1)$ and $(\mathcal{V}_2, \mathcal{E}_2)$ satisfy $(\mathcal{V}_1, \mathcal{E}_1) \prec (\mathcal{V}_2, \mathcal{E}_2)$ if and only if $(\mathcal{V}_1, \mathcal{E}_1)$ is a subgraph of $(\mathcal{V}_2, \mathcal{E}_2)$. Based on this partial order relation, $\mathcal{F}$ can be rendered graphically as a *Hasse diagram* ([26]). In this diagram, each element of $\mathcal{F}$ is represented by a node on a plane at a certain position such that the node corresponding to $(\mathcal{V}_1, \mathcal{E}_1)$ is placed at a lower position than the node corresponding to $(\mathcal{V}_2, \mathcal{E}_2)$ if $(\mathcal{V}_1, \mathcal{E}_1) \prec (\mathcal{V}_2, \mathcal{E}_2)$, and a line segment is drawn upward from node $(\mathcal{V}_1, \mathcal{E}_1)$ to node $(\mathcal{V}_2, \mathcal{E}_2)$ if and only if $(\mathcal{V}_1, \mathcal{E}_1) \prec (\mathcal{V}_2, \mathcal{E}_2)$ and there exists no other $(\mathcal{V}, \mathcal{E}) \in \mathcal{F}$ such that $(\mathcal{V}_1, \mathcal{E}_1) \prec (\mathcal{V}, \mathcal{E})$ and $(\mathcal{V}, \mathcal{E}) \prec (\mathcal{V}_2, \mathcal{E}_2)$. As an example, Figure 2.1 plots the Hasse diagram of $\mathcal{F}$ in the case $M = \mathbb{R}^2$ and $k = 3$.

Now we can define the OFS problem.

Problem 2 (Optimal Formation Switching (OFS)) *Let $\tilde{\mathcal{F}}$ be a subset of $\mathcal{F}$ such that the formation patterns of both $\langle a_i \rangle_{i=1}^k$ and $\langle b_i \rangle_{i=1}^k$ belong to $\tilde{\mathcal{F}}$. Among all collision-free $\gamma = \langle \gamma_i \rangle_{i=1}^k$ that start from $\langle a_i \rangle_{i=1}^k$ at time t_0 and end at $\langle b_i \rangle_{i=1}^k$ at time t_1, find the ones minimizing the cost (2.2) and satisfying the constraint that the formation pattern of γ at any time $t \in [t_0, t_1]$ belongs to $\tilde{\mathcal{F}}$.*

The OFS problem is a natural generalization of the OCA problem: the OFS problem reduces to the OCA problem if $\tilde{\mathcal{F}} = \mathcal{F}$. Regarded as a curve in $M^{(k)} \setminus W$, a solution γ to the OFS problem can only lie in a *subset* of $M^{(k)} \setminus W$ obtained by piecing together cells of various dimensions, one for each formation pattern in $\tilde{\mathcal{F}}$. Depending on $\tilde{\mathcal{F}}$, this union of cells can be highly complicated. In the example shown in Figure 2.1, one can choose $\tilde{\mathcal{F}}$ to consist of formation patterns 1, 2, 3, and 4, thus requiring that every two agents "contact" each other either directly or indirectly via the third agent at all time. This makes sense in practical situations where the three agents have to share data among one another and information exchange is possible only at the minimum allowed distance. As another example, $\tilde{\mathcal{F}}$ can be chosen to consist of formation patterns 1, 3, 4, and 7. In this case agent 1 and agent 2 are required to be bound together during the whole time interval $[t_0, t_1]$; and the OFS problem can be viewed as the OCA problem between agent 3 and this two-agent subsystem.

Remark 2 *Solutions to the OCA and OFS problems may not exist. The OCA problem of two agents on a line trying to switch positions is one such example. As another example, consider the OFS problem in Figure 2.1, with $\tilde{\mathcal{F}}$ consisting of only formation pattern 8. Regarded as a curve in $M^{(k)} \setminus W = \mathbb{R}^6 \setminus W$, a solution γ has to lie in the interior of $\mathbb{R}^6 \setminus W$. If the starting and destination positions correspond to two points in int$(\mathbb{R}^6 \setminus W)$ that are 'invisible' to each other, i.e., if the line segment connecting them intersects the obstacle W, then the OFS problem does not admit a solution. In general, to ensure that a solution to the OFS problem exists, it is sufficient (though not necessary) to require that the subset of $M^{(k)} \setminus W$ corresponding to $\tilde{\mathcal{F}}$ is closed and that the two points corresponding to the starting and destination positions are in the same connected component of this subset. The first requirement translates into the following property of $\tilde{\mathcal{F}}$: for each $(\mathcal{V}, \mathcal{E}) \in \tilde{\mathcal{F}}$, any formation pattern $(\mathcal{V}_1, \mathcal{E}_1)$ such that $(\mathcal{V}, \mathcal{E}) \prec (\mathcal{V}_1, \mathcal{E}_1)$ is also an element of $\tilde{\mathcal{F}}$. This is automatically satisfied in the OCA problem because $\tilde{\mathcal{F}} = \mathcal{F}$. The second requirement is satisfied if there exists at least one collision-free $\langle \gamma_i \rangle_{i=1}^k$ from the starting to the destination position whose formation pattern is always in $\tilde{\mathcal{F}}$.*

Remark 3 *Solutions to the OFS problem are less regular than those to the OCA problem. For example, in [12] it was proved that when M is an Euclidean space and $L = \frac{1}{2}\|\cdot\|^2$, solutions to the OCA problem*

are always C^1 (though not C^2 in general), while solutions to the OFS problem that are not C^1 can be easily constructed. See [15] for one such example.

In this paper, we focus on the OCA and OFS problems on certain Riemannian manifolds satisfying the following assumptions, whose implications will be detailed next.

Assumption 1 (Symmetry) *There is a Lie group G such that*

1. *G acts on M from the left by isometries (denote by $\Phi : G \times M \to M$ this C^∞ action);*
2. *the Lagrangian function L is G-invariant.*

We now explain the meaning of these assumptions. For brevity, we write gq for $\Phi(g, q)$, $g \in G$ and $q \in M$. For each $g \in G$, define $\Phi_g : M \to M$ to be the map $\Phi_g : q \mapsto gq$, $\forall q \in M$. Similarly, for each $q \in M$, define $\Phi^q : G \to M$ to be the map $\Phi^q : g \mapsto gq$, $\forall g \in G$. Both Φ_g and Φ^q are C^∞ maps since Φ is C^∞. Φ being a left action on M is equivalent to that i) $\Phi_{g_1 g_2} = \Phi_{g_1} \circ \Phi_{g_2}$ for $g_1, g_2 \in G$, where $\circ$ denotes composition of maps, and ii) $\Phi_e(q) = q$, $\forall q \in M$, where e is the identity element of G. For each $g \in G$, the first assumption implies that Φ_g is an isometry of M, i.e., $\Phi_g : M \to M$ is a map preserving the metric $\langle \cdot, \cdot \rangle$ (hence the distance) on M, while the second assumption implies that $L \circ d\Phi_g = L$, where $d\Phi_g : TM \to TM$ is the tangent map of Φ_g. If in particular $L = \frac{1}{2}\| \cdot \|^2$, then the second assumption is a direct consequence of the first one.

We now give a few simple examples of M and G satisfying the above assumptions. More examples will be presented later.

Example 1 (Euclidean space) A classical example is the Euclidean space $M = \mathbb{R}^n$, which is the manifold of interest in many of the practical applications mentioned in Section 1 such as, for instance, aircraft conflict resolution. Elements of $\mathbb{R}^n$ are thought of as column vectors. The tangent space of $\mathbb{R}^n$ at any point can be identified with $\mathbb{R}^n$ itself and is equipped with the standard Euclidean metric. Let $L = \frac{1}{2}\| \cdot \|^2$. Then the cost of a k-tuple of curves $\gamma = \langle \gamma_i \rangle_{i=1}^k$ in $\mathbb{R}^n$ is $J(\gamma) = \frac{1}{2} \sum_{i=1}^k \lambda_i \int_{t_0}^{t_1} \|\dot{\gamma}_i(t)\|^2 \, dt$. There are many choices of G for which Assumption 1 holds. For example, G can be chosen to be $\mathbb{R}^n$ itself, with the group operation being vector addition. The action $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is simply the group operation. As another example, consider the group of orientation-preserving n-by-n orthogonal matrices, $G = \mathbf{SO}_n \triangleq \{A \in \mathbb{R}^{n \times n} : A^T A = I_n, \det A = 1\}$. The matrix multiplication defines an action of $\mathbf{SO}_n$ on $\mathbb{R}^n$ that also satisfies Assumption 1. □

Example 2 (Sphere in $\mathbb{R}^n$) Let $M = \mathbf{S}^{n-1} \triangleq \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\}$ be the unit $(n-1)$-sphere for some $n \geq 2$. For each $q \in \mathbf{S}^{n-1}$, the tangent space $T_q \mathbf{S}^{n-1} = \{v \in \mathbb{R}^n : v^T q = 0\} \simeq \mathbb{R}^{n-1}$ is equipped with the standard Euclidean metric. This metric is the one $\mathbf{S}^{n-1}$ inherits from $\mathbb{R}^n$ as a submanifold. Let $L = \frac{1}{2}\| \cdot \|^2$. Then the action of the matrix group $G = \mathbf{SO}_n$ on $\mathbf{S}^{n-1}$ by matrix multiplication satisfies Assumption 1. □

Example 3 (Lie group) More abstractly, let $M = G$ be a Lie group with a left invariant Riemannian metric, in other words, the left multiplication by g defines an isometry of G for each $g \in G$. Let $L : TG \to \mathbb{R}$ be a left invariant Lagrangian function. Such L correspond in a one-to-one way with nonnegative and convex functions $T_e G \to \mathbb{R}$, where e is the identity element of G. Then the group multiplication $G \times G \to G$ is a left action of G on itself satisfying Assumption 1.

In the particular case when the metric on G is also right invariant, i.e., it is *bi-invariant*, the results in this paper turn out to be especially simple. Lie groups with bi-invariant metric include all compact Lie groups and semi-simple Lie groups (see [6, 10]). Example 1 with $G = M = \mathbb{R}^n$ is an example of a Lie group with a bi-invariant metric. As another example, consider $G = \mathbf{SO}_n$. Its Lie algebra, namely, the tangent space of $\mathbf{SO}_n$ at the identity element, is $\mathfrak{so}_n = \{X \in \mathbb{R}^{n \times n} : X + X^T = 0\}$, the set of skew symmetric n-by-n matrices. Denote by $\langle \cdot, \cdot \rangle_F$ the Frobenius inner product on $\mathbb{R}^{n \times n}$ defined by $\langle Y, Z \rangle_F = \mathrm{tr}(Y^T Z)$ for $Y, Z \in \mathbb{R}^{n \times n}$. A left invariant metric on $\mathbf{SO}_n$ can be established by first specifying its restriction on the fiber $\mathfrak{so}_n$ to be $\frac{1}{2}\langle \cdot, \cdot \rangle_F$, and then extending it to all other fibers so that each left multiplication is an isometry. It is easy to see that the metric thus defined is also right invariant, hence bi-invariant.

This last example finds application in surveillance systems. Consider a cluster of cameras monitoring, for instance, a chamber in a museum. Suppose that each camera has a limited angle of view, and is mounted on a ball head that can rotate freely. The configuration space of each camera is $\mathbf{SO}_3$, and we can define two cameras to be in a "collision" if their visibility regions ever overlap. Efficient coordination of the surveillance cameras can then be reformulated as an OCA (or OFS) problem on $\mathbf{SO}_3$. The results proved in this paper still apply in the case when the visibility regions of the cameras have possibly an irregular shape (see Section 5.1). Similar applications can be found in multiple satellites covering the earth for surveillance/communication purposes. □

3. Necessary Conditions for Optimality. In this section, we derive necessary conditions for continuous and piecewise smooth curves to be optimal solutions to the OCA and OFS problems on a Riemannian manifold M satisfying Assumption 1. It should be pointed out that some of the results, more specifically those in Section 3.2.1, can be proved using the Hamiltonian or symplectic approach. In this paper, however, we adopt the more direct (though less elegant) Lagrangian viewpoint for two reasons: it is easier to deal with the nonsmooth nature of the problems addressed; and, as a byproduct, further optimality conditions such as those in Sections 3.2.2 and 3.3 can be obtained.

3.1. Variations of Curves in the Lie Group G. We first review some notions and results on smooth variations of curves in G that are useful in later sections. All of the results in this section are well known in the literature and can be found, e.g., in [6, 23].

Definition 1 *Let $h_0 : [t_0, t_1] \to G$ be a C^∞ curve in G. A (smooth)* variation *of h_0 is a C^∞ map $h : (-\epsilon, \epsilon) \times [t_0, t_1] \to G$ such that $h(0, \cdot) = h_0(\cdot)$, ϵ being some small positive real number. If in addition $h(\cdot, t_0) \equiv h_0(t_0)$ and $h(\cdot, t_1) \equiv h_0(t_1)$, then the variation h is called* proper.

Let h be a variation of h_0 as in Definition 1. For each $s \in (-\epsilon, \epsilon)$, $h(s, \cdot) : [t_0, t_1] \to G$ is a curve in G which we denote by $h_s(\cdot)$ (note that this is consistent with Definition 1, since at $s = 0$ we obtain h_0). The variation h can then be equivalently specified by a smoothly varying family of curves $\{h_s\}_{s\in(-\epsilon,\epsilon)}$. Also, the condition that h is a proper variation is equivalent to that all curves in this family have the same starting and ending points.

For each $(s, t) \in (-\epsilon, \epsilon) \times [t_0, t_1]$, we define $\dot{h}(s, t) \triangleq \frac{\partial h}{\partial t}(s, t)$, $h'(s, t) \triangleq \frac{\partial h}{\partial s}(s, t)$, using dot and prime to indicate differentiation with respect to t and s, respectively. Both $\dot{h}(s, t)$ and $h'(s, t)$ belong to the tangent space of G at $h(s, t)$. We can pull them back via left multiplication to the tangent space of G at the identity element e, i.e., the Lie algebra $\mathfrak{g} = T_eG$ of G. Thus we define

$$\xi(s, t) \triangleq h(s, t)^{-1}\dot{h}(s, t) \in \mathfrak{g}, \qquad \eta(s, t) \triangleq h(s, t)^{-1}h'(s, t) \in \mathfrak{g}.$$

Here to simplify the notation we use $h(s, t)^{-1}\dot{h}(s, t)$ to denote $dm_{h(s,t)^{-1}}[\dot{h}(s, t)]$ (m is the action of G on itself defined from the group operation, so that for any $g \in G$, $m_g : G \to G$ stands for the left multiplication by g, while $dm_g : TG \to TG$ is its tangent map). Similarly for $h(s, t)^{-1}h'(s, t)$. This kind of notational simplification will be carried out in the following without further explanation.

Define $\dot{\xi}(s, t) = \frac{\partial \xi}{\partial t}(s, t)$ and $\xi'(s, t) = \frac{\partial \xi}{\partial s}(s, t)$, both of which belong to $T_{\xi(s,t)}\mathfrak{g}$. Since $\mathfrak{g}$ is a vector space, we can identify $T_{\xi(s,t)}\mathfrak{g}$ with $\mathfrak{g}$. Hence $\dot{\xi}(s, t)$ and $\xi'(s, t)$ belong to $\mathfrak{g}$. Similarly we can define $\dot{\eta}(s, t)$, $\eta'(s, t) \in \mathfrak{g}$. Denote by $[\cdot, \cdot]$ the Lie bracket of $\mathfrak{g}$. Then

Lemma 1 *At any $(s, t) \in (-\epsilon, \epsilon) \times [t_0, t_1]$,*

$$\xi'(s, t) = \dot{\eta}(s, t) + [\xi(s, t), \eta(s, t)].$$

See Appendix A for a concise proof of Lemma 1. Proofs can also be found in [3], and in the case of matrix Lie groups, in [22].

In line with our previous notation, we define $\dot{h}_s(\cdot) = \dot{h}(s, \cdot)$ and $h'_s(\cdot) = h'(s, \cdot)$, for each $s \in (-\epsilon, +\epsilon)$. We shall also write $\xi_s(\cdot) = \xi(s, \cdot)$, $\dot{\xi}_s(\cdot) = \dot{\xi}(s, \cdot)$, and $\xi'_s(\cdot) = \xi'(s, \cdot)$. Similarly for $\eta_s(\cdot)$, $\dot{\eta}_s(\cdot)$ and $\eta'_s(\cdot)$. Thus

the statement in Lemma 1 can be rewritten as

$$\xi_s' = \dot{\eta}_s + [\xi_s, \eta_s], \quad \text{for all } s. \tag{3.1}$$

We now apply Lemma 1 to a very special case. Denote by c_e the constant map that maps every $t \in [t_0, t_1]$ to the identity e in G, i.e., $c_e(\cdot) \equiv e$. Suppose that h is a *proper* variation of $h_0 = c_e$. Then $\dot{h}_0(\cdot) \equiv 0$ since $h_0(\cdot) \equiv e$, and therefore $\xi_0(\cdot) \equiv 0$. Since h is a proper variation, we have $h'(\cdot, t_0) = h'(\cdot, t_1) \equiv 0$, hence $\eta(\cdot, t_0) = \eta(\cdot, t_1) \equiv 0$. Define $\chi : [t_0, t_1] \to \mathfrak{g}$ by

$$\chi = \xi_0'. \tag{3.2}$$

χ is a C^∞ map which, by Lemma 1, satisfies

$$\chi = \dot{\eta}_0 + [\xi_0, \eta_0] = \dot{\eta}_0, \tag{3.3}$$

where the second equality follows from $\xi_0(\cdot) \equiv 0$. Therefore, $\int_{t_0}^{t_1} \chi(t)\, dt = \int_{t_0}^{t_1} \dot{\eta}_0(t)\, dt = \eta_0(t_1) - \eta_0(t_0) = 0$. Conversely, given any C^∞ map $\chi : [t_0, t_1] \to \mathfrak{g}$ satisfying $\int_{t_0}^{t_1} \chi(t)\, dt = 0$, define $h(s,t) = \exp[s \int_{t_0}^{t} \chi(t)\, dt]$, $\forall (s,t) \in (-\epsilon, \epsilon) \times [t_0, t_1]$, where ϵ is a positive real number small enough and exp is the exponential map of G. One can verify that h is indeed a proper variation of c_e such that ξ_0' coincides with χ. Therefore,

Lemma 2 *A necessary and sufficient condition for a C^∞ map $\chi : [t_0, t_1] \to \mathfrak{g}$ to be realized as $\chi = \xi_0'$, where $\xi = h^{-1}\dot{h}$ for some C^∞ proper variation h of $h_0 : [t_0, t_1] \to G$ given by $h_0(\cdot) \equiv e$, is*

$$\int_{t_0}^{t_1} \chi(t)\, dt = 0.$$

Remark 4 *The result in Lemma 2 can also be derived from Proposition 1.14.1 in [7].*

3.2. Variational Analysis. Suppose that $\gamma = \langle \gamma_i \rangle_{i=1}^k$ is an optimal solution to the OCA (or OFS) problem that starts from $\langle a_i \rangle_{i=1}^k$ at time t_0 and ends at $\langle b_i \rangle_{i=1}^k$ at time t_1. Necessary conditions on γ can be derived in the following way. Let $h : (-\epsilon, \epsilon) \times [t_0, t_1] \to G$ be a C^∞ proper variation of the constant map $c_e(\cdot) \equiv e$ for some small $\epsilon > 0$. According to the notations introduced in Section 3.1, for each $s \in (-\epsilon, \epsilon)$, $h_s : [t_0, t_1] \to G$ is a C^∞ curve in G both starting and ending at e, hence can be used to define a k-tuple of perturbed curves $\gamma_s = \langle \gamma_{s,i} \rangle_{i=1}^k$ in M by

$$\gamma_{s,i}(\cdot) = h_s(\cdot)\gamma_i(\cdot), \quad i = 1, \ldots, k,$$

which by the fact that $h_s(t_0) = h_s(t_1) = e$ also starts from $\langle a_i \rangle_{i=1}^k$ at time t_0 and ends at $\langle b_i \rangle_{i=1}^k$ at time t_1. Note that $\gamma_0 = \gamma$ since $h_0(\cdot) \equiv 0$. Moreover, since by Assumption 1 $\Phi_{h_s(t)}$ is an isometry of M, γ_s is collision-free, and has the same formation pattern as γ at any time $t \in [t_0, t_1]$. Define

$$J(s) \triangleq J(\gamma_s), \quad \forall s \in (-\epsilon, \epsilon). \tag{3.4}$$

$J(s)$ is a C^∞ function since h is a C^∞ variation.

For each $(s,t) \in (-\epsilon, \epsilon) \times [t_0, t_1]$, and each $i = 1, \ldots, k$, we have[1]

$$L[\dot{\gamma}_{s,i}] = L[\dot{h}_s \gamma_i + h_s \dot{\gamma}_i] = L[h_s \xi_s \gamma_i + h_s \dot{\gamma}_i] = L[h_s(\xi_s \gamma_i + \dot{\gamma}_i)] = L[\xi_s \gamma_i + \dot{\gamma}_i].$$

Here $\dot{h}_s \gamma_i$ denotes $d\Phi^{\gamma_i}(\dot{h}_s)$, and $h_s \dot{\gamma}_i$ denotes $d\Phi_{h_s}(\dot{\gamma}_i)$, both of which belong to $T_{h_s \gamma_i} M$. In the second equality we use the fact that $\dot{h}_s \gamma_i = h_s \xi_s \gamma_i$, a consequence of the property that $(g_1 g_2) q = g_1(g_2 q)$, $\forall g_1, g_2 \in G$, $q \in M$. The last equality follows from the G-invariance of L. The cost of γ_s is then

$$J(s) = \sum_{i=1}^k \lambda_i \int_{t_0}^{t_1} L[\xi_s \gamma_i + \dot{\gamma}_i]\, dt. \tag{3.5}$$

A necessary condition for γ to be optimal is that $J(s)$ assumes its minimum at $s = 0$. In particular, this implies that the derivatives $J'(0) = 0$ and $J''(0) \geq 0$. The implications of these two conditions will be studied in the next two subsections.

[1] Since γ_i is only piecewise C^∞, this and all equations that follow should be understood to hold only at those t where $\dot{\gamma}_i$'s are well defined. In addition, the parameter t is implicit in these equations for brevity.

3.2.1. First Variation. For any vector space V, denote by $(\cdot,\cdot) : V^* \times V \to \mathbb{R}$ the natural pairing between V and its dual V^*, i.e., $\forall \alpha \in V^*$, $v \in V$, $(\alpha, v) = \alpha(v)$ is the value of α on v. Differentiating (3.5) with respect to s, we have

$$J'(s) = \sum_{i=1}^{k} \lambda_i \int_{t_0}^{t_1} (\mathbb{D}L_{\xi_s\gamma_i+\dot\gamma_i}, \xi_s'\gamma_i)\, dt. \tag{3.6}$$

Here we identify the tangent space at $\xi_s\gamma_i + \dot\gamma_i$ of $T_{\gamma_i}M$ with $T_{\gamma_i}M$ itself, so $\xi_s'\gamma_i \in T_{\gamma_i}M$; $\mathbb{D}L_{\xi_s\gamma_i+\dot\gamma_i}$ is the fiberwise differential of L, or more precisely, the differential of L restricted on $T_{\gamma_i}M$ and evaluated at $\xi_s\gamma_i + \dot\gamma_i \in T_{\gamma_i}M$. $\mathbb{D}L_{\xi_s\gamma_i+\dot\gamma_i}$ can be thought of as an element of $T^*_{\gamma_i}M$. At $s = 0$, we have $\xi_0 \equiv 0$ and $\xi_0' = \chi$. Therefore, recalling that $\xi_s'\gamma_i = d\Phi^{\gamma_i}(\xi_s')$, we get

$$J'(0) = \sum_{i=1}^{k} \lambda_i \int_{t_0}^{t_1} (\mathbb{D}L_{\dot\gamma_i}, d\Phi^{\gamma_i}(\chi))\, dt = \int_{t_0}^{t_1} (\sum_{i=1}^{k} \lambda_i (d\Phi^{\gamma_i})^* \mathbb{D}L_{\dot\gamma_i}, \chi)\, dt, \tag{3.7}$$

where $(d\Phi^{\gamma_i})^* : T^*_{\gamma_i}M \to \mathfrak{g}^*$ is the dual of $d\Phi^{\gamma_i} : \mathfrak{g} \to T_{\gamma_i}M$ defined by

$$((d\Phi^{\gamma_i})^*\alpha, \zeta) = (\alpha, d\Phi^{\gamma_i}(\zeta)), \quad \forall \alpha \in T^*_{\gamma_i}M,\ \zeta \in \mathfrak{g}. \tag{3.8}$$

From (3.7) and Lemma 2, the condition that $J'(0) = 0$ for all proper variations h of the constant map $c_e(\cdot) \equiv e$ is equivalent to

$$\int_{t_0}^{t_1} (\sum_{i=1}^{k} \lambda_i (d\Phi^{\gamma_i})^* \mathbb{D}L_{\dot\gamma_i}, \chi)\, dt = 0 \tag{3.9}$$

for all C^∞ map $\chi : [t_0, t_1] \to \mathfrak{g}$ such that $\int_{t_0}^{t_1} \chi(t)\, dt = 0$. Since $\sum_{i=1}^{k} \lambda_i (d\Phi^{\gamma_i})^* \mathbb{D}L_{\dot\gamma_i}$ is piecewise C^∞ (though not necessarily continuous) in $\mathfrak{g}^*$, condition (3.9) implies that $\sum_{i=1}^{k} \lambda_i (d\Phi^{\gamma_i})^* \mathbb{D}L_{\dot\gamma_i}$ is constant for all $t \in [t_0, t_1]$ where $\dot\gamma_i$'s are well defined, because otherwise one can choose a χ with $\int_{t_0}^{t_1} \chi(t)\, dt = 0$ such that (3.9) fails to hold. This concludes the proof of the following theorem.

Theorem 1 *Suppose that $\gamma = \langle\gamma_i\rangle_{i=1}^k$ is an optimal solution to the OCA (or OFS) problem. Then there exists a constant $\nu_0 \in \mathfrak{g}^*$ such that*

$$\sum_{i=1}^{k} \lambda_i (d\Phi^{\gamma_i})^* \mathbb{D}L_{\dot\gamma_i} \equiv \nu_0 \tag{3.10}$$

for all $t \in [t_0, t_1]$ where $\dot\gamma_i$'s are well defined.

In the following we shall denote by ν the quantity on the left hand side of (3.10).

Remark 5 *The action of G on M induces an action of G on $M^{(k)}$ naturally, which can be cotangent lifted to an action of G on $T^*(M^{(k)})$. ν is the momentum map for this last action evaluated along the curve $(\gamma_1, \ldots, \gamma_k, \lambda_1\mathbb{D}L_{\dot\gamma_1}, \ldots, \lambda_k\mathbb{D}L_{\dot\gamma_k})$ in $T^*(M^{(k)})$. Theorem 1 thus generalizes the classical Noether theorem ([1, 23]) to the nonsmooth case.*

If $L = \frac{1}{2}\|\cdot\|^2$, then the conclusion of Theorem 1 can be simplified by canonically identifying each $v \in T_{\gamma_i}M$ with the element in $T^*_{\gamma_i}M$ defined by $u \mapsto \langle v, u\rangle$, $\forall u \in T_{\gamma_i}M$. Thus $\mathbb{D}L_{\dot\gamma_i}$ is identified with $\dot\gamma_i$, and (3.10) becomes

$$\sum_{i=1}^{k} \lambda_i (d\Phi^{\gamma_i})^* \dot\gamma_i \equiv \nu_0 \in \mathfrak{g}^*, \tag{3.11}$$

where $(d\Phi^{\gamma_i})^* : T_{\gamma_i}M \to \mathfrak{g}^*$ is now defined by

$$((d\Phi^{\gamma_i})^* v, \zeta) = \langle v, d\Phi^{\gamma_i}(\zeta)\rangle, \quad \forall v \in T_{\gamma_i}M,\ \zeta \in \mathfrak{g}. \tag{3.12}$$

Furthermore, there is occasionally a natural choice for a metric on $\mathfrak{g}$, which can be used to identify $\mathfrak{g}$ and $\mathfrak{g}^*$. In this case, the conserved quantity ν can be thought of taking values in $\mathfrak{g}$.

Example 4 ($G = \mathbf{SO}_n$, $M = \mathbb{R}^n$) Consider Example 1 with $G = \mathbf{SO}_n$, $M = \mathbb{R}^n$, and $L = \frac{1}{2}\|\cdot\|^2$. The Lie algebra $\mathfrak{g}$ of $\mathbf{SO}_n$ is $\mathfrak{so}_n = \{X \in \mathbb{R}^{n\times n} : X + X^T = 0\}$. Suppose that a k-tuple of curves in $\mathbb{R}^n$, $\gamma = \langle\gamma_i\rangle_{i=1}^k$, is an optimal solution to the OCA (or OFS) problem defined on $[t_0, t_1]$. At each time $t \in [t_0, t_1]$, let $v \in T_{\gamma_i}\mathbb{R}^n \simeq \mathbb{R}^n$ and $X \in \mathfrak{so}_n$ be arbitrary. Then

$$\langle v, d\Phi^{\gamma_i}(X)\rangle = \langle v, X\gamma_i\rangle = v^T X\gamma_i = \mathrm{tr}(\gamma_i v^T X) = \langle v\gamma_i^T, X\rangle_F = \frac{1}{2}\langle v\gamma_i^T - \gamma_i v^T, X\rangle_F,$$

where we recall that $\langle\cdot,\cdot\rangle_F$ is the Frobenius inner product on $\mathbb{R}^{n\times n}$ defined in Example 3. The last equality follows from the skew-symmetry of X. So by (3.12),

$$((d\Phi^{\gamma_i})^* v, X) = \frac{1}{2}\langle v\gamma_i^T - \gamma_i v^T, X\rangle_F, \quad \forall X \in \mathfrak{so}_n.$$

Note that $v\gamma_i^T - \gamma_i v^T \in \mathfrak{so}_n$. So, if $\mathfrak{so}_n$ is identified with $\mathfrak{so}_n^*$ using the metric $\frac{1}{2}\langle\cdot,\cdot\rangle_F$, the above equation implies that $(d\Phi^{\gamma_i})^* v = v\gamma_i^T - \gamma_i v^T$. Hence (3.11) becomes

$$\sum_{i=1}^k \lambda_i(\dot\gamma_i\gamma_i^T - \gamma_i\dot\gamma_i^T) \equiv \nu_0 \in \mathfrak{so}_n. \tag{3.13}$$

Or equivalently, $\sum_{i=1}^k \lambda_i(\dot\gamma_i \wedge \gamma_i)$ is constant, where $\wedge$ is the wedge product defined on $\mathbb{R}^n$. In particular, if $n = 3$ ($G = \mathbf{SO}_3$, $M = \mathbb{R}^3$), equation (3.13) is equivalent to $\sum_{i=1}^k \lambda_i(\gamma_i \times \dot\gamma_i) \equiv \Omega_0$ for some $\Omega_0 \in \mathbb{R}^3$, where $\times$ is the vector product. Therefore, if γ is thought of as the trajectories of k particles moving in $\mathbb{R}^3$ with mass $\lambda_1, \ldots, \lambda_k$, respectively, then their total angular momentum is preserved along an optimal solution to the OCA (or OFS) problem. The conserved quantity when $n > 3$ can also be thought of as the generalized angular momentum for the particle system. □

Example 5 ($G = \mathbf{SO}_n$, $M = \mathbf{S}^{n-1}$) Consider Example 2, where $G = \mathbf{SO}_n$, $M = \mathbf{S}^{n-1}$, and $L = \frac{1}{2}\|\cdot\|^2$. Since $\mathbf{S}^{n-1}$ is a submanifold of $\mathbb{R}^n$, by following the same steps as in Example 4, we conclude that (3.13) still holds for optimal solutions γ to the OCA (or OFS) problem on $\mathbf{S}^{n-1}$. □

Example 6 (Lie Group with a Bi-Invariant Metric) Suppose that in Example 3, $M = G$ is a Lie group with a bi-invariant Riemannian metric, and $L = \frac{1}{2}\|\cdot\|^2$. Let $\gamma = \langle\gamma_i\rangle_{i=1}^k$ be a solution to the OCA (or OFS) problem on G. Then at each time t, $\forall v \in T_{\gamma_i}G$, $\zeta \in \mathfrak{g}$,

$$\langle v, d\Phi^{\gamma_i}(\zeta)\rangle = \langle v, \zeta\gamma_i\rangle = \langle v\gamma_i^{-1}\gamma_i, \zeta\gamma_i\rangle = \langle v\gamma_i^{-1}, \zeta\rangle \quad \Rightarrow \quad ((d\Phi^{\gamma_i})^* v, \zeta) = \langle v\gamma_i^{-1}, \zeta\rangle. \tag{3.14}$$

Under the canonical identification of $\mathfrak{g}$ with $\mathfrak{g}^*$ via $\langle\cdot,\cdot\rangle$, the right hand side is equivalent to $(d\Phi^{\gamma_i})^* v = v\gamma_i^{-1} \in \mathfrak{g}$. Therefore, the conservation law (3.11) is

$$\sum_{i=1}^k \lambda_i\dot\gamma_i\gamma_i^{-1} \equiv \nu_0 \in \mathfrak{g}. \tag{3.15}$$

In the particular case when $M = G = \mathbb{R}^n$, (3.15) implies $\sum_{i=1}^k \lambda_i\dot\gamma_i \equiv \nu_0 \in \mathbb{R}^n$. In other words, if k particles with mass $\lambda_1, \ldots, \lambda_k$ follow the trajectories of γ, then their total linear momentum is preserved. This condition, together with the one on the conservation of total angular momentum obtained in Example 4, can be used in determining the optimal solutions to the OCA and OFS problem on $\mathbb{R}^n$, particularly when k is small. See [12, 14] for more details, and [13] for application in aircraft conflict resolution. If we consider $G = \mathbf{SO}_n$ with the bi-invariant metric defined in Example 3, and $L = \frac{1}{2}\|\cdot\|^2$, then (3.15) holds for solutions $\gamma = \langle\gamma_i\rangle_{i=1}^k$ to the OCA (or OFS) problem on $\mathbf{SO}_n$, and $\nu_0 \in \mathfrak{so}_n$ is now a constant skew symmetric n-by-n matrix. □

A large class of examples can be derived from Example 6 by considering the quotient spaces of G under certain subgroups H, i.e., the symmetric spaces G/H. We give two of these examples here. The first one is the OCA (or OFS) problem on the Grassmann manifold and has it origin in multi-user wireless communication [32]. In such a scenario, a communication channel is shared by multiple users. Specifically, each user is allocated a k-dimensional subspace in the n-dimensional signal space used for data transmission. Separation among these subspaces should be maintained to minimize crosstalk and hence guaranteeing a satisfactory signal-to-noise ratio (SNR). Due to the possible changes of user locations and channel conditions, the signal subspaces might need to be re-allocated from time to time in an incremental way, resulting each time in an OCA problem on the Grassmann manifold.

Example 7 (Grassmann Manifold) Suppose that $\mathbf{SO}_n$ has the bi-invariant metric described in Example 3. Let p be an integer, $1 \le p \le n$. Denote by H_p the subgroup $\begin{bmatrix} \mathbf{SO}_p & 0 \\ 0 & \mathbf{SO}_{n-p} \end{bmatrix} \simeq \mathbf{SO}_p \times \mathbf{SO}_{n-p}$ of $\mathbf{SO}_n$. Define $G_{n,p} \triangleq \mathbf{SO}_n/H_p$ to be the set of left cosets of H_p in $\mathbf{SO}_n$, and let $\pi : \mathbf{SO}_n \to G_{n,p}$ be the natural projection. Elements of $G_{n,p}$ are $\pi(A) = AH_p$, $\forall A \in \mathbf{SO}_n$. For each $\pi(A) \in G_{n,p}$, the subspace of $\mathbb{R}^n$ spanned by the first p column vectors of A is the same for all $A \in \pi(A)$, hence there is a one-to-one correspondence between $G_{n,p}$ and set of p-dimensional subspaces of $\mathbb{R}^n$. Since H_p is a closed subgroup of $\mathbf{SO}_n$, $G_{n,p}$ admits a natural differential structure, and is called a *Grassmann manifold*. At each $A \in \mathbf{SO}_n$, the tangent space of $\mathbf{SO}_n$ has the orthogonal decomposition [8]:

$$T_A\mathbf{SO}_n = \mathrm{vert}_A\mathbf{SO}_n \oplus \mathrm{hor}_A\mathbf{SO}_n.$$

The *vertical space* $\mathrm{vert}_A\mathbf{SO}_n = A\begin{bmatrix} \mathfrak{so}_p & 0 \\ 0 & \mathfrak{so}_{n-p} \end{bmatrix}$ is the tangent space of AH_p at A; the *horizontal space* $\mathrm{hor}_A\mathbf{SO}_n$ consists of all those matrices of the form $A\begin{bmatrix} 0 & -X^T \\ X & 0 \end{bmatrix}$ for some $X \in \mathbb{R}^{(n-p)\times p}$. Note that $d\pi : \mathrm{hor}_A\mathbf{SO}_n \to T_{\pi(A)}G_{n,p}$ is a vector space isomorphism. The restriction of the metric on $T_A\mathbf{SO}_n$ defines a metric on $\mathrm{hor}_A\mathbf{SO}_n$ as

$$\langle A\begin{bmatrix} 0 & -X_1^T \\ X_1 & 0 \end{bmatrix}, A\begin{bmatrix} 0 & -X_2^T \\ X_2 & 0 \end{bmatrix}\rangle = \langle X_1, X_2\rangle_F, \quad \forall X_1, X_2 \in \mathbb{R}^{(n-p)\times p}. \tag{3.16}$$

An important observation is that there is a unique Riemannian metric on $G_{n,p}$ that makes $d\pi : \mathrm{hor}_A\mathbf{SO}_n \to T_{\pi(A)}G_{n,p}$ an isometry for each $\pi(A) \in G_{n,p}$, regardless of the choice of $A \in \pi(A)$. Such a metric exists because the metric on $\mathbf{SO}_n$ is right invariant. Moreover, this metric on $G_{n,p}$ is invariant under the induced left action of $\mathbf{SO}_n$ on $G_{n,p}$ since the metric on $\mathbf{SO}_n$ is left invariant. In the terminology of [25], $\pi : \mathbf{SO}_n \to G_{n,p}$ is a *Riemannian submersion*, and when viewed as a principal H_p-bundle over $G_{n,p}$, $\mathbf{SO}_n$ has a metric of *constant bi-invariant type*. See [25] for more details on (sub-Riemannian) metrics of principal bundles.

Suppose that $L = \frac{1}{2}\|\cdot\|^2$. Let $\gamma = \langle\gamma_i\rangle_{i=1}^k$ be a k-tuple of curves in $G_{n,p}$ that is a solution to the OCA (or OFS) problem. For each $i = 1, \ldots, k$, let A_i be a lifting of γ_i in $\mathbf{SO}_n$ in the sense that $\pi(A_i) = \gamma_i$. In other words, the first p column vectors of $A_i \in \mathbf{SO}_n$ span the subspace $\gamma_i \in G_{n,p}$. We can choose A_i to be continuous and piecewise C^∞. At each time t, choose arbitrary $X \in \mathfrak{so}_n$ and $v \in T_{\gamma_i}G_{n,p}$, and let V be the unique element of $\mathrm{hor}_{A_i}\mathbf{SO}_n$ such that $d\pi(V) = v$. Then, using the fact that $d\pi : \mathrm{hor}_{A_i}\mathbf{SO}_n \to T_{\gamma_i}G_{n,p}$ is an isometry, we have

$$\langle v, d\Phi^{\gamma_i}(X)\rangle_{T_{\gamma_i}G_{n,p}} = \langle V, P_{A_i}(XA_i)\rangle_{\mathrm{hor}_{A_i}\mathbf{SO}_n} = \langle V, XA_i\rangle_{T_{A_i}\mathbf{SO}_n} = \langle VA_i^T, X\rangle_{\mathfrak{so}_n},$$

where P_{A_i} is the orthogonal projection $T_{A_i}\mathbf{SO}_n \to \mathrm{hor}_{A_i}\mathbf{SO}_n$. Here for clarity we indicate in subscript the associated tangent space of each inner product. Therefore, $(d\Phi^{\gamma_i})^*v = VA_i^T \in \mathfrak{so}_n \simeq \mathfrak{so}_n^*$. Finally, notice that $\dot\gamma_i = d\pi[P_{A_i}(\dot A_i)]$. Then (3.11) becomes

$$\sum_{i=1}^k \lambda_i (d\Phi^{\gamma_i})^*\dot\gamma_i = \sum_{i=1}^k \lambda_i P_{A_i}(\dot A_i)A_i^T \equiv \nu_0 \in \mathfrak{so}_n. \tag{3.17}$$

□

Example 8 (Stiefel Manifold) Denote by K_p the subgroup $\begin{bmatrix} I_p & 0 \\ 0 & \mathbf{SO}_{n-p} \end{bmatrix} \simeq \mathbf{SO}_{n-p}$ of $\mathbf{SO}_n$. Then the quotient space $V_{n,p} \triangleq \mathbf{SO}_n/K_p$ is called a *Stiefel manifold*. Elements in $V_{n,p}$ correspond in a one-to-one way to the orthonormal p-frames of $\mathbb{R}^n$. At each $A \in \mathbf{SO}_n$, the vertical space is now $A \begin{bmatrix} 0 & 0 \\ 0 & \mathfrak{so}_{n-p} \end{bmatrix}$, while the horizontal space consists of matrices of the form $A \begin{bmatrix} Y & -X^T \\ X & 0 \end{bmatrix}$ for $X \in \mathbb{R}^{(n-p)\times p}$, $Y \in \mathbb{R}^{p\times p}$, $Y + Y^T = 0$. The metric on $\mathbf{SO}_n$ restricts to a metric on the horizontal space as

$$\langle A \begin{bmatrix} Y_1 & -X_1^T \\ X_1 & 0 \end{bmatrix}, A \begin{bmatrix} Y_2 & -X_2^T \\ X_2 & 0 \end{bmatrix} \rangle = \langle X_1, X_2 \rangle_F + \frac{1}{2}\langle Y_1, Y_2 \rangle_F,$$

which can be used to define a Riemannian metric on $V_{n,p}$ such that $d\pi$ is an isometry from the horizontal space at each $A \in \mathbf{SO}_n$ to $T_{\pi(A)}V_{n,p}$ (π is now the natural projection from $\mathbf{SO}_n$ to $V_{n,p}$). The metric thus defined is invariant under the induced left action of $\mathbf{SO}_n$ on $V_{n,p}$.

Let $L = \frac{1}{2}\|\cdot\|^2$. By similar arguments as in Example 7, we can show that if $\gamma = \langle \gamma_i \rangle_{i=1}^k$ is a solution to the OCA (or OFS) problem on $V_{n,p}$, and if A_i is a lifting of γ_i in $\mathbf{SO}_n$, $1 \le i \le k$, then

$$\sum_{i=1}^k \lambda_i \hat{P}_{A_i}(\dot{A}_i) A_i^T \equiv \nu_0 \in \mathfrak{so}_n. \tag{3.18}$$

Here $\hat{P}_{A_i}$ is the orthogonal projection from $T_{A_i}\mathbf{SO}_n$ to the horizontal space of $\mathbf{SO}_n$ at A_i. □

3.2.2. Second Variation. Let $J(s)$ be defined as in (3.5). Differentiating equation (3.6) with respect to s at $s = 0$, we have

$$\begin{aligned} J''(0) &= \sum_{i=1}^k \lambda_i \int_{t_0}^{t_1} [(\mathbb{D}L_{\dot{\gamma}_i}, \xi_0''\gamma_i) + \mathbb{D}^2 L_{\dot{\gamma}_i}(\xi_0'\gamma_i, \xi_0'\gamma_i)]\, dt \\ &= \int_{t_0}^{t_1} (\sum_{i=1}^k \lambda_i (d\Phi^{\gamma_i})^* \mathbb{D}L_{\dot{\gamma}_i}, \xi_0'')\, dt + \int_{t_0}^{t_1} \sum_{i=1}^k \lambda_i \mathbb{D}^2 L_{\dot{\gamma}_i}(\xi_0'\gamma_i, \xi_0'\gamma_i)\, dt. \end{aligned} \tag{3.19}$$

Here $\xi_0'' \in \mathfrak{g}$, and $\mathbb{D}^2 L_{\dot{\gamma}_i} : T_{\gamma_i}M \times T_{\gamma_i}M \to \mathbb{R}$ is the fiberwise second order derivative (Hessian) of L on $T_{\gamma_i}M$ evaluated at $\dot{\gamma}_i$. By Theorem 1, the first term in (3.19) can be written as $\int_{t_0}^{t_1} (\nu_0, \xi_0'')\, dt$, which can in turn be simplified as

$$\begin{aligned} \int_{t_0}^{t_1} (\nu_0, \xi_0'')\, dt &= \frac{d}{ds}\Big|_{s=0} (\nu_0, \int_{t_0}^{t_1} \xi_s'\, dt) \qquad = \frac{d}{ds}\Big|_{s=0} (\nu_0, \int_{t_0}^{t_1} (\dot{\eta}_s + [\xi_s, \eta_s])\, dt) \\ &= \frac{d}{ds}\Big|_{s=0} (\nu_0, \int_{t_0}^{t_1} [\xi_s, \eta_s]\, dt) = (\nu_0, \int_{t_0}^{t_1} ([\xi_0', \eta_0] + [\xi_0, \eta_0'])\, dt) \\ &= (\nu_0, \int_{t_0}^{t_1} [\dot{\eta}_0, \eta_0]\, dt), \end{aligned} \tag{3.20}$$

where we have used Lemma 1 and the following facts: $\eta_s(t_0) = \eta_s(t_1) = 0$; $\xi_0(\cdot) \equiv 0$; and $\xi_0' = \chi = \dot{\eta}_0$ by equation (3.3). As for the second term in (3.19), define

$$\mathbb{I}_t(\zeta_1, \zeta_2) \triangleq \sum_{i=1}^k \lambda_i \mathbb{D}^2 L_{\dot{\gamma}_i(t)}(\zeta_1\gamma_i(t), \zeta_2\gamma_i(t)), \quad \forall \zeta_1, \zeta_2 \in \mathfrak{g}, \tag{3.21}$$

for each $t \in [t_0, t_1]$. Then $\mathbb{I}_t(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is a nonnegative definite quadratic form on $\mathfrak{g}$, since $\lambda_i > 0$ and L is convex on each fiber of TM by assumption. If in particular $L = \frac{1}{2}\|\cdot\|^2$, then $\mathbb{D}^2 L_v(\cdot,\cdot) = \langle \cdot, \cdot \rangle_q$ for any $v \in T_qM$, $q \in M$. Hence $\mathbb{I}_t$ in this case is given by

$$\mathbb{I}_t(\zeta_1, \zeta_2) \triangleq \sum_{i=1}^k \lambda_i \langle \zeta_1\gamma_i, \zeta_2\gamma_i \rangle, \quad \forall \zeta_1, \zeta_2 \in \mathfrak{g}. \tag{3.22}$$

In mechanics, $\mathbb{I}_t$ defined in (3.22) is called the *moment of inertia tensor* ([23, 25]) for the action of G on $M^{(k)}$ (with the metric $\prod_{i=1}^k \lambda_i\langle\cdot,\cdot\rangle$) evaluated at $(\gamma_1,\ldots,\gamma_k)$. By substituting (3.20) and (3.21) into (3.19) with $\xi_0' = \dot\eta_0$ by (3.2) and (3.3), we obtain

$$J''(0) = (\nu_0, \int_{t_0}^{t_1} [\dot\eta_0, \eta_0]\, dt) + \int_{t_0}^{t_1} \mathbb{I}_t(\dot\eta_0, \dot\eta_0)\, dt.$$

In order for γ to be optimal, we must have $J''(0) \geq 0$ for all feasible η_0. Therefore,

Theorem 2 *Suppose that $\langle\gamma_i\rangle_{i=1}^k$ is an optimal solution to the OCA (or OFS) problem. Let $\nu_0 \in \mathfrak{g}^*$ be defined as in Theorem 1. Then for any C^∞ curve $\eta_0 : [t_0, t_1] \to \mathfrak{g}$ such that $\eta_0(t_0) = \eta_0(t_1) = 0$,*

$$(\nu_0, \int_{t_0}^{t_1} [\dot\eta_0, \eta_0]\, dt) + \int_{t_0}^{t_1} \mathbb{I}_t(\dot\eta_0, \dot\eta_0)\, dt \geq 0. \tag{3.23}$$

Remark 6 *Consider the OCA problem with $L = \frac{1}{2}\|\cdot\|^2$. If $k = 1$, then solutions γ to this problem are geodesics of M. If the action Φ is transitive, then any local proper variation of γ in M can be generated as $h\gamma$ by some proper variation h of c_e in G. So in this case Theorem 2 characterizes the first* conjugate point *along γ. If $k > 1$, then solutions γ are geodesics in $M^{(k)} \setminus W$, a manifold with boundary whose dimension is usually much larger than that of G. The variations of γ in the form of $h\gamma$ can only perturb the k components of γ uniformly by multiplying all of them from the left by the* same *elements of G. Hence the condition in Theorem 2 is in general only necessary for the local optimality of γ when $k > 1$. See Section 4 for an example.*

It is often difficult to apply Theorem 2 directly. In the following we shall derive some of its implications that are easier to check. Note that if $\dim(\mathfrak{g}) = 1$ (or if $\mathfrak{g}$ is abelian), condition (3.23) holds trivially, since the first term is zero and the integrand of the second term is nonnegative. So we shall assume that $\dim(\mathfrak{g}) > 1$.

Choose an arbitrary inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$, whose corresponding norm is denoted by $\|\cdot\|_{\mathfrak{g}}$. In many cases there is a natural choice for $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$. At each time t, define the *spectral radius* of $\mathbb{I}_t$ as

$$\rho(\mathbb{I}_t) = \inf\{\lambda \in \mathbb{R} : \lambda\langle\cdot,\cdot\rangle_{\mathfrak{g}} - \mathbb{I}_t(\cdot,\cdot) \text{ is nonnegative definite on } \mathfrak{g}\}.$$

Then $\rho(\mathbb{I}_t) \geq 0$ is the largest eigenvalue of the symmetric matrix representing $\mathbb{I}_t$ in any orthonormal basis of $\mathfrak{g}$. For any subspace $\mathfrak{h}$ of $\mathfrak{g}$, the restriction $\mathbb{I}_t|_{\mathfrak{h}}$ is still nonnegative definite. Define

$$\rho(\mathbb{I}_t; \mathfrak{h}) = \rho(\mathbb{I}_t|_{\mathfrak{h}}) = \inf\{\lambda \in \mathbb{R} : \lambda\langle\cdot,\cdot\rangle_{\mathfrak{g}} - \mathbb{I}_t(\cdot,\cdot) \text{ is nonnegative definite on } \mathfrak{h}\}. \tag{3.24}$$

An immediate result of definition (3.24) is

$$\mathbb{I}_t(\zeta_1, \zeta_2) \leq \rho(\mathbb{I}_t; \mathfrak{h})\, \langle\zeta_1, \zeta_2\rangle_{\mathfrak{g}}, \quad \forall \zeta_1, \zeta_2 \in \mathfrak{h}.$$

Pick a two dimensional subspace $\mathfrak{h}$ of $\mathfrak{g}$, and let $\{\zeta_1, \zeta_2\}$ be an orthonormal basis of $\mathfrak{h}$. Denote

$$\zeta_0 = [\zeta_1, \zeta_2]. \tag{3.25}$$

Now consider condition (3.23) in the special case when η_0 as a curve in $\mathfrak{g}$ is contained entirely in $\mathfrak{h}$. Then there exist C^∞ functions $x_1, x_2 : [t_0, t_1] \to \mathbb{R}$ such that $\eta_0 = x_1\zeta_1 + x_2\zeta_2$. The constraints that $\eta_0(t_0) = \eta_0(t_1) = 0$ imply that $x_1(t_0) = x_1(t_1) = 0$ and $x_2(t_0) = x_2(t_1) = 0$. Moreover,

$$[\dot\eta_0, \eta_0] = [\dot x_1\zeta_1 + \dot x_2\zeta_2, x_1\zeta_1 + x_2\zeta_2] = (\dot x_1 x_2 - x_1\dot x_2)\zeta_0.$$

Therefore, on the left hand side of inequality (3.23), the first term becomes

$$(\nu_0, \int_{t_0}^{t_1} [\dot\eta_0, \eta_0]\, dt) = \nu_0(\zeta_0) \int_{t_0}^{t_1} (\dot x_1 x_2 - x_1\dot x_2)\, dt = -2\nu_0(\zeta_0) S_{\eta_0},$$

where S_{η_0} is the (oriented) planar area encircled by η_0 in $\mathfrak{h}$. The second term is dominated by

$$\int_{t_0}^{t_1} \mathbb{I}_t(\dot{\eta}_0, \dot{\eta}_0)\, dt \leq \int_{t_0}^{t_1} \rho(\mathbb{I}_t; \mathfrak{h}) \|\dot{\eta}_0\|_{\mathfrak{g}}^2\, dt \leq \sup_{t_0 \leq t \leq t_1} \rho(\mathbb{I}_t; \mathfrak{h}) \int_{t_0}^{t_1} \|\dot{\eta}_0\|_{\mathfrak{g}}^2\, dt = 2E_{\eta_0} \sup_{t_0 \leq t \leq t_1} \rho(\mathbb{I}_t; \mathfrak{h}),$$

where $E_{\eta_0} = \frac{1}{2}\int_{t_0}^{t_1} \|\dot{\eta}_0\|_{\mathfrak{g}}^2\, dt$ is the energy of the curve η_0. As a result, (3.23) implies

$$\nu_0(\zeta_0) S_{\eta_0} \leq E_{\eta_0} \sup_{t_0 \leq t \leq t_1} \rho(\mathbb{I}_t; \mathfrak{h}). \tag{3.26}$$

By reversing the parameterization of η_0 in (3.26), the sign of the left hand side is flipped, while the right hand side remains unchanged. Therefore,

$$|\nu_0(\zeta_0)|\, |S_{\eta_0}| \leq E_{\eta_0} \sup_{t_0 \leq t \leq t_1} \rho(\mathbb{I}_t; \mathfrak{h}). \tag{3.27}$$

Since (3.27) holds for all η_0 with $\eta_0(t_0) = \eta_0(t_1) = 0$, and $\sup_{t_0 \leq t \leq t_1} \rho(\mathbb{I}_t; \mathfrak{h})$ is independent of the choice of η_0, we have

$$|\nu_0(\zeta_0)| \leq \sup_{t_0 \leq t \leq t_1} \rho(\mathbb{I}_t; \mathfrak{h}) \inf_{\eta_0} \frac{E_{\eta_0}}{|S_{\eta_0}|}, \tag{3.28}$$

where the infimum is taken over all closed curves η_0 in $\mathfrak{h}$ with $\eta_0(t_0) = \eta_0(t_1) = 0$ and $S_{\eta_0} \neq 0$.

Denote by $l_{\eta_0} = \int_{t_0}^{t_1} \|\dot{\eta}_0\|_{\mathfrak{g}}\, dt$ the arc length of η_0. It is well known [24] that

$$E_{\eta_0} \geq \frac{l_{\eta_0}^2}{2(t_1 - t_0)},$$

with equality if and only if η_0 has constant speed. Since $|S_{\eta_0}|$ is independent of the parameterizations of η_0, we can always choose η_0 with constant speed, which implies that (3.28) is equivalent to

$$|\nu_0(\zeta_0)| \leq \frac{1}{2(t_1 - t_0)} \sup_{t_0 \leq t \leq t_1} \rho(\mathbb{I}_t; \mathfrak{h}) \inf_{\eta_0} \frac{l_{\eta_0}^2}{|S_{\eta_0}|}. \tag{3.29}$$

From an ancient theorem stated below without proof (Theorem 3), $l_{\eta_0}^2/|S_{\eta_0}|$ achieves its infimum when η_0 draws a circle in $\mathfrak{h}$ of arbitrary radius through the origin, and the infimum is 4π. So (3.29) can be written as

$$|\nu_0(\zeta_0)| \leq \frac{2\pi}{t_1 - t_0} \sup_{t_0 \leq t \leq t_1} \rho(\mathbb{I}_t; \mathfrak{h}).$$

Theorem 3 (Isoperimetric Problem, [11]) *Using a string of fixed length, one can encircle the maximal area by arranging the string into a circle. Or equivalently, among all the closed curves that enclose a fixed area, the one with the shortest length is a circle.*

Recalling the expression for ζ_0 in (3.25), we have

Corollary 1 *Suppose that $\gamma = \langle \gamma_i \rangle_{i=1}^k$ is an optimal solution to the OCA (or OFS) problem, and ν_0 is defined as in Theorem 1. Let $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ be an arbitrary inner product on $\mathfrak{g}$. Then*

$$|\nu_0([\zeta_1, \zeta_2])| \leq \frac{2\pi}{t_1 - t_0} \sup_{t_0 \leq t \leq t_1} \rho(\mathbb{I}_t; \mathfrak{h}), \tag{3.30}$$

for any orthonormal pair $\zeta_1, \zeta_2 \in \mathfrak{g}$. Here $\mathfrak{h} = span\{\zeta_1, \zeta_2\}$, and $\rho(\mathbb{I}_t; \mathfrak{h})$ is defined in (3.24).

Remark 7 *The choice of the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ affects both the choices of ζ_1, ζ_2 and the values of $\rho(\mathbb{I}_t, \mathfrak{h})$, so in this sense the conclusion of Corollary 1 is not intrinsic.*

In certain cases, Corollary 1 takes an especially simple form. Suppose for instance that $L = \frac{1}{2}\|\cdot\|^2$ and that the inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ is chosen such that

$$\langle \zeta_1 q, \zeta_2 q\rangle = \langle \zeta_1, \zeta_2\rangle_{\mathfrak{g}}, \quad \forall \zeta_1, \zeta_2 \in \mathfrak{g}, q \in M. \tag{3.31}$$

Then at each t, $\mathbb{I}_t(\zeta_1,\zeta_2) = \sum_{i=1}^k \lambda_i \langle \zeta_1\gamma_i, \zeta_2\gamma_i\rangle = (\sum_{i=1}^k \lambda_i)\langle \zeta_1,\zeta_2\rangle_{\mathfrak{g}}$, $\forall \zeta_1,\zeta_2 \in \mathfrak{g}$, which implies that

$$\rho(\mathbb{I}_t;\mathfrak{h}) = \rho(\mathbb{I}_t) = \sum_{i=1}^k \lambda_i$$

for any t and any two dimensional subspace $\mathfrak{h}$ of $\mathfrak{g}$. Therefore,

Corollary 2 *If in addition to the hypotheses of Corollary 1, we have $L = \frac{1}{2}\|\cdot\|^2$, and $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ satisfying condition (3.31), then for any orthonormal pair $\zeta_1,\zeta_2 \in \mathfrak{g}$,*

$$|\nu_0([\zeta_1,\zeta_2])| \le \frac{2\pi \sum_{i=1}^k \lambda_i}{t_1 - t_0}. \tag{3.32}$$

Example 9 (Lie Group with a Bi-Invariant Metric) Consider a Lie group G with a bi-invariant metric $\langle\cdot,\cdot\rangle$, and $L = \frac{1}{2}\|\cdot\|^2$. Choose the metric $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ to be the restriction of $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$. This metric obviously satisfies condition (3.31). Hence Corollary 2 holds, where $\nu_0 \in \mathfrak{g}$ is given by (3.15). In particular, let $G = \mathbf{SO}_3$ be equipped with the bi-invariant metric described in Example 6. Due to the well known fact that $\mathfrak{so}_3$ is isomorphic to $(\mathbb{R}^3,\times)$ as a Lie algebra ([10]), the set of $[\zeta_1,\zeta_2]$ for orthonormal pairs $\zeta_1,\zeta_2 \in \mathfrak{so}_3$ is the unit sphere in $\mathfrak{so}_3$, hence (3.32) implies

$$\|\nu_0\|_{\mathfrak{so}_3} \le \frac{2\pi \sum_{i=1}^k \lambda_i}{t_1 - t_0}. \tag{3.33}$$

If $k = 1$, $\lambda_1 = 1$, then solutions γ to the OCA problem are geodesics in $\mathbf{SO}_3$. Consider the example $\gamma(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\forall t \in [t_0,t_1]$. Then $\nu_0 = \dot\gamma\gamma^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus (3.33) becomes $t_1 - t_0 \le 2\pi$, which indeed characterizes the first conjugate point along the geodesic γ in $\mathbf{SO}_3$, as is pointed out in Remark 6. □

Example 10 (Grassmann Manifold) We continue the discussion in Example 7, where $G = \mathbf{SO}_n$, $M = G_{n,p}$, and $L = \frac{1}{2}\|\cdot\|^2$. Suppose that $\gamma = \langle\gamma_i\rangle_{i=1}^k$ is an optimal solution to the OCA (or OFS) problem defined on $[t_0,t_1]$, and that $\langle A_i\rangle_{i=1}^k$ is a lifting of γ in $\mathbf{SO}_n$. At each time t, we have

$$\begin{aligned}\mathbb{I}_t(X,X) &= \sum_{i=1}^k \lambda_i \langle d\Phi^{\gamma_i}(X), d\Phi^{\gamma_i}(X)\rangle_{T_{\gamma_i}G_{n,p}} = \sum_{i=1}^k \lambda_i \langle P_{A_i}(XA_i), P_{A_i}(XA_i)\rangle_{T_{A_i}\mathbf{SO}_n} \\ &\le \sum_{i=1}^k \lambda_i \langle XA_i, XA_i\rangle_{T_{A_i}\mathbf{SO}_n} = \sum_{i=1}^k \lambda_i \|X\|^2_{\mathfrak{so}_n}, \quad \forall X \in \mathfrak{so}_n.\end{aligned} \tag{3.34}$$

where equality holds if and only if $XA_i \in \mathrm{hor}_{A_i}\mathbf{SO}_n$ for each i. Hence $\rho(\mathbb{I}_t) \le \sum_{i=1}^k \lambda_i$, and Corollary 1 implies that

$$|\nu_0([X_1,X_2])| \le \frac{2\pi \sum_{i=1}^k \lambda_i}{t_1 - t_0} \tag{3.35}$$

for all orthonormal pairs $X_1, X_2 \in \mathfrak{so}_n$, where ν_0 is defined in (3.17). In particular, if $n = 3$, then the set of possible $[X_1,X_2]$ is the unit sphere in $\mathfrak{so}_3$. Hence (3.35) reduces to

$$\|\nu_0\|_{\mathfrak{so}_3} \le \frac{2\pi \sum_{i=1}^k \lambda_i}{t_1 - t_0}.$$

□

Example 11 (Stiefel Manifold) For the Stiefel manifold $V_{n,p}$ studied in Example 8, a similar argument shows that (3.35) still holds, with ν_0 now defined in (3.18). □

3.3. A Topological Optimality Condition. In this section we focus on OCA and OFS problems on a Riemannian manifold M satisfying Assumption 1, with the Lagrangian function given by $L = \frac{1}{2}\|\cdot\|^2$. Let $\gamma = \langle\gamma_i\rangle_{i=1}^k$ be an optimal solution to the OCA (or OFS) problem defined on $[t_0, t_1]$. Based on the first variational analysis, we have proved in Theorem 1 that the quantity ν_0 is conserved along γ. In this section, we derive additional optimality conditions based on topological properties of M. Roughly speaking, we shall prove in Theorem 4 below that for every possible way of embedding a circle in G, ν_0 is bounded when evaluated along the corresponding direction in $\mathfrak{g}$, for otherwise one can get a better solution generated by "going the other way" around the circle.

Consider $\mathbf{T}^1 = \mathbb{R}/2\pi\mathbb{Z} = \{\theta \mod 2\pi : \theta \in \mathbb{R}\}$ with the quotient metric of $\mathbb{R}$. $\mathbf{T}^1$ is a Lie group under addition modulo 2π, and its Lie algebra is isomorphic to $\mathbb{R}$ under the correspondence $\lambda\frac{\partial}{\partial\theta} \in T_0\mathbf{T}^1 \mapsto \lambda \in \mathbb{R}$. Hence we shall denote it by $\mathbb{R}$. Suppose that there exists a Lie group homomorphism $\varphi : \mathbf{T}^1 \to G$. Then $d\varphi : \mathbb{R} \to \mathfrak{g}$ is a Lie algebra homomorphism.

Let $h_0 : [t_0, t_1] \to \mathbf{T}^1$ be a continuous and piecewise C^∞ curve in $\mathbf{T}^1$ starting from and ending at 0. Then $\varphi(h_0)$ is a curve in G that starts from and ends at e, and hence the k-tuple of curves in M define by $\varphi(h_0)\gamma = \langle\varphi(h_0)\gamma_i\rangle_{i=1}^k$ has the same starting and destination positions as γ. Since φ is a homomorphism, the following diagram commutes at each $t \in [t_0, t_1]$:

$$\begin{array}{ccc} T_0\mathbf{T}^1 & \xrightarrow{d\varphi} & T_eG \\ {\scriptstyle dm_{h_0^{-1}}}\uparrow & & \uparrow{\scriptstyle dm_{\varphi(h_0^{-1})}} \\ T_{h_0}\mathbf{T}^1 & \xrightarrow{d\varphi} & T_{\varphi(h_0)}G \end{array}$$

where we recall that m_g stands for left group multiplication by $g \in G$. We then have

$$\frac{d}{dt}[\varphi(h_0)\gamma_i] = d\varphi(\dot{h}_0)\gamma_i + \varphi(h_0)\dot{\gamma}_i = \varphi(h_0)[\varphi(h_0^{-1})d\varphi(\dot{h}_0)\gamma_i + \dot{\gamma}_i] = \varphi(h_0)[d\varphi(h_0^{-1}\dot{h}_0)\gamma_i + \dot{\gamma}_i].$$

Therefore, the cost of $\varphi(h_0)\gamma$ is

$$J[\varphi(h_0)\gamma] = \sum_{i=1}^k \lambda_i \int_{t_0}^{t_1} \frac{1}{2}\|\varphi(h_0)[d\varphi(\xi_0)\gamma_i + \dot{\gamma}_i]\|^2\, dt, \tag{3.36}$$

where we set

$$\xi_0(t) = h_0(t)^{-1}\dot{h}_0(t), \quad \forall t \in [t_0, t_1].$$

Note that ξ_0 is a piecewise C^∞ curve in the Lie algebra $\mathbb{R}$ such that $\int_{t_0}^{t_1} \xi_0(t)dt = 2m\pi$ for some $m \in \mathbb{Z}$. Conversely, every curve ξ_0 in $\mathbb{R}$ satisfying $\frac{1}{2\pi}\int_{t_0}^{t_1} \xi_0(t)dt \in \mathbb{Z}$ can be realized as $h_0^{-1}\dot{h}_0$ for some curve h_0 in $\mathbf{T}^1$ that starts from and ends at 0.

Since $\varphi(h_0)$ is an isometry on M, equation (3.36) can be further simplified to

$$\begin{aligned} J[\varphi(h_0)\gamma] &= \sum_{i=1}^k \lambda_i \int_{t_0}^{t_1} \frac{1}{2}\|d\varphi(\xi_0)\gamma_i + \dot{\gamma}_i\|^2\, dt \\ &= \int_{t_0}^{t_1} \left[\frac{1}{2}\sum_{i=1}^k \lambda_i\|\dot{\gamma}_i\|^2 + \frac{1}{2}\sum_{i=1}^k \lambda_i\|d\varphi(\xi_0)\gamma_i\|^2 + \sum_{i=1}^k \lambda_i((d\Phi^{\gamma_i})^*\dot{\gamma}_i, d\varphi(\xi_0))\right] dt \\ &= J(\gamma) + \int_{t_0}^{t_1} \frac{1}{2}\{\mathbb{I}_t[d\varphi(\xi_0), d\varphi(\xi_0)] + (\nu_0, d\varphi(\xi_0))\}\, dt, \end{aligned}$$

where in the last step we have used Theorem 1, and the definition (3.22) of $\mathbb{I}_t$. Denote by $\varphi^*\mathbb{I}_t$ the pull back of $\mathbb{I}_t$ via φ defined by $\varphi^*\mathbb{I}_t(x_1, x_2) \triangleq \mathbb{I}_t[d\varphi(x_1), d\varphi(x_2)]$, $\forall x_1, x_2 \in \mathbb{R}$. Then $\varphi^*\mathbb{I}_t : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a quadratic function, and is obviously of the form

$$\varphi^*\mathbb{I}_t(x_1, x_2) = \|\varphi^*\mathbb{I}_t\|x_1x_2, \quad \forall x_1, x_2 \in \mathbb{R},$$

where $\|\varphi^*\mathbb{I}_t\| \geq 0$ is the spectral radius of $\varphi^*\mathbb{I}_t$ given by

$$\|\varphi^*\mathbb{I}_t\| \triangleq \varphi^*\mathbb{I}_t(1,1) = \mathbb{I}_t[d\varphi(1), d\varphi(1)]. \tag{3.37}$$

Similarly, denote by $\varphi^*\nu_0 \in \mathbb{R}^* \simeq \mathbb{R}$ the pull back of ν_0 via φ such that $(\nu_0, d\varphi(x)) = (\varphi^*\nu_0)x$, $\forall x \in \mathbb{R}$. From the above equations, the difference between the cost of $\varphi(h_0)\gamma$ and γ is given by

$$\Delta J(\xi_0) \triangleq J[\varphi(h_0)\gamma] - J(\gamma) = \int_{t_0}^{t_1} \left[\frac{1}{2}\|\varphi^*\mathbb{I}_t\|\xi_0^2 + (\varphi^*\nu_0)\xi_0\right] dt. \tag{3.38}$$

A necessary condition for γ to be optimal is that $\Delta J(\xi_0) \geq 0$ for all possible ξ_0. By (3.38), this implies

$$\int_{t_0}^{t_1} \left[\frac{1}{2}\|\varphi^*\mathbb{I}_t\|\xi_0^2 + (\varphi^*\nu_0)\xi_0\right] dt \geq 0, \tag{3.39}$$

for all curves ξ_0 in $\mathbb{R}$ such that $\frac{1}{2\pi}\int_{t_0}^{t_1} \xi_0(t)dt \in \mathbb{Z}$.

Fix an $m \in \mathbb{Z}$. To find the ξ_0 that minimizes $\Delta J(\xi_0)$ subject to the constraint that $\int_{t_0}^{t_1} \xi_0(t)dt = 2m\pi$, we use the Lagrangian multiplier approach. Assume that $\|\varphi^*\mathbb{I}_t\| > 0$ for almost all $t \in [t_0, t_1]$. Define

$$\mathcal{L}(\xi_0, \lambda) \triangleq \Delta J(\xi_0) + \lambda(\int_{t_0}^{t_1} \xi_0 dt - 2m\pi) = \int_{t_0}^{t_1} \left[\frac{1}{2}\|\varphi^*\mathbb{I}_t\|\xi_0^2 + (\lambda + \varphi^*\nu_0)\xi_0\right] dt - 2\lambda m\pi$$

for $\lambda \in \mathbb{R}$. Note that $\mathcal{L}(\xi_0, \lambda)$ is minimized when $\xi_0 = -(\lambda + \varphi^*\nu_0)/\|\varphi^*\mathbb{I}_t\|$. The constraint that $\int_{t_0}^{t_1} \xi_0(t)dt = 2m\pi$ implies that $\lambda + \varphi^*\nu_0 = -2m\pi/[\int_{t_0}^{t_1} \frac{1}{\|\varphi^*\mathbb{I}_t\|} dt]^{-1}$. Hence

$$\xi_0 = \frac{2m\pi}{\|\varphi^*\mathbb{I}_t\| \int_{t_0}^{t_1} \frac{1}{\|\varphi^*\mathbb{I}_t\|} dt} \tag{3.40}$$

minimizes $\Delta J(\xi_0)$ among ξ_0 such that $\int_{t_0}^{t_1} \xi_0(t)dt = 2m\pi$. Substituting (3.40) into (3.39), we have

$$m^2\pi + m(\varphi^*\nu_0)\int_{t_0}^{t_1} \frac{1}{\|\varphi^*\mathbb{I}_t\|} dt \geq 0.$$

Note that the above inequality must hold for all $m \in \mathbb{Z}$, which implies

$$|\varphi^*\nu_0| \leq \pi\left[\int_{t_0}^{t_1} \frac{1}{\|\varphi^*\mathbb{I}_t\|} dt\right]^{-1}.$$

Therefore, we have

Theorem 4 *Suppose that $\gamma = \langle\gamma_i\rangle_{i=1}^k$ is a solution to the OCA (or OFS) problem with $L = \frac{1}{2}\|\cdot\|^2$. Let ν_0 and $\mathbb{I}_t$ be defined in (3.11) and (3.22) respectively. Then, for any Lie group homomorphism $\varphi : \mathbf{T}^1 \to G$ such that $\|\varphi^*\mathbb{I}_t\| > 0$ for almost all $t \in [t_0, t_1]$,*

$$|\varphi^*\nu_0| \leq \pi\left[\int_{t_0}^{t_1} \frac{1}{\|\varphi^*\mathbb{I}_t\|} dt\right]^{-1}. \tag{3.41}$$

Example 12 ($G = M = \mathbf{T}^n$) Let $G = \mathbf{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the flat n-torus with the metric inherited from $\mathbb{R}^n$. $\mathbf{T}^n$ is a Lie group under componentwise modulo $\mathbb{Z}$ addition, and its metric is bi-invariant. Its Lie algebra is $\mathbb{R}^n$ with trivial Lie bracket, and is equipped with the standard metric. Let $\gamma = \langle\gamma_i\rangle_{i=1}^k$ be a solution to the OCA (or OFS) problem. In Example 6 we have shown that

$$\sum_{i=1}^k \lambda_i\dot{\gamma}_i\gamma_i^{-1} = \sum_{i=1}^k \lambda_i\dot{\gamma}_i \equiv \nu_0 \in \mathbb{R}^n.$$

Pick any $z \in \mathbb{Z}^n$, $z \neq 0$. The map $\varphi(\theta \mod 2\pi) = \frac{\theta}{2\pi} z \mod \mathbb{Z}^n$, $\forall \theta \in \mathbb{R}$, is a homomorphism from $\mathbf{T}^1$ to $\mathbf{T}^n$ with $d\varphi(1) = z/2\pi$. So $\varphi^* \nu_0 = \langle \nu_0, d\varphi(1) \rangle = \langle \nu_0, z \rangle / 2\pi$, and $\|\varphi^* \mathbb{I}_t\| = \mathbb{I}_t[d\varphi(1), d\varphi(1)] = \sum_{i=1}^k \lambda_i \|z/2\pi\|^2$. As a result, Theorem 4 implies that

$$|\langle \nu_0, \frac{z}{\|z\|^2} \rangle| \leq \frac{\sum_{i=1}^k \lambda_i}{2(t_1 - t_0)}, \quad \text{for all } z \in \mathbb{Z}^n,\ z \neq 0. \tag{3.42}$$

In particular, if $\nu_0 = (\nu_{0,1}, \ldots, \nu_{0,n})$ in coordinates, and $z = e_j$ is the element in $\mathbb{R}^n$ with the j-th coordinate 1 and the rest 0, then a necessary condition of (3.42) is

$$|\nu_{0,j}| \leq \frac{\sum_{i=1}^k \lambda_i}{2(t_1 - t_0)}, \quad j = 1, \ldots, n. \tag{3.43}$$

It can be verified that (3.43) is also sufficient for (3.42). □

Example 13 ($G = \mathbf{SO}_2$, $M = \mathbb{R}^2$) Suppose $G = \mathbf{SO}_2$, $M = \mathbb{R}^2$ with the standard metric, and G acts on M by matrix multiplication. As before, choose the metric on $\mathfrak{so}_2$ to be $\frac{1}{2}\langle \cdot, \cdot \rangle_F$. By following the same arguments as in Example 4, we conclude that for any solution $\gamma = \langle \gamma_i \rangle_{i=1}^k$ to the OCA (or OFS) problem,

$$\sum_{i=1}^k \lambda_i (\dot{\gamma}_i \gamma_i^T - \gamma_i \dot{\gamma}_i^T) \equiv \nu_0 \in \mathfrak{so}_2.$$

Note that $\mathbf{SO}_2 \simeq \mathbf{T}^1$ under the isomorphism $\varphi(\theta \mod 2\pi) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $\forall \theta \in \mathbb{R}$, and $d\varphi(1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Hence $\varphi^* \nu_0 = \langle \nu_0, d\varphi(1) \rangle_{\mathfrak{so}_2} = \sum_{i=1}^k \lambda_i (\gamma_{i,1} \dot{\gamma}_{i,2} - \gamma_{i,2} \dot{\gamma}_{i,1})$ if we write each γ_i in coordinates, and $\|\varphi^* \mathbb{I}_t\| = \sum_{i=1}^k \lambda_i \|d\varphi(1)\gamma_i\|^2 = \sum_{i=1}^k \lambda_i \|\gamma_i\|^2$. Therefore, by Theorem 4,

$$\left| \sum_{i=1}^k \lambda_i (\gamma_{i,1} \dot{\gamma}_{i,2} - \gamma_{i,2} \dot{\gamma}_{i,1}) \right| \leq \pi \left[\int_{t_0}^{t_1} \frac{dt}{\sum_{i=1}^k \lambda_i \|\gamma_i\|^2} \right]^{-1}. \tag{3.44}$$

In [14], equation (3.44) is derived through an elementary approach. For a concrete example, consider the case of two agents ($k = 2$) with equal priorities ($\lambda_1 = \lambda_2 = 1$), and $r = 2$. Let $t_0 = 0$, $t_1 = T$, and $a_1 = -a_2 = (1, 0)$, $b_1 = -b_2 = (\cos T, \sin T)$. Then (3.44) implies that $\gamma = \langle \gamma_i \rangle_{i=1}^2$ with $\gamma_1(t) = -\gamma_2(t) = (\cos t, \sin t)$, $\forall t \in [0, T]$, is not an optimal solution for the OCA problem if $T > \pi$, which is obvious since the two agents could rotate at constant speed in reversed direction around the origin with a smaller cost. □

Example 14 ($G = M = \mathbf{SO}_n$) Let $G = \mathbf{SO}_n$ be equipped with the bi-invariant metric defined in Example 6. So the conserved quantity is $\nu_0 = \sum_{i=1}^k \lambda_i \dot{\gamma}_i \gamma_i^{-1} \in \mathfrak{so}_n$ along a solution $\gamma = \langle \gamma_i \rangle_{i=1}^k$ to the OCA (or OFS) problem. Let $X \in \mathfrak{so}_n$ be such that $e^{2\pi X} = I_n$, $X \neq 0$. Define a Lie group homomorphism $\varphi_X : \mathbf{T}^1 \to \mathbf{SO}_n$ by $\varphi_X(\theta \mod 2\pi) = e^{\theta X}$, $\forall \theta \in \mathbb{R}$. Then $d\varphi_X(1) = X$, $\varphi_X^* \nu_0 = \langle \nu_0, X \rangle_{\mathfrak{so}_n}$, and $\|\varphi_X^* \mathbb{I}_t\| = \sum_{i=1}^k \lambda_i \|X\|_{\mathfrak{so}_n}^2$. Theorem 4 thus implies that

$$|\langle \nu_0, X \rangle_{\mathfrak{so}_n}| \leq \pi \sum_{i=1}^k \lambda_i \|X\|_{\mathfrak{so}_n}^2 / (t_1 - t_0)$$

for all $X \in \mathfrak{so}_n$ such that $e^{2\pi X} = I_n$. By Lemma 9 in Appendix B, this is equivalent to

$$\|\nu_0\|_2 \leq \frac{\pi}{t_1 - t_0} \sum_{i=1}^k \lambda_i, \tag{3.45}$$

where $\|\nu_0\|_2$ is the L^2 norm of the matrix ν_0. Or equivalently,

$$\text{the maximum of the singular values of } \nu_0 \leq \frac{\pi}{t_1 - t_0} \sum_{i=1}^k \lambda_i.$$

□

Example 15 (Grassmann Manifold) Let $G = \mathbf{SO}_n$, $M = G_{n,p}$ be as in Example 7. So for any solution $\gamma = \langle \gamma_i \rangle_{i=1}^k$ to the OCA (or OFS) problem in $G_{n,p}$ and its lifting $\langle A_i \rangle_{i=1}^k$ in $\mathbf{SO}_n$, $\sum_{i=1}^k \lambda_i P_{A_i}(\dot{A}_i) A_i^T \equiv \nu_0 \in \mathfrak{so}_n$. Let $X \in \mathfrak{so}_n$ be such that $e^{2\pi X} = I_n$, $X \neq 0$. Define the homomorphism $\varphi_X : \mathbf{T}^1 \to \mathbf{SO}_n$ as in Example 14. Then $\varphi_X^* \nu_0 = \langle \nu_0, X \rangle_{\mathfrak{so}_n}$, and by (3.34), $\|\varphi_X^* \mathbb{I}_t\| = \mathbb{I}_t(X, X) \leq \sum_{i=1}^k \lambda_i \|X\|_{\mathfrak{so}_n}^2$. Theorem 4 then implies that

$$|\langle \nu_0, X \rangle_{\mathfrak{so}_n}| \leq \pi \sum_{i=1}^k \lambda_i \|X\|_{\mathfrak{so}_n}^2 / (t_1 - t_0), \tag{3.46}$$

for all $X \in \mathfrak{so}_n$ such that $e^{2\pi X} = I_n$. Therefore, by Lemma 9 in Appendix B, bound (3.45) still holds. However, it is possible to improve this bound by considering an additional symmetry of $G_{n,p}$. Suppose $X \in \mathfrak{so}_n$ is chosen such that $e^{\pi X} = -I_n$ (such X exists only if n is even). Consider $\{\pm I_n\}$, a discrete subgroup of $\mathbf{SO}_n$. The action of each of $\{\pm I_n\}$ on $G_{n,p}$ is the identity map, so Φ induces naturally an action of the quotient group $\mathbf{SO}_n / \{\pm I_n\}$ on $G_{n,p}$, which also satisfies Assumption 1 in Section 2. Since $\{\pm I_n\}$ is discrete, the Lie algebra of $\mathbf{SO}_n / \{\pm I_n\}$ is $\mathfrak{so}_n$, and the conserved quantity ν_0 and the map $\mathbb{I}_t$ remain the same for this induced action. Now the map $\varphi(\theta \mod 2\pi) = e^{\theta X/2}$, $\forall \theta \in \mathbb{R}$, is a homomorphism from T^1 to $\mathbf{SO}_n / \{\pm I_n\}$ with $d\varphi(1) = X/2$. So $\varphi^* \nu_0 = \langle \nu_0, X/2 \rangle_{\mathfrak{so}_n}$, and $\|\varphi^* \mathbb{I}_t\| \leq \sum_{i=1}^k \lambda_i \|X/2\|_{\mathfrak{so}_n}^2$. Applying Theorem 4, we have

$$|\langle \nu_0, X \rangle_{\mathfrak{so}_n}| \leq \frac{\pi}{2} \sum_{i=1}^k \lambda_i \|X\|_{\mathfrak{so}_n}^2 / (t_1 - t_0)$$

for all $X \in \mathfrak{so}_n$ such that $e^{\pi X} = -I_n$. By Lemma 10 in Appendix B, this implies that

$$\text{the average of the singular values of } \nu_0 \leq \frac{\pi}{2(t_1 - t_0)} \sum_{i=1}^k \lambda_i.$$

□

Example 16 (Stiefel Manifold) Consider $G = \mathbf{SO}_n$ and $M = V_{n,p}$ defined in Example 8. Let $\gamma = \langle \gamma_i \rangle_{i=1}^k$ be a solution to the OCA (or OFS) problem on $V_{n,p}$. Then, in Example 8 it is shown that $\sum_{i=1}^k \lambda_i \hat{P}_{A_i}(\dot{A}_i) A_i^T \equiv \nu_0 \in \mathfrak{so}_n$, where $\langle A_i \rangle_{i=1}^k$ is a lifting of γ_i in $\mathbf{SO}_n$. By following the same steps as in the previous example, we conclude that (3.46), hence (3.45), holds. □

4. An Example. The necessary conditions derived in Section 3 apply to solutions of both the OCA problem and the OFS problem under any choice of $\tilde{\mathcal{F}}$. The price for this general applicability, however, is that these conditions are generally far from being sufficient, especially when the number of agents is large. This fact is illustrated here by a simple example, which is also of its own interest.

Consider the OCA problem with $M = \mathbb{R}^2$, $\lambda_1 = \ldots = \lambda_k = 1$, $r = 1$, $t_0 = 0$, and $t_1 = T > 0$. Suppose that the starting position $\langle a_i \rangle_{i=1}^k$ is given by $\langle (\frac{2i-k-1}{2}, 0) \rangle_{i=1}^k$. In other words, at time $t = 0$ the k agents are aligned on the x-axis with a minimal allowed separation between consecutive ones and with their centroid at the origin. For each $t \geq 0$, denote by $R_t : \mathbb{R}^2 \to \mathbb{R}^2$ the rotation of $\mathbb{R}^2$ by an angle t counterclockwise. Suppose that the destination position is $\langle b_i \rangle_{i=1}^k = \langle R_T(a_i) \rangle_{i=1}^k$. By Theorem 1, any optimal solution $\gamma = \langle \gamma_i \rangle_{i=1}^k$ to the OCA problem must satisfy that $\sum_{i=1}^k \dot{\gamma}_i$ is constant, which after integration implies that $\sum_{i=1}^k \gamma_i \equiv 0$ since the centroid of both the starting and destination positions is at the origin, i.e., $\sum_{i=1}^k a_i = \sum_{i=1}^k b_i = 0$. Therefore, γ as a curve in $M^{(k)} = \mathbb{R}^{2k}$ lies in the subspace $V = \{(q_1, \ldots, q_k) \in \mathbb{R}^{2k} : \sum_{i=1}^k q_i = 0\}$. Taking into consideration the separation constraints, γ is contained in $(\mathbb{R}^{2k} \setminus W) \cap V$, where W is defined in (2.3). Furthermore, by the remarks in Section 2, γ is a distance-minimizing geodesic in $(\mathbb{R}^{2k} \setminus W) \cap V$ connecting $a = (a_1, \ldots, a_k)$ to $b = (b_1, \ldots, b_k)$.

Now as a candidate for the optimal solution, consider γ given by $\gamma_i(t) = R_t(a_i)$, $\forall t \in [0, T]$, $i = 1, \ldots, k$, i.e., the joint trajectory under which each agent rotates at constant angular velocity 1 around the origin during $[0, T]$. It can be checked that Theorem 4 implies that γ is not optimal if $T > \pi$. However, we shall improve

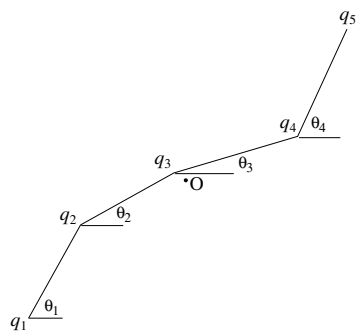

FIG. 4.1. *Coordinates of N when $k = 5$.*

this bound by showing that γ fails to be optimal once $T > \tau_k$, where $\tau_k \to 0$ as $k \to \infty$. In fact, note that γ is contained in a smooth component N of $(\mathbb{R}^{2k} \setminus W) \cap V$ given by

$$\{(q_1, \ldots, q_k) \in \mathbb{R}^{2k} : \|q_i - q_{i+1}\| = 1,\ i = 1, \ldots, k-1,\ \text{and } \|q_i - q_j\| > 1 \text{ for all other } i \neq j\} \cap V.$$

N is a $k-1$ dimensional smooth submanifold of $\mathbb{R}^{2k}$, and admits (global) coordinates $(\theta_1, \ldots, \theta_{k-1})$, where θ_i is the angle $q_{i+1} - q_i \in \mathbb{R}^2$ makes with respect to the positive x-axis (see Figure 4.1). The coordinate map $f : (\theta_1, \ldots, \theta_{k-1}) \to (q_1, \ldots, q_k) \in N$ is defined by

$$q_i = q_1 + \sum_{j=1}^{i-1} \begin{bmatrix} \cos\theta_j \\ \sin\theta_j \end{bmatrix}, \quad i = 2, \ldots, k,$$

where q_1 is the unique element of $\mathbb{R}^2$ such that $\sum_{i=1}^{k} q_i = 0$ for q_i thus defined, namely,

$$q_1 = -\frac{1}{k} \sum_{j=1}^{k-1} (k-j) \begin{bmatrix} \cos\theta_j \\ \sin\theta_j \end{bmatrix}.$$

In these coordinates, γ corresponds to $\theta_i(t) = t$, $t \in [0, T]$, $i = 1, \ldots, k-1$. Using these coordinates, we shall prove in the rest of this section that γ is a geodesic of N and compute the time instance τ_k at which γ encounters its first conjugate point. Since a geodesic is not distance-minimizing beyond its first conjugate point, this will show that γ is no longer an optimal solution of the OCA problem once $T > \tau_k$. It turns out that there is a very nice formula for τ_k.

To see this, note that at any $q \in N$, $\frac{\partial}{\partial\theta_1}, \ldots, \frac{\partial}{\partial\theta_{k-1}}$ form a basis of T_qN. In this basis, the Riemannian metric $\langle \cdot, \cdot \rangle$ of N that N inherits from $\mathbb{R}^{2k}$ as a submanifold can be computed as

$$g_{ij} \triangleq \langle \frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j} \rangle = \langle \frac{\partial f}{\partial\theta_i}, \frac{\partial f}{\partial\theta_j} \rangle_{\mathbb{R}^{2k}} = \Delta_{ij} \cos(\theta_i - \theta_j), \quad 1 \leq i, j \leq k-1,$$

where Δ_{ij} are the components of a matrix $\Delta = (\Delta_{ij})_{1 \leq i,j \leq k-1} \in \mathbb{R}^{(k-1)\times(k-1)}$ defined by

$$\Delta_{ij} = \begin{cases} \frac{i(k-j)}{k} & \text{if } i \leq j, \\ \frac{(k-i)j}{k} & \text{if } i > j. \end{cases} \tag{4.1}$$

Denote by $(g^{ij})_{1 \leq i,j \leq k-1}$ the inverse matrix of $(g_{ij})_{1 \leq i,j \leq k-1}$. The covariant derivative with respect to the Levi-Civita connection on N is given by $\nabla_{\frac{\partial}{\partial\theta_i}} \frac{\partial}{\partial\theta_j} = \sum_{m=1}^{k-1} \Gamma_{ij}^m \frac{\partial}{\partial\theta_m}$, where

$$\Gamma_{ij}^m = \frac{1}{2} \sum_{l=1}^{k-1} \{ \frac{\partial g_{jl}}{\partial\theta_i} + \frac{\partial g_{li}}{\partial\theta_j} - \frac{\partial g_{ij}}{\partial\theta_k} \} g^{lm}, \quad 1 \leq i, j, m \leq k-1,$$

are the Christoffel symbols. From the above definition, one can compute that for $1 \le i, j, m \le k-1$,

$$\Gamma_{ij}^m = \begin{cases} 0 & \text{if } i \neq j, \\ \sum_{l=1}^{k-1} \Delta_{il} \sin(\theta_l - \theta_i) g^{lm} & \text{if } i = j. \end{cases} \tag{4.2}$$

Notice that $\theta_i = \theta_j$ for all i, j along γ. Therefore,

Lemma 3 *Along γ we have $\Gamma_{ij}^m = 0$, for all $1 \le i, j, m \le k-1$, hence $\nabla_{\frac{\partial}{\partial \theta_i}} \frac{\partial}{\partial \theta_j} = 0$, for all $1 \le i, j \le k-1$.*

Since $\dot\gamma = \frac{\partial}{\partial \theta_1} + \cdots + \frac{\partial}{\partial \theta_{k-1}}$, by Lemma 3, $\nabla_{\dot\gamma} \frac{\partial}{\partial \theta_i} = 0$, i.e., $\frac{\partial}{\partial \theta_i}$ is parallel along γ, for each $i = 1, \ldots, k-1$. As a result, one can see that γ is a geodesic in N, since $\nabla_{\dot\gamma}\dot\gamma = 0$. The curvature tensor of N is given by $R(\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j})\frac{\partial}{\partial \theta_l} = \sum_{m=1}^{k-1} R_{ijl}^m \frac{\partial}{\partial \theta_m}$, where ([6])

$$R_{ijl}^m = \sum_{\beta=1}^{k-1} \Gamma_{il}^\beta \Gamma_{j\beta}^m - \sum_{\beta=1}^{k-1} \Gamma_{jl}^\beta \Gamma_{i\beta}^m + \frac{\partial \Gamma_{il}^m}{\partial \theta_j} - \frac{\partial \Gamma_{jl}^m}{\partial \theta_i}, \quad 1 \le i, j, l, m \le k-1.$$

A Jacobi field X along γ satisfies the equation $\nabla_{\dot\gamma}\nabla_{\dot\gamma} X + R(\dot\gamma, X)\dot\gamma = 0$. Write $X = \sum_{i=1}^{k-1} x_i \frac{\partial}{\partial \theta_i}\big|_\gamma$ for some smooth $x_i : [0, T] \to \mathbb{R}$. Then $\nabla_{\dot\gamma}\nabla_{\dot\gamma} X = \sum_{i=1}^{k-1} \ddot x_i \frac{\partial}{\partial \theta_i}\big|_\gamma$, and $R(\dot\gamma, X)\dot\gamma = \sum_{i,j,l,m=1}^{k-1} R_{ijl}^m x_j \frac{\partial}{\partial \theta_m}\big|_\gamma$. So by denoting $x = (x_1, \ldots, x_{k-1}) \in \mathbb{R}^{k-1}$, the Jacobi equation along γ is reduced to

$$\ddot x + B_k x = 0, \tag{4.3}$$

where $B_k = (b_{mj})_{1 \le m, j \le k-1} \in \mathbb{R}^{(k-1)\times(k-1)}$ is defined by $b_{mj} = \sum_{i,l=1}^{k-1} R_{ijl}^m$. Note that here and in the rest of this section, all terms (such as R_{ijl}^m, the Christoffel symbols and their derivatives) in the equations are evaluated along γ. Then, by (4.2) and Lemma 3,

$$b_{mj} = \sum_{i,l} R_{ijl}^m = \sum_{i,l} \frac{\partial \Gamma_{il}^m}{\partial \theta_j} - \sum_{i,l} \frac{\partial \Gamma_{jl}^m}{\partial \theta_i} = \sum_i \frac{\partial \Gamma_{ii}^m}{\partial \theta_j} - \sum_i \frac{\partial \Gamma_{jj}^m}{\partial \theta_i}, \tag{4.4}$$

where the summations are all from 1 to $k-1$. The first term can be simplified to

$$\begin{aligned} \sum_i \frac{\partial \Gamma_{ii}^m}{\partial \theta_j} &= \sum_{i \neq j} \frac{\partial \Gamma_{ii}^m}{\partial \theta_j} + \frac{\partial \Gamma_{jj}^m}{\partial \theta_j} \\ &= \sum_{i \neq j} \frac{\partial}{\partial \theta_j}\left[\sum_l \Delta_{il} \sin(\theta_l - \theta_i) g^{lm}\right] + \frac{\partial}{\partial \theta_j}\left[\sum_{l \neq j} \Delta_{jl} \sin(\theta_l - \theta_j) g^{lm}\right] \\ &= \sum_{i \neq j} \Delta_{ij} g^{jm} - \sum_{l \neq j} \Delta_{jl} g^{lm}, \end{aligned}$$

where we have used the fact that $\theta_1 = \ldots = \theta_{k-1}$ on γ. Similarly,

$$\begin{aligned} \sum_i \frac{\partial \Gamma_{jj}^m}{\partial \theta_i} &= \sum_{i \neq j} \frac{\partial}{\partial \theta_i}\left[\sum_l \Delta_{jl} \sin(\theta_l - \theta_j) g^{lm}\right] + \frac{\partial}{\partial \theta_j}\left[\sum_{l \neq j} \Delta_{jl} \sin(\theta_l - \theta_j) g^{lm}\right] \\ &= \sum_{i \neq j} \Delta_{ji} g^{im} - \sum_{l \neq j} \Delta_{jl} g^{lm}. \end{aligned}$$

Hence (4.4) can be rewritten as

$$b_{mj} = \sum_{i \neq j} \Delta_{ij} g^{jm} - \sum_{i \neq j} \Delta_{ji} g^{im} = (\sum_i \Delta_{ij}) g^{jm} - \sum_i \Delta_{ji} g^{im}.$$

Since $g_{ji} = \Delta_{ji}$ on γ, $\sum_i \Delta_{ji} g^{im} = \sum_i g_{ji} g^{im} = \delta_{mj}$ by the definition of g^{im}. Moreover, $\sum_i \Delta_{ij} = \sum_{i \le j} \frac{i(k-j)}{2} + \sum_{i>j} \frac{(k-i)j}{2} = \frac{j(k-j)}{2}$. Therefore,

$$b_{mj} = \frac{j(k-j)}{2} g^{jm} - \delta_{mj} = \frac{j(k-j)}{2} g^{mj} - \delta_{mj}, \tag{4.5}$$

by the symmetry of g^{mj}. Note that $(g^{mj})_{1 \le m,j \le k-1} = [(g_{mj})_{1 \le m,j \le k-1}]^{-1} = \Delta^{-1}$ on γ. So by (4.5),

Lemma 4 *$B_k = \Delta^{-1}\Lambda - I_k$, where $\Lambda = diag(\frac{k-1}{2}, \ldots, \frac{i(k-i)}{2}, \ldots, \frac{k-1}{2})_{1 \le i \le k-1} \in \mathbb{R}^{(k-1)\times(k-1)}$.*

Remark 8 *B_k is a constant matrix independent of t since the metric of N is homogeneous along γ, or more precisely, for each $\tau > 0$, the map $(q_1, \ldots, q_k) \in N \mapsto (R_\tau(q_1), \ldots, R_\tau(q_k)) \in N$ is an isometry of N mapping $\gamma(t)$ to $\gamma(t+\tau)$ whose differential map takes $\frac{\partial}{\partial \theta_i}\big|_{\gamma(t)}$ to $\frac{\partial}{\partial \theta_i}\big|_{\gamma(t+\tau)}$ for each $i = 1, \ldots, k-1$, provided that both t and $t+\tau$ belong to $[0, T]$.*

To compute the eigenvalues of B_k, some preliminary results are needed. For each $l = 1, \ldots, k-1$, define $u_l = (1, \ldots, i^{l-1}, \ldots, (k-1)^{l-1}) \in \mathbb{R}^{k-1}$. Then $U \triangleq [u_1 | \ldots | u_{k-1}] \in \mathbb{R}^{(k-1)\times(k-1)}$ is a Vandermonde matrix, hence nonsingular. The following two lemmas can be verified directly.

Lemma 5 *Let $\Delta \in \mathbb{R}^{(k-1)\times(k-1)}$ be defined in (4.1). Then*

$$\Delta^{-1} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

Lemma 6 *For each $l = 1, \ldots, k-1$,*

$$\Delta^{-1}\Lambda u_l = \frac{l(l+1)}{2} u_l + \textit{linear combination of } u_1, \ldots, u_{l-1}. \tag{4.6}$$

In matrix form, (4.6) is equivalent to $\Delta^{-1}\Lambda U = U\Sigma$, i.e., $U^{-1}\Delta^{-1}\Lambda U = \Sigma$, where Σ is an upper triangular matrix whose elements on the main diagonal are $\frac{l(l+1)}{2}$, $l = 1, \ldots, k-1$. This implies that $\Delta^{-1}\Lambda$ and Σ have the same set of eigenvalues, namely, $\frac{l(l+1)}{2}$, $1 \le l \le k-1$, and that an eigenvector v_l of $\Delta^{-1}\Lambda$ corresponding to eigenvalue $\frac{l(l+1)}{2}$ is of the form $u_l +$ linear combination of $u_1, \ldots, u_{l-1}$. Hence, from Lemma 4, 5, and 6, we conclude that

Proposition 1 *B_k has $k-1$ distinctive eigenvalues $\mu_l = \frac{l(l+1)}{2} - 1$, for $l = 1, \ldots, k-1$.*

The exact expressions of the eigenvectors v_l are complicated, except for l at the two extremes, e.g., $l = 1, 2, 3$ and $l = k-2, k-1$.

Lemma 7 *B_k has the following eigenvectors:*

- $v_1 = (1, \ldots, 1)$ *for* $\mu_1 = 0$;
- $v_2 = (2-k, \ldots, 2i-k, \ldots, k-2)$ *for* $\mu_2 = 2$;
- $v_3 = [4k^2 - 10k + 9, \ldots, 5(2i-k)^2 - k^2 + 4, \ldots, 4k^2 - 10k + 9]$ *for* $\mu_3 = 5$;
- $v_{k-2} = [(2-k)\binom{k}{1}, \ldots, (-1)^{i+1}(2i-k)\binom{k}{i}, \ldots, (-1)^k(k-2)\binom{k}{k-1}]$ *for* $\mu_{k-2} = \frac{(k-1)(k-2)}{2} - 1$;
- $v_{k-1} = [\binom{k}{1}, \ldots, (-1)^{i+1}\binom{k}{i}, \ldots, (-1)^k\binom{k}{k-1}]$ *for* $\mu_{k-1} = \frac{k(k-1)}{2} - 1$.

Lemma 7 can be verified by direct computation. It is not hard to show that the i-th component of v_l is a polynomial function of $2i-k$ of degree $l-1$ consisting of even order terms only when l is odd, and odd order terms only when l is even.

Remark 9 *B_k always has an eigenvalue 0 with the corresponding eigenvector $(1,\dots,1)$, a consequence of the fact that $\dot\gamma = \frac{\partial}{\partial\theta_1}+\cdots+\frac{\partial}{\partial\theta_{k-1}}$ is parallel along γ since γ is a geodesic of N.*

Note that $v_1,\dots,v_{k-1}$ form a basis of $\mathbb{R}^{k-1}$. If we write $x=\sum_{l=1}^{k-1} y_l v_l$ in this basis, then the Jacobi equation (4.3) is equivalent to $\ddot y_l+\mu_l y_l=0$, $1\le l\le k-1$. Assume that X, hence x, vanishes at $t=0$. Then $y_l(0)=0$, and solutions to the above equations are of the form $y_1(t)=c_1 t$, $y_l(t)=c_l\sin(\sqrt{\mu_l}t)$, $l=2,\dots,k-1$, for some constants $c_1,\dots,c_{k-1}$. Therefore, the smallest $\tau_k>0$ for which there is a nontrivial solution x such that $x(0)=x(\tau_k)=0$ is

$$\tau_k=\frac{\pi}{\sqrt{\max_{1\le l\le k-1}\mu_l}}=\frac{\pi}{\sqrt{\mu_{k-1}}}=\frac{\pi\sqrt{2}}{\sqrt{(k-2)(k+1)}}.$$

In other words, the first conjugate point of $\gamma(0)$ along γ in N is $\gamma(\tau_k)$. Since a geodesic is no longer distance-minimizing after passing its first conjugate point, we have

Proposition 2 *γ is not an optimal solution to the OCA problem if $T>\tau_k$.*

Note that $\tau_k\sim\frac{1}{k}$ as $k\to\infty$. The result for the case $k=3$ was first proved in [14].

We will next explain how solutions better than γ look, at least infinitesimally, once $T>\tau_k$, and, in particular, when $T>\pi/\sqrt{\mu_l}$ for some $2\le l\le k-1$. Let $\{\gamma_s\}_{-\epsilon<s<\epsilon}$ be a C^∞ proper variation of γ with variation field $X\triangleq\frac{\partial\gamma_s}{\partial s}\big|_{s=0}$. By the variation of energy formulas ([6]), if we denote by $E(s)$ the energy of γ_s (which coincides with $J(s)$ defined as the cost function of the joint trajectory corresponding to γ_s), then $E'(0)=0$, and

$$\frac{1}{2}E''(0)=-\int_0^T\langle X,\nabla_{\dot\gamma}\nabla_{\dot\gamma}X+R(\dot\gamma,X)\dot\gamma\rangle\,dt.$$

Write $X=\sum_{i=1}^{k-1}x_i\frac{\partial}{\partial\theta_i}\big|_\gamma$ in the basis $\frac{\partial}{\partial\theta_1},\dots,\frac{\partial}{\partial\theta_{k-1}}$ along γ. Then $x=(x_1,\dots,x_{k-1})$ vanishes at 0 and T since $\{\gamma_s\}_{-\epsilon<s<\epsilon}$ is a proper variation. The above equation is thus reduced to

$$\frac{1}{2}E''(0)=-\int_0^T x^T\Delta(\ddot x+B_k x)\,dt.$$

Suppose now that $T>\pi/\sqrt{\mu_l}$ for some $l\in\{2,\dots,k-1\}$. Then, by choosing $\{\gamma_s\}_{-\epsilon<s<\epsilon}$ such that $x(t)=v_l\sin(\pi t/T)$ where v_l is an eigenvector of B_k for eigenvalue μ_l, we have

$$\frac{1}{2}E''(0)=-(\mu_l-\frac{\pi^2}{T^2})\int_0^T(v_l^T\Delta v_l)\sin^2(\pi t/T)\,dt<0, \tag{4.7}$$

since $v_l^T\Delta v_l>0$ and $\mu_l-\pi^2/T^2>0$. Therefore, the arc length of γ_s is smaller than that of γ for s sufficiently close to 0. To sum up, the above analysis shows that if $T>\pi/\sqrt{\mu_l}$, a solution better than γ can be obtained by infinitesimally perturbing γ in such a way that, at each $t\in[0,T]$, $(\theta_1,\dots,\theta_{k-1})$ is incremented by an amount of $v_l\sin(\pi t/T)ds$. In particular, the signs of the components of v_l determine the shape of the $(k-1)$-rod link during such perturbations. For example, the alternating signs of the components of v_{k-1} indicate a perturbation where the $k-1$ rods are first folded into a saw-like shape during the first half of the time interval $[0,T]$, with the degree of folding of each rod depending on its position (in fact, proportional to $\binom{k}{l}$ for the l-th rod from the edge, $l=1,\dots,k-1$), and then straightened up during the later half of the time interval. In contrast, v_2 indicates the $k-1$ rods to bend into a bow-like shape, whereas the shape specified by v_{k-2} is a mixing (product) of the bending specified by v_2 and the folding specified by v_{k-1}. The efficiency of the perturbations specified by different v_l (provided $T>\pi/\sqrt{\mu_l}$) can be studied by comparing their respective $E''(0)$ under the requirement that $\int_0^T\|X\|^2\,dt=\int_0^T(v_l^T\Delta v_l)\sin^2(\pi t/T)\,dt$ be constant, such that a smaller $E''(0)$ corresponds to a more efficient perturbation. In this respect, by (4.7), the larger the eigenvalue μ_l, the more efficient the perturbation specified by its corresponding eigenvector v_l. The most efficient perturbation is thus the one given by v_{k-1}.

5. Two Extensions. In this section, we consider two classes of problems to which the results obtained in Section 3 can be directly generalized, namely, the OCA problem of bodies (Section 5.1), and the optimal control problem of a special class of hybrid system called switched Lagrangian systems (Section 5.2).

5.1. Collision Avoidance of Bodies. The OCA and OFS problems studied in Section 3 can be thought of as optimal motion planning problems for k agents moving on a Riemannian manifold, with each agent represented by a disk of radius $r/2$. The arguments in Section 3 can be generalized to the situation where agents have shape other than disks. To be precise, let M be a Riemannian manifold.

Definition 2 (Shape of Body) *The shape of a body on M is specified by a map $\mathcal{S} : M \to 2^M$ that assigns to each $q \in M$ a subset $\mathcal{S}(q) \subset M$ corresponding to the subset of M the body occupies if it is at q. $\mathcal{S}$ is called the* shape (map) *of the body.*

Consider k bodies on M whose shapes are given by the maps $\mathcal{S}_i$, $i = 1, \ldots, k$, respectively. Suppose that during the time interval $[t_0, t_1]$ their trajectories are given by a k-tuple of curves $\gamma = \langle \gamma_i \rangle_{i=1}^k$ in M. γ is called *collision-free* if $\mathcal{S}_i(\gamma_i(t))$, $i = 1, \ldots, k$, are disjoint at any time $t \in [t_0, t_1]$. Fix the starting position $\langle a_i \rangle_{i=1}^k$ and the destination position $\langle b_i \rangle_{i=1}^k$ of the k bodies such that $\mathcal{S}_i(a_i)$, $i = 1, \ldots, k$, and $\mathcal{S}_i(b_i)$, $i = 1, \ldots, k$, are disjoint, respectively. Let $L : TM \to \mathbb{R}$ be a Lagrangian function and let the cost of γ, $J(\gamma)$, be defined by (2.2). Then the OCA problem for bodies is

Problem 3 (OCA of Bodies) *Among all collision-free γ that start from $\langle a_i \rangle_{i=1}^k$ at time t_0 and end at $\langle b_i \rangle_{i=1}^k$ at time t_1, find the one (or ones) minimizing $J(\gamma)$.*

The OFS problem of bodies can be similarly formulated. However, it is omitted here for brevity. In analogy to Assumption 1 in Section 3, we consider the following special case.

Assumption 2 *There is a C^∞ action $\Phi : G \times M \to M$ of a Lie group G on M such that*

1. *the shapes of the bodies are G-invariant. Namely, $\Phi_g \circ \mathcal{S}_i = \mathcal{S}_i \circ \Phi_g$, $\forall g \in G$, $i = 1, \ldots, k$;*
2. *the Lagrangian function L is G-invariant.*

Under Assumption 2, if G acts on M transitively, then each $\mathcal{S}_i$ is completely determined by $\mathcal{S}_i(q)$ at an arbitrary point $q \in M$. In general, one needs to specify $\mathcal{S}_i(q)$ for one q in each G-orbit of M to fully determine $\mathcal{S}_i$.

A key implication of Assumption 2 is that, if $\gamma = \langle \gamma_i \rangle_{i=1}^k$ is collision-free, so is $h_0\gamma = \langle h_0\gamma_i \rangle_{i=1}^k$ for any continuous and piecewise C^∞ curve $h_0 : [t_0, t_1] \to G$, since at any time t, $1 \le i < j \le k$,

$$\mathcal{S}_i(h_0\gamma_i) \cap \mathcal{S}_j(h_0\gamma_j) = \Phi_{h_0}[\mathcal{S}_i(\gamma_i)] \cap \Phi_{h_0}[\mathcal{S}_j(\gamma_j)] = \Phi_{h_0}[\mathcal{S}_i(\gamma_i) \cap \mathcal{S}_j(\gamma_j)],$$

hence $\mathcal{S}_i(h_0\gamma_i) \cap \mathcal{S}_j(h_0\gamma_j) = \emptyset$ if and only if $\mathcal{S}_i(\gamma_i) \cap \mathcal{S}_j(\gamma_j) = \emptyset$. This property enables one to apply the variational approach in Section 3 without modification. Therefore,

Theorem 5 *For the OCA and the OFS problems of bodies, all the necessary conditions derived in Section 3 remain true, including Theorem 1, 2, 4, and all their corollaries.*

As an example, consider $\mathbf{SE}_2$, the group of orientation-preserving isometries of $\mathbb{R}^2$. Elements of $\mathbf{SE}_2$ are of the form

$$A(x, y, \theta) \triangleq \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}, \quad \forall x, y, \theta \in \mathbb{R}.$$

$\mathbf{SE}_2$ acts on $\mathbb{R}^2 \simeq \mathbb{R}^2 \times \{1\} \subset \mathbb{R}^3$ by left matrix multiplication. Then $A(x, y, \theta)$ corresponds to the rigid body motion in $\mathbb{R}^2$ of a rotation by θ counterclockwise followed by a translation by (x, y). The Lie algebra

of $\mathbf{SE}_2$, $\mathfrak{se}_2$, is the set of all matrices of the form

$$\zeta(u,v,w) \triangleq \begin{bmatrix} 0 & -w & u \\ w & 0 & v \\ 0 & 0 & 0 \end{bmatrix}, \quad \forall u,v,w \in \mathbb{R}.$$

Define an inner product on $\mathfrak{se}_2$ by

$$\langle \zeta(u_1,v_1,w_1), \zeta(u_2,v_2,w_2)\rangle \triangleq u_1u_2 + v_1v_2 + \kappa w_1 w_2,$$

where $\kappa > 0$ is a constant, and extend it to a left invariant Riemannian metric $\langle\cdot,\cdot\rangle$ on $\mathbf{SE}_2$ through left translation. Consider Problem 3 with $M = \mathbf{SE}_2$ and the Lagrangian function $L = \frac{1}{2}\|\cdot\|^2$. Let $G = \mathbf{SE}_2$ and the action Φ be the group multiplication. Suppose that the shapes of the bodies are given by

$$\mathcal{S}_i[A(x,y,\theta)] = \{A(\hat{x},\hat{y},\hat{\theta}) \in \mathbf{SE}_2 : (\hat{x},\hat{y}) \in A(x,y,\theta)D_i\}, \quad \forall A(x,y,\theta) \in \mathbf{SE}_2, \tag{5.1}$$

where D_i is a subset of $\mathbb{R}^2$ containing the origin, for $i = 1,\ldots,k$. It is easy to verify that all $\mathcal{S}_i$ are G-invariant, hence Assumption 2 is satisfied.

Remark 10 *To justify the choice of $\mathcal{S}_i$ in (5.1), note that $\mathbf{SE}_2$ is the configuration space of a rigid body moving on $\mathbb{R}^2$, in the sense that each element $A(x,y,\theta) \in \mathbf{SE}_2$ can be thought of as a configuration of the rigid body whose pivot point is at $(x,y) \in \mathbb{R}^2$ and whose orientation is in the direction that makes an angle θ with the positive x-axis. The shape of the rigid body can be specified by the region $D \subset \mathbb{R}^2$ it occupies when it is in configuration $A(0,0,0)$, i.e., when it has its pivotal point at the origin and points at the positive x-axis. The region it occupies in any other configuration $A(x,y,\theta)$ is obtained by applying on D the rigid body motion that transforms configuration $A(0,0,0)$ to $A(x,y,\theta)$, hence the definition in (5.1). In this perspective, the problem can be alternatively formulated as the optimal motion planning problem for k rigid bodies in $\mathbb{R}^2$, such that no two of them can overlap at any time, and that the cost $\sum_{i=1}^2 \frac{1}{2}\lambda_i \int_{t_0}^{t_1} (\dot{x}_i^2 + \dot{y}_i^2 + \kappa\dot{\theta}_i^2)\,dt$ is minimized.*

Let $\gamma = \langle\gamma_i\rangle_{i=1}^k = \langle A(x_i,y_i,\theta_i)\rangle_{i=1}^k$ be an optimal solution to Problem 3, where x_i, y_i, θ_i are continuous and piecewise C^∞ curves in $\mathbb{R}$ defined on $[t_0,t_1]$. Then it is easy to show that $\sum_{i=1}^k \lambda_i (d\Phi^{\gamma_i})^*\dot{\gamma}_i = \zeta(\sum_{i=1}^k \lambda_i\dot{x}_i, \sum_{i=1}^k \lambda_i\dot{y}_i, \sum_{i=1}^k \lambda_i[\dot{\theta}_i + (x_i\dot{y}_i - \dot{x}_iy_i)/\kappa]) \in \mathfrak{se}_2$, which by Theorem 1 should be constant for all t. In other words, the following quantities are conserved:

$$\sum_{i=1}^k \lambda_i\dot{x}_i, \quad \sum_{i=1}^k \lambda_i\dot{y}_i, \quad \sum_{i=1}^k \lambda_i[\dot{\theta}_i + (x_i\dot{y}_i - \dot{x}_iy_i)/\kappa].$$

In some simple cases, it is possible to construct the optimal solutions from these necessary conditions. We next discuss one of these cases. Consider $k = 2$, and denote by $\langle A(x_i^0,y_i^0,\theta_i^0)\rangle_{i=1}^2$ and $\langle A(x_i^1,y_i^1,\theta_i^1)\rangle_{i=1}^2$ the starting and the destination positions respectively. Integrating the first two conserved quantities, we get

$$\sum_{i=1}^2 \lambda_i \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} = \frac{t_1 - t}{t_1 - t_0}\sum_{i=1}^2 \lambda_i \begin{bmatrix} x_i^0 \\ y_i^0 \end{bmatrix} + \frac{t - t_0}{t_1 - t_0}\sum_{i=1}^2 \lambda_i \begin{bmatrix} x_i^1 \\ y_i^1 \end{bmatrix}, \quad \forall t \in [t_0,t_1]. \tag{5.2}$$

Hence the weighted center of the two-body system moves at constant speed from the weighted center of their starting position to the weighted center of their destination position. Another fact we need is summarized in the following lemma.

Lemma 8 *Suppose that $\gamma = \langle A(x_i,y_i,\theta_i)\rangle_{i=1}^k$ is an optimal solution to the OCA problem of k bodies on $\mathbf{SE}_2$ with starting position $\langle A(x_i^0,y_i^0,\theta_i^0)\rangle_{i=1}^k$ and destination position $\langle A(x_i^1,y_i^1,\theta_i^1)\rangle_{i=1}^k$. Then for any $(x,y) \in \mathbb{R}^2$, $\tilde{\gamma} = \langle A(x_i + \frac{t-t_0}{t_1-t_0}x, y_i + \frac{t-t_0}{t_1-t_0}y, \theta_i)\rangle_{i=1}^k$ is an optimal solution of the OCA problem of the same k bodies on $\mathbf{SE}_2$ with starting position $\langle A(x_i^0,y_i^0,\theta_i^0)\rangle_{i=1}^k$ and destination position $\langle A(x_i^1 + x, y_i^1 + y, \theta_i^1)\rangle_{i=1}^k$.*

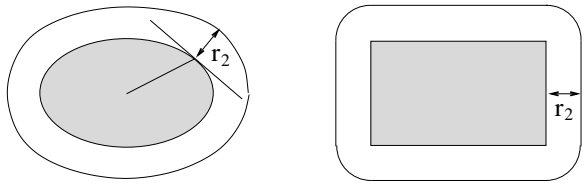

FIG. 5.1. *Two examples of outgrowing a set (shown by shaded areas) by r_2. Left: an ellipse; Right: a rectangle*

Proof: Note that $\tilde{\gamma}$ is collision-free if and only if γ is, and that the costs of γ and $\tilde{\gamma}$ are related by $J(\tilde{\gamma}) = J(\gamma)+$ some constant independent of γ. Hence the conclusion. ■

By Lemma 8, we may assume without loss of generality that

$$\sum_{i=1}^{2} \lambda_i \begin{bmatrix} x_i^0 \\ y_i^0 \end{bmatrix} = \sum_{i=1}^{2} \lambda_i \begin{bmatrix} x_i^1 \\ y_i^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{5.3}$$

So by (5.2), for all $t \in [t_0, t_1]$,

$$\sum_{i=1}^{2} \lambda_i \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{i.e.,} \quad \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} = -\frac{\lambda_1}{\lambda_2} \begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix}. \tag{5.4}$$

Assume in addition that D_2 is an open disk of radius r_2 centered at the origin. Then $\dot{\theta}_2$ must be constant since γ is collision-free under any reparameterization of θ_2 and the one with constant velocity minimizes the term $\int_{t_0}^{t_1} \dot{\theta}_2^2 \, dt$, which is the contribution of θ_2 to the cost of γ_1 (see Remark 10). Hence θ_2 moves at constant speed from θ_2^0 to the nearest point in $\theta_2^1 + 2\pi\mathbb{Z}$ in $\mathbb{R}$.

Since x_2, y_2 are related to x_1, y_1 as in (5.4), it remains only to specify x_1, y_1, θ_1. The set of feasible (x_1, y_1, θ_1) is

$$\begin{aligned} F =& \{(x,y,\theta) : A(x,y,\theta)D_1 \cap (-\frac{\lambda_1}{\lambda_2}\begin{bmatrix} x \\ y \end{bmatrix} + D_2) = \emptyset\} \\ =& \{(x,y,\theta) : (A(0,0,\theta)D_1 + \begin{bmatrix} x \\ y \end{bmatrix}) \cap (-\frac{\lambda_1}{\lambda_2}\begin{bmatrix} x \\ y \end{bmatrix} + D_2) = \emptyset\} \\ =& \{(x,y,\theta) : A(0,0,\theta)D_1 \cap (-\frac{\lambda_1+\lambda_2}{\lambda_2}\begin{bmatrix} x \\ y \end{bmatrix} + D_2) = \emptyset\} \\ =& \{(x,y,\theta) : \text{distance of } \begin{bmatrix} x \\ y \end{bmatrix} \text{ to } -\frac{\lambda_2}{\lambda_1+\lambda_2}A(0,0,\theta)D_1 \text{ is at least } \frac{\lambda_2 r_2}{\lambda_1+\lambda_2}\}, \end{aligned}$$

which defines a *static* obstacle in $\mathbb{R}^3$. Denote by $F_\theta = \{(x,y) : (x,y,\theta) \in F\}$ a section of F. Then it is easy to check that $F_\theta = A(0,0,\theta)F_0$, where F_0 can be obtained by first "outgrowing" $-D_1$ by r_2, and then scaling the resultant set by a factor of $\frac{\lambda_2}{\lambda_1+\lambda_2}$. See Figure 5.1 for two examples of how to outgrow a set D_1, and see Figure 5.2 for two examples of F.

The optimal solution corresponds to a curve (x_1, y_1, θ_1) in $\mathbb{R}^3 \setminus F$ that starts from $(x_1^0, y_1^0, \theta_1^0)$ at time t_0 and ends in $(x_1^1, y_1^1, \theta_1^1 + 2m\pi)$ for some integer m at time t_1, while minimizing the cost

$$J(\gamma) = \frac{1}{2}\sum_{i=1}^{2} \lambda_i \int_{t_0}^{t_1} (\dot{x}_i^2 + \dot{y}_i^2 + \kappa\dot{\theta}_i^2)\, dt = \frac{\lambda_1(\lambda_1+\lambda_2)}{2\lambda_2} \int_{t_0}^{t_1} \left[\dot{x}_1^2 + \dot{y}_1^2 + \kappa_1\dot{\theta}_1^2\right] dt + C,$$

where $\kappa_1 = \frac{\lambda_2}{\lambda_1+\lambda_2}\kappa$, and C is a constant. By scaling the θ_1-axis by a factor of $\sqrt{\kappa_1}$, the integral above coincides with the usual definition of curve energy, and the problem is then reduced to finding the shortest curve between two points in the scaled feasible set. Except for very simple cases (for example, when D_1 is a disk of radius r_1 centered at the origin, then the problem is reduced to Problem 1 on $\mathbb{R}^2$ with $r = r_1 + r_2$, and solutions can be constructed geometrically), analytic solutions are not available. However, given the geometrical interpretation, there are various numerical algorithms to solve it approximately such as, for example, the fast marching algorithm proposed in [27].

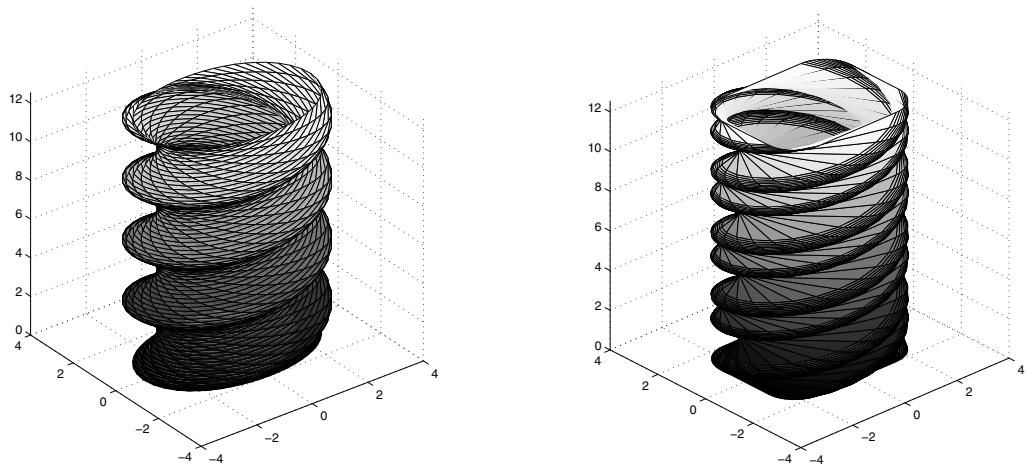

FIG. 5.2. *Plots of F when D_1 is an ellipse (left) and a rectangle (right).*

5.2. Optimal Control of a Class of Hybrid Systems. As another generalization, we study the optimal control problem for a special class of *hybrid systems*. The concept of hybrid systems has been proposed in the control community to model dynamic systems that possess both continuous dynamics and discrete transitions, which are often encountered in modern computer control systems. See [21] for a general definition of hybrid systems, and [29] for a general treatment on optimal control problem of hybrid systems.

Definition 3 (Switched Lagrangian System) *A switched Lagrangian system $\mathcal{H}$ is given by:*

1. *a set Γ of discrete modes;*
2. *for each $l \in \Gamma$, a domain M_l which is a manifold, possibly with boundary ∂M_l, and a Lagrangian function $L_l : TM_l \to \mathbb{R}$ that is nonnegative and convex on each fiber. We assume that the domains M_l, $l \in \Gamma$, are disjoint, and write $M = \cup_{l\in\Gamma} M_l$;*
3. *a set of discrete transitions $E_d \subset \Gamma \times \Gamma$;*
4. *for each $(l_1, l_2) \in E_d$, a subset $D_{(l_1,l_2)} \subset M_{l_1}$, called the* guard *associated with the discrete transition (l_1, l_2), and a continuous transition relation $E_c(l_1, l_2) \subset D_{(l_1,l_2)} \times M_{l_2}$ such that for each $q_1 \in D_{(l_1,l_2)}$, there exists at least one $q_2 \in M_{l_2}$ with $(q_1, q_2) \in E_c(l_1, l_2)$. In other words, $E_c(l_1, l_2)$ specifies a one-to-many map from $D_{(l_1,l_2)}$ to M_{l_2}.*

Definition 4 (Hybrid Path) *Consider a switched Lagrangian system $\mathcal{H}$. A* hybrid path *(or simply a* path*) of $\mathcal{H}$ defined on $[t_0, t_1]$ is a collection of arcs $\gamma_1, \ldots, \gamma_m$ in M together with a partition of $[t_0, t_1]$, $t_0 = \tau_0 \le \ldots \le \tau_{m+1} = t_1$, and a succession of discrete modes $l_0, \ldots, l_m \in \Gamma$ for some integer $m > 0$ satisfying*

- *$(l_j, l_{j+1}) \in E_d$ for $j = 0, \ldots, m-1$;*
- *$\gamma_j : [\tau_j, \tau_{j+1}] \to M_{l_j}$ is a continuous and piecewise C^∞ curve in M_{l_j} for $j = 0, \ldots, m$;*
- *for each $j = 0, \ldots, m-1$, $\gamma_j(\tau_{j+1}) \in D_{(l_j,l_{j+1})}$, and $(\gamma_j(\tau_{j+1}), \gamma_{j+1}(\tau_{j+1})) \in E_c(l_j, l_{j+1})$.*

We call $\gamma_0, \ldots, \gamma_m$ the segments *of γ.*

The *cost* of the path γ is defined by

$$J(\gamma) = \sum_{j=0}^{m} \int_{\tau_j}^{\tau_{j+1}} L_{l_j}(\dot{\gamma}_j)\, dt.$$

Intuitively speaking, a path is the trajectory of a point moving in M such that whenever the point is in M_{l_1} and it reaches a point in a guard, say, $q_1 \in D_{(l_1,l_2)}$, it has the option of jumping to a point q_2 in M_{l_2}

according to the continuous transition relation $(q_1, q_2) \in E_c(l_1, l_2)$, and continuing its motion in M_{l_2}, and so on. We call such a jump a *transition*, which consists of a discrete transition $l_1 \to l_2$ and a continuous transition $q_1 \to q_2$. Note that during a transition, there is possibly more than one position the point can jump to due to two reasons: guards for different discrete transitions may intersect; and for a single discrete transition, the continuous transition relation is a one-to-many map. It is allowed in the definition of the path γ that some of $\tau_0, \ldots, \tau_{m+1}$ are identical, implying that more than one transition may occur at the same epoch, though in a sequential way. This can cause trouble since it is possible that all the legitimate transitions from a certain point lead to infinite number of jumps occurring at the same epoch, thus blocking the system from further evolving in time. To avoid this difficulty, we make the following assumption.

Assumption 3 (Connectness) *Assume that the switched Lagrangian system $\mathcal{H}$ is connected in the sense that for any two points $a, b \in M$, there exists at least one path γ connecting them.*

The problem of interest to us can be formulated as

Problem 4 (Optimal Control on $\mathcal{H}$) *Given $a, b \in M$ and a time interval $[t_0, t_1]$, find the path (or paths) from a to b defined on $[t_0, t_1]$ with minimal cost.*

We focus on a special class of switched Lagrangian systems.

Assumption 4 (Symmetry) *There exists a Lie group G such that*

- *G acts on the domain M_l from the left for each $l \in \Gamma$. We shall denote this smooth action uniformly by Φ for all l;*
- *for each $l \in \Gamma$, the Lagrangian function L_l is G-invariant.*
- *for each discrete transition $(l_1, l_2) \in E_d$, the guard $D_{(l_1,l_2)}$ and the continuous transition relation $E_c(l_1, l_2)$ are both invariant under the action of G, i.e., for any $g \in G$, $\Phi(g, D_{(l_1,l_2)}) = D_{(l_1,l_2)}$, and $(q_1, q_2) \in E_c(l_1, l_2)$ if and only if $(\Phi(g, q_1), \Phi(g, q_2)) \in E_c(l_1, l_2)$;*

From the above assumption, we have

Proposition 3 *If γ is a path of $\mathcal{H}$ defined on $[t_0, t_1]$, then for any continuous and piecewise C^∞ curve $h_0 : [t_0, t_1] \to G$, $h_0\gamma$ is also a path of $\mathcal{H}$ defined on $[t_0, t_1]$.*

To be precise, $h_0\gamma$ is defined in the following way. Let $\tau_0, \ldots, \tau_{m+1}$ and $l_0, \ldots, l_m$ be as in the definition of γ, and let $\gamma_0, \ldots, \gamma_m$ be the corresponding segments of γ. Then $h_0\gamma$ is a sequence of arcs, $h_0\gamma_j \triangleq \Phi(h_0, \gamma_j)$, $j = 0, \ldots, m$, whose intervals of definition and corresponding discrete modes coincide with those of γ_j, $j = 0, \ldots, m$, respectively. The proof of Proposition 3 is straightforward, hence omitted. As a result of Proposition 3, all of the results in Section 3 have their corresponding versions for Problem 4. We state only the first one of them.

Theorem 6 *Suppose that γ is an optimal solution to Problem 4, and let $\gamma_0, \ldots, \gamma_m$ be its segments. Then there exists a constant $\nu_0 \in \mathfrak{g}^*$ such that*

$$(d\Phi^{\gamma_j})^*(\mathbb{D}L_{l_j})_{\dot{\gamma}_j} \equiv \nu_0, \quad \forall t \in [\tau_j, \tau_{j+1}],\ j = 0, \ldots, m, \tag{5.5}$$

where $l_0, \ldots, l_m$ and $\tau_0, \ldots, \tau_{m+1}$ are defined as in Definition 4.

A simple way of writing equation (5.5) is $(d\Phi^\gamma)^*\mathbb{D}L_{\dot{\gamma}} \equiv \nu_0$.

We illustrate the above concepts by two simple examples.

Example 17 Shown in Figure 5.3 is an example of $\mathcal{H}$ that is $\mathbf{T}^1$-symmetric. Here $\Gamma = \{0, 1\}$, and the domains of the two discrete modes are two disjoint surfaces of revolution M_0 and M_1 in $\mathbb{R}^3$, each with a metric inherited from $\mathbb{R}^3$ and a boundary obtained as the cross section of the surface with a plane perpendicular to

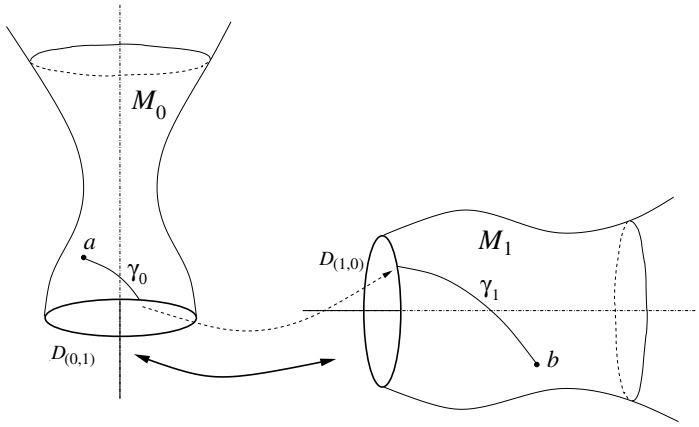

FIG. 5.3. *An example of an* $\mathbf{T}^1$*-symmetric switched Lagrangian system.*

its rotational axis. Let $E_d = \{(0,1),(1,0)\}$, $D_{(0,1)} = \partial M_0$, $D_{(1,0)} = \partial M_1$. Define the continuous transition relation $E_c(0,1)$ as the graph of a map from ∂M_0 to ∂M_1 that rotates the circle ∂M_0 by a certain angle and "fits" it into ∂M_1 (after a scaling, if necessary). Define $E_c(1,0)$ to be the graph of the inverse of the same map. Suppose that for each $l \in \Gamma$, $L_l = \frac{1}{2}\|\cdot\|^2$, and that each $\theta \mod 2\pi \in \mathbf{T}^1$ acts on M_l by rotating it along its axis by an angle θ. By properly choosing the directions of rotation in the above definition, one can check that the resulting $\mathcal{H}$ satisfies Assumption 3 and 4 with $G = \mathbf{T}^1$. Consider two points $a \in M_0$, $b \in M_1$, and an optimal path γ connecting them. It can be shown that for any segment γ_0 of γ in M_0, the conserved quantity is the component along the rotational axis of M_0 of the angular momentum of a unit particle following γ_0 in $M_0 \subset \mathbb{R}^3$. Similarly we can define the conserved quantity for segments of γ in M_1. Theorem 6 states that these two quantities are identical. If the optimal solutions in this problem are thought of as the paths that light travels in a heterogeneous media, then the above characterization is in fact a generalization of the *Snell Law* in optics. □

Example 18 Let $\Gamma = \{0,1\}$. For each $l \in \Gamma$, $M_l = \{l\} \times (\mathbb{R}^2 \setminus \{0\}) \subset \mathbb{R}^3$ is a plane with the origin removed, and the Lagrangian function L_l is defined in the polar coordinates (r,θ) of $M_l \simeq \mathbb{R}^2 \setminus \{0\}$ as: $L_l(v) = \frac{1}{2}v^T A_l(r) v$, for $v = (\dot{r}, \dot{\theta}) \in T_q M_l$, $q \in M_l$. Here $A_l(r)$ is a 2-by-2 positive definite matrix whose entries are smooth functions of r, and $\|A_l(r)\|_2 \to \infty$ as $r \to 0$. Let $E_d = \{(0,1),(1,0)\}$, $D_{(0,1)} = M_0$, $D_{(1,0)} = M_1$. Choose $E_c(0,1) = \{((0,x),(1,x)) : x \in \mathbb{R}^2, x \neq 0\}$, $E_c(1,0) = \{((1,x),(0,x)) : x \in \mathbb{R}^2, x \neq 0\}$. Therefore, a point moving in $\mathcal{H}$ can freely switch between two copies of $\mathbb{R}^2 \setminus \{0\}$ with different Lagrangian functions, both of which are invariant with respect to rotations around the origin. So $\mathcal{H}$ satisfies Assumption 3 and 4 with $G = \mathbf{SO}_2 \simeq \mathbf{T}^1$. For an optimal solution γ to Problem 4 connecting two points $a, b \in M$, Theorem 6 implies that the conserved quantity for the segments of γ on each M_l is the angular momentum of a unit particle following γ. □

6. Conclusions. The problems of optimal collision avoidance and optimal formation control are studied for multiple agents moving on a Riemannian manifold with a group of symmetries. Some necessary conditions are given for the optimal solutions. In certain simple cases, these necessary conditions can be used to characterize the optimal solutions, whereas in general, they are not sufficient.

As a future direction of research, it will be interesting to see how the derived necessary conditions can help to find a numerical solution of these problems.

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Appendix A. Proof of Lemma 1. Define a $\mathfrak{g}$-valued left invariant 1-form ω on G by $\omega(v) = g^{-1}v$, $\forall v \in T_g G$, $g \in G$. By the Maurer-Cartan structure equation [28], $d\omega = -[\omega, \omega]$. Pulling this back via the map $h : (-\epsilon, \epsilon) \times [t_0, t_1] \to G$ yields $h^*(d\omega) = -[h^*\omega, h^*\omega]$. Evaluating both sides at the vector fields $\frac{\partial}{\partial s}$ and

$\frac{\partial}{\partial t}$, and noting that $\omega(\dot{h}) = \xi$ and $\omega(h') = \eta$ by definition, we obtain

$$-[h^*\omega(\frac{\partial}{\partial s}), h^*\omega(\frac{\partial}{\partial t})] = -[\omega(dh\frac{\partial}{\partial s}), \omega(dh\frac{\partial}{\partial t})] = -[\omega(h'), \omega(\dot{h})] = -[\eta, \xi] = [\xi, \eta],$$
$$h^*(d\omega)(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}) = d(h^*\omega)(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}) = \frac{\partial}{\partial s}[h^*\omega(\frac{\partial}{\partial t})] - \frac{\partial}{\partial t}[h^*\omega(\frac{\partial}{\partial s})] - h^*\omega([\frac{\partial}{\partial s}, \frac{\partial}{\partial t}]) = \xi' - \dot{\eta}.$$

The desired conclusion follows by equating the above two equations.

Appendix B. Two Lemmas used in Section 3.3.

Assume that $\mathfrak{so}_n$ $(n \geq 2)$ is equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{so}_n} = \frac{1}{2}\langle \cdot, \cdot \rangle_F$.

Lemma 9 *Suppose that $Y \in \mathfrak{so}_n$ and $\lambda > 0$ are constants. Then the following are equivalent:*

1. *$|\langle Y, X \rangle_{\mathfrak{so}_n}| \leq \lambda \|X\|^2_{\mathfrak{so}_n}$ for all $X \in \mathfrak{so}_n$ such that $e^{2\pi X} = I_n$;*
2. *The L^2-norm of Y, $\|Y\|_2$, is bounded by λ.*

Proof: $1 \to 2$: For any unit vector $v_1 \in \mathbb{R}^n$ such that $Yv_1 \neq 0$, define $v_2 = Yv_1/\|Yv_1\|$, which is a unit vector orthogonal to v_1 by the skew symmetry of Y. Let $v_1, v_2, \ldots, v_n$ be an orthonormal basis of $\mathbb{R}^n$, hence $A = [v_1|v_2|\ldots|v_n] \in \mathbf{O}_n$. Define $X = AZA^T$, where $Z = (z_{ij}) \in \mathfrak{so}_n$ is such that $z_{21} = -z_{12} = 1$, and $z_{ij} = 0$ otherwise. Then $X \in \mathfrak{so}_n$, and $e^{2\pi X} = I_n$. Hence $|\langle Y, X \rangle_{\mathfrak{so}_n}| \leq \lambda \|X\|^2_{\mathfrak{so}_n} = \lambda$. But $\langle Y, X \rangle_{\mathfrak{so}_n} = \langle Y, AZA^T \rangle_{\mathfrak{so}_n} = \langle A^TYA, Z \rangle_{\mathfrak{so}_n} = v_2^TYv_1 = \|Yv_1\|$. Therefore, $\|Yv_1\| \leq \lambda$. That this holds for every unit vector $v_1 \in \mathbb{R}^n$ implies that $\|Y\|_2 \leq \lambda$. $2 \to 1$: For each $X \in \mathfrak{so}_n$ with $e^{2\pi X} = I_n$, there exist $A \in \mathbf{O}_n$ and $Z \in \mathfrak{so}_n$ such that $X = AZA^T$, where $Z = \mathrm{diag}(\begin{bmatrix} 0 & -m_1 \\ m_1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & -m_l \\ m_l & 0 \end{bmatrix}, 0, \ldots, 0)$ for some $m_1, \ldots, m_l \in \mathbb{Z}$ $(2l \leq n)$. Write $A = [u_1|v_1|\ldots|u_l|v_l|w_1|\ldots|w_{n-2l}]$ in column vectors. Then

$$|\langle Y, X \rangle_{\mathfrak{so}_n}| = |\langle A^TYA, Z \rangle_{\mathfrak{so}_n}| = |\sum_{j=1}^{l} m_j v_j^T Y u_j| \leq \sum_{j=1}^{l} |m_j| \cdot |v_j^T Y u_j| \leq \|Y\|_2 \sum_{j=1}^{l} m_j^2 \leq \lambda \|X\|^2_{\mathfrak{so}_n},$$

since $|v_j^T Y u_j| \leq \|v_j\| \|Y\|_2 \|u_j\| = \|Y\|_2$, and $\|X\|^2_{\mathfrak{so}_n} = \|Z\|^2_{\mathfrak{so}_n} = \sum_{j=1}^{l} m_j^2$. ∎

Lemma 10 *Suppose that $n = 2l$ is even, and that $Y \in \mathfrak{so}_n$ and $\lambda > 0$ are constants such that $|\langle Y, X \rangle_{\mathfrak{so}_n}| \leq \lambda \|X\|^2_{\mathfrak{so}_n}$ for all $X \in \mathfrak{so}_n$ satisfying $e^{\pi X} = -I_n$. Then*

$$\frac{1}{n} \sum_{j=1}^{n} \lambda_j \leq \lambda,$$

where $\lambda_1, \ldots, \lambda_n$ are the singular values of Y.

Proof: Since $Y \in \mathfrak{so}_n$, there exist $A \in \mathbf{O}_n$ and $\omega_1, \ldots, \omega_l \geq 0$ such that $Y = AZA^T$, where $Z = \mathrm{diag}(\begin{bmatrix} 0 & -\omega_1 \\ \omega_1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & -\omega_l \\ \omega_l & 0 \end{bmatrix}) \in \mathfrak{so}_n$. Hence the singular values $\lambda_1, \ldots, \lambda_n$ of Y are simply $\omega_1, \omega_1, \ldots, \omega_l, \omega_l$. Define $X = A \cdot \mathrm{diag}(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}) \cdot A^T$. Then $X \in \mathfrak{so}_n$ and $e^{\pi X} = -I_n$. So by hypothesis, $|\langle Y, X \rangle_{\mathfrak{so}_n}| = \sum_{j=1}^{l} \omega_j \leq \lambda \|X\|^2_{\mathfrak{so}_n} = \lambda l$, which is the desired conclusion. ∎

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