Geodesics in Riemannian Manifolds with Boundary

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1 Introduction

Many problems arising in practical situations have boundary constraints and can only be described in the setting of Riemannian manifolds with boundary. This justifies our focus in this report on various geometric aspects of manifolds with boundary. In particular, we are interested in those results concerning the properties of geodesics in such manifolds. A related problem is the geometry of wavefront propagation around an obstacle in an isotropic medium, since the orthogonal trajectories of the wavefronts are geodesics in the appropriate Riemannian manifold with boundary. In the last section, we will give as an example a problem of multiple aircraft conflict resolution, which can also be reduced to the problem of finding the shortest geodesics between two points in certain manifold with boundary. Therefore, although the results concerning manifolds with boundary are sometimes less elegant than their counterparts for manifolds without boundary, the study of this area is of both theoretical and practical interest.

In the following, M will denote a C^∞ Riemannian n dimensional manifold with boundary B, an $n-1$ dimensional manifold. Unless otherwise stated, B will be assumed to be smooth.

2 Geodesics

2.1 Regularity

As in the case of manifolds without boundary, a *geodesic* in M is a curve which is locally distance minimizing and parametrized by arc length. The existence and regularity of geodesics between any two points in M are studied in [9] from the general viewpoint of elliptic variational problems with constraints. A more geometrical approach is adopted in [2]. Due to the existence of the boundary, one can no longer write a single second order differential equation governing the evolution of geodesics. Even for manifolds with smooth boundary, geodesics are in general not C^2. For example, consider the shortest paths in the Euclidean plane with the open unit disc removed. They are, however, C^1 by the following theorem proved in [2].

Theorem 1 *Let M be a Riemannian C^3-manifold-with-C^1-boundary and γ be an arbitrary geodesic in M. Then γ is C^1, and at any point where it touches the boundary, γ has an osculating plane normal to the boundary.*

It is shown independently in [10] that any geodesic in M is C^1 even if one weakens the hypothesis that $B = \partial M$ is C^1 by require only that every point of B has a neighborhood in M which is C^2-diffeomorphic to a convex set in $\mathbb{R}^n$. A large class of examples belonging to this category can be constructed by removing from $\mathbb{R}^n$ the union of a finite number of open convex sets with non-empty interiors and C^2 boundaries. For example, consider M obtained by removing from $\mathbb{R}^2$ the union of two intersecting disks. The boundary of M is not C^1, yet geodesics in M are C^1.

One can look deeper into the structure of the geodesics in M. In the terminology of [3], a geodesic segment can be decomposed into

1. Geodesic segments of the interior of M, whose accelerations vanish.
2. Geodesic segments of the boundary B, whose accelerations are outwardly normal to B.
3. Switch points, where geodesic switches from a boundary segment to an interior segment or vice-versa.
4. Intermittent points, which are the accumulation points of switch points.

In [3] it is shown that the acceleration at an intermittent point exists and must be zero. Thus a geodesic fails to have acceleration only at the switch points, and at those points the velocities are continuous, and one-sided accelerations exist.

The existence of intermittent points makes the variational analysis of geodesics in M difficult. In [1] an example is given of a geodesic whose intermittent points constitute a Cantor set of positive measure. On the other hands, by focusing on those M obtained from $\mathbb{R}^n$ by removing a locally analytic obstacle, *i.e.* an obstacle with boundary locally of the form $x_n = f(x_1, \cdots, x_{n-1})$ for a real analytic function f, [1] shows that a geodesic can have, in any segment of fine arc length, only a finite number of distinct switch points, hence no intermittent points at all. This is summarized in the following theorem.

Theorem 2 *Let $n > 1$ and let M be an $(n+2)$-dimensional analytic manifold-with-boundary embedded in $\mathbb{R}^{n+2}$ and equipped with the induced Riemannian structure. Denote the boundary surface of M by B and let γ be a geodesic on M parametrized by arc length s, with $\gamma(0) = p \in B$. Then there exists an $\epsilon > 0$ such that γ has no switch point for $0 < s < \epsilon$.*

The conclusion of Theorem 2 is trivial when $n = 1$, *i.e.* when the ambient space is $\mathbb{R}^2$. In fact, for an analytic boundary B in $\mathbb{R}^2$, a geodesic cannot have an accumulation of switch points, since between any two switch points there must be a point of B of zero Euclidean curvature. An infinite set of points of curvature zero, necessarily clustering at a point p,

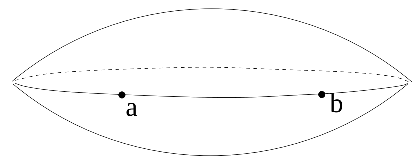

Figure 1: Example of manifold with boundary where bipointwise uniqueness fails

implies that all the derivatives after the first of B vanish at p and hence by analyticity the boundary B is a straight line. This argument fails in $\mathbb{R}^n$ for $n > 2$ since we may have vanishing directional second derivatives, but in different directions. It is not clear whether the conclusion of Theorem 2 can be generalized to the case of nonflat M.

The regularity analysis is carried one step further in [3] by considering geodesics near the boundary B. Let x_n be the distance from B. Starting from arbitrary (local) coordinates $x_1, \cdots, x_{n-1}$ of B, we can extend them locally to be constant on geodesics normal to B. Let γ be an arbitrary geodesic in M with coordinates $(x_1, \cdots, x_n)$. The curve on B with coordinates $(x_1, \cdots, x_{n-1})$ is the normal projection of γ to B and called the *tangential part* of γ. The *normal* part of γ is simply x_n. it is proved in [3] that the tangential part of a geodesic is C^2 with locally Lipschitz second derivative. Some convexity condition is also given for the normal part.

2.2 Uniqueness

Unless some bound is imposed on the curvature of the boundary B, there is no hope of getting bipointwise uniqueness of geodesics in M, even locally. Figure 1 shows an example due to [4], in which two spherical caps are glued together along a common circle (not great in either sphere). Between points a and b there are numerous geodesic segments, most of which oscillate back and forth across the edge. The segment along the edge is not a geodesic since it is nowhere locally distance minimizing but it is a limit of geodesics. Although the boundary is nonsmooth in this case, we can smooth the sharp edge to make the surface C^1 but with infinite normal curvature, and the above observations remain valid.

To get bipointwise uniqueness, the notion of tubular radius is introduced in [4]. Suppose M can be isometrically embedded in some Euclidean space N of the same dimension.

Definition 1 (Tubular radius) *A positive number r is a tubular radius for M in N if every point at distance r or less from M is the center of a closed ball which meets M at a single point.*

For M with tubular radius r, the Euclidean curvature of any geodesic γ on the boundary B is bounded above by $1/r$, *i.e.* $\ddot{\gamma} = kN_\gamma$ for some $k < 1/r$, where N_γ is the unit normal to B at γ.

Using a tubular radius, we can obtain an estimate of how fast two different geodesics starting from the same point converge. Suppose $r = 1/k$ is a tubular radius for M. Let γ and σ be geodesics in M having speed no more than one. Let $f(t) = \|\gamma(t) - \sigma(t)\|$ be the Euclidean displacement between corresponding points. Then except at the countably many points where f'' fails to exists, we have a differential inequality:

$$f'' \geq -k^2 f \tag{1}$$

with strict inequality when $f > 0$. Since $g(t) = A\sin(kt + b)$ is a solution of $g'' = -k^2 g$, it is not hard to see that if f and g agree at $t = t_0$ and $t = t_f$ for properly chosen A and b, then g dominates f. In particular,

Theorem 3 ([4]) *If r is a tubular radius of M, then two different geodesics in M starting from the same point must each travel more than πr before they can meet again.*

Considering the example of $\mathbb{R}^3$ with a ball of radius r removed, then πr is a sharp estimate.

As a result of the preceding theorem, one can prove the following version of bipointwise uniqueness of geodesics for manifold with positive tubular radius.

Theorem 4 (Bipointwise uniqueness neighborhood) *Suppose M has positive tubular radius. Then every point of M has a neighborhood U such that for every p, q in U,*

1. *there is a unique minimal geodesic segment joining p and q, and*
2. *there is no other geodesic segment joining p and q and lying in U.*

Although it was claimed in [4] that a stronger result was obtained by making U convex, *i.e.* the unique segments in condition 1 are contained in U, a formal proof has never appeared.

In another paper [3], Cauchy uniqueness is also examined. For manifolds without boundary, we have Cauchy uniqueness, which means that for arbitrary $p \in M$ and $v \in T_pM$, there is a unique geodesic (up to reparameterization) γ such that $\gamma(0) = p$ and $\gamma'(0) = v$. This is not the case for manifold M with boundary. For example, whenever there is a boundary direction in which the boundary bends away from the interior, there will be a one-parameter family of distinct geodesics of a given sufficiently small length which start in that direction. They are all involutes in the following sense.

Definition 2 (Involute) *An involute of a geodesic β in M is another geodesic with the same initial position, initial velocity and length as β, which consists of a segment in common with β followed by a nontrivial segment of the interior.*

It is shown in [3] that this is the only case Cauchy uniqueness can fail in manifold with boundary, at least locally.

Theorem 5 (Cauchy uniqueness for manifold with boundary) *Every boundary point of M has a neighborhood in which: if two geodesic segments with the same initial point, initial tangent vector and length do not coincide, then one of them has its right endpoint in the interior and is an involute of the other.*

2.3 Convergence

Convergence of geodesics is studied in [4]. Suppose that M has tubular radius r. Consider a sequence $\gamma_1, \gamma_2, \ldots$ of unit speed geodesic segments parametrized by $[0, l]$ such that $l < \pi r$. Moreover, suppose that the γ_i's are contained in a compact region.

Lemma 1 *If $\gamma_i(0)$ and $\gamma_i(l)$ converge, then γ_i and $\dot{\gamma}_i$ converge uniformly on $[0, l]$ to a geodesic segment γ and its velocity field $\dot{\gamma}$.*

Uniform convergence of γ_i to γ follows from equation (1) and the discussion thereafter in Section 2.2, while the convergence of $\dot{\gamma}_i$ is proved by using the estimate

$$\|(\gamma(s) - \gamma(u))/(s - u) - \dot{\gamma}(u)\| \leq |s - u|/2$$

and the triangle inequality. In general the speed of convergence for $\|\gamma_i - \gamma\|$ is asymptotically quadratic compared to that for $\|\dot{\gamma}_i - \dot{\gamma}\|$. From this lemma, we get the global result:

Theorem 6 (Convergence of geodesics) *If a sequence of geodesics γ_i converge pointwise, then the limit function is a geodesic γ, and the convergence of both γ_i and $\dot{\gamma}_i$ to γ and $\dot{\gamma}$ is uniform on closed bounded segments.*

3 Jacobi field

In manifold without boundary, A Jacobi field along a geodesic is obtained as the variation field of a deformation by a family of geodesics. Jacobi fields for manifolds with boundary are obtained in much the same way in [5].

We follows the notation of [5]. A vector field J along a geodesic γ in M will be called a *Jacobi field* if there is a sequence of geodesics γ_i converging to γ in the uniform topology, and a sequence of positive numbers u_i approaching 0 for which

$$\|J\| = \lim u_i^{-1} \, d(\gamma, \gamma_i),$$

and the unit vector in the direction of $J(t)$ is the limit of the initial unit vectors of the minimizing geodesics from $\gamma(t)$ to $\gamma_i(t)$. We say that the parametrized sequence (γ_i, u_i) approaches γ tangentially to J. Compared with the classic definition of Jacobi field, the one presented here is a "snapshot" of the differentiable deformation at a discrete sequence of epochs approaching 0. It is obvious that from a differentiable deformation by geodesics, we can always obtain (γ_i, u_i) satisfying the above conditions by taking snapshots, and the reverse is also true, *i.e.* every such (γ_i, u_i) can be "embedded" in a differentiable deformation. So this version of definition of Jacobi field is indeed equivalent to the classic one.

3.1 Existence

The main results in [5] regarding the existence of Jacobi fields in M can be summarized as follows: If a parametrized sequence of geodesics (γ_i, u_i) approaches a geodesic γ tangentially to well-defined vectors at either endpoint, and if no subsequence approaches with infinite speed at any intermediate point (*i.e.* if $d(\gamma(t), \gamma_i(t)) \leq Cu_i$ for some finite constant C), then some subsequence approaches γ tangentially to a Jacobi field.

3.2 First variation formula

The first variation formula for manifolds with boundary is derived in [5]. Assume without loss of generality that all the geodesics are defined on the interval $[0, 1]$.

Proposition 1 *For any geodesic γ of M and any parametrized sequence (γ_i, u_i) of geodesics converging to γ tangentially along a Jacobi field J,*

$$\lim u_i^{-1}[l(\gamma_i) - l(\gamma)] = l(\gamma)^{-1}[\langle J(1), \dot{\gamma}(1)\rangle - \langle J(0), \dot{\gamma}(0)\rangle]$$

where l denotes arc length.

Using the first variation formula, one can show that

Corollary 1 *The tangential component of a Jacobi field J is linear. Its normal component is again a Jacobi field (except possibly if the base geodesic γ meets J in an acute angle at a boundary end point of γ).*

3.3 K-convexity

Definition 3 (K-convexity) *A Jacobi field J along γ is K-convex if it satisfies the differential inequality $\|J\|'' \geq -Kv^2\|J\|$, where v is the speed of γ.*

K-convexity condition can be interpreted as: if $K > 0$, then on any parameter subinterval of length less than $\pi v/K$, the sinusoid $a\sin(\sqrt{K}vt - b)$ that coincides with $\|J\|$ at the endpoints is an upper bound for $\|J\|$. If $K \leq 0$, then the appropriate linear function or hyperbolic sinusoid is used instead of the sinusoid, with no bound on the parameter subinterval.

The following results is claimed in [5] regarding the regularity of Jacobi fields in M: A Jacobi field J in M is K-convex for some sufficiently large positive constant K. Moreover, J is continuous on the interior of its interval of definition and $\|J\|$ is upper semicontinuous at the endpoints. It follows from K-convexity and continuity that $\|J\|$ has the regularity properties of a convex function on the interior of its domain. In particular, $\|J\|$ has left and right derivatives everywhere; there are only countably many points where $\|J\|'$ fails to exist and at these points $\|J\|'$ has a positive jump; and $\|J\|''$ exists almost everywhere.

There is a concept closely related to K-convexity. A space has *curvature bounded above by K*, in the sense of Alexandrov, if every point has a neighborhood in which any minimizing geodesic triangle with vertices in the neighborhood has perimeter less than $2\pi/\sqrt{K}$ (if $K > 0$), and has each of its angles at most equal to the corresponding angle in a triangle with the same side lengths in the standard surface S_K of constant curvature K.

The following theorem is proved in [5].

Theorem 7 (Characterization Theorem) *Let M be a Riemannian manifold with boundary B. Then the following conditions are equivalent:*

1. *M has curvature bounded above by K in Alexandrov's sense.*
2. *All normal Jacobi fields in M are K-convex.*
3. *The sectional curvatures of the interior of M and the outward sectional curvatures of the boundary B do not exceed K (where an outward sectional curvature of B is one that corresponds to a tangent section all of whose normal curvature vectors point outward).*

When the boundary is empty, the above theorem corresponds to the Alexandrov's basic theorem equating upper bounds on Alexandrov curvature to those on sectional curvature [6], and is its extension to manifolds with boundary.

The characterization theorem places manifolds with boundary in the setting of Alexandrov's theory of spaces of curvature bounded above, and its extensions within Gromov's theory of hyperbolic groups. By a theorem of Gromov which extends the Hadamard-Cartan theorem to geodesic metric space with curvature bounded above by 0, one has the following immediate implication.

Corollary 2 *If for a simply connected, complete, connected Riemannian manifold with boundary, the sectional curvatures of the interior and the outward principle curvatures of the boundary are nonpositive, then any two points are joined by a unique geodesic, and the distance between any two geodesics is convex.*

Compared with Theorem 4, the condition in this corollary is intrinsic in the sense that M does not necessarily need to be embedded in a Euclidean space of the same dimension.

4 An interesting example

The following problem has its origin in aircraft conflict resolution. Given a set of n points $a_1, \ldots, a_n$ in some Riemannian manifold M such that the distance (in the Riemannian metric) between any two of them is greater than or equal to some positive constant r (we say that they satisfy the r-separation condition). Let $b_1, \ldots, b_n$ be another set of n points satisfying the r-separation condition. The problem is to find a set of n piecewise smooth

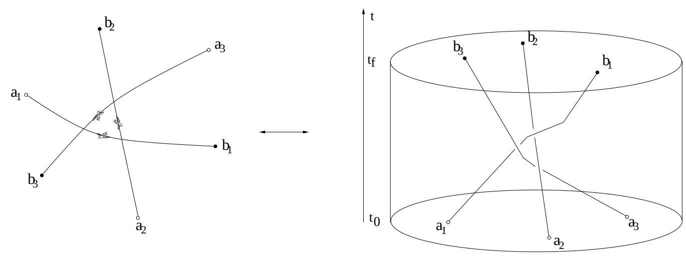

Figure 2: Correspondence between resolution maneuvers and braids

curves $\gamma_1, \ldots, \gamma_n$ in M such that $\gamma_i(0) = a_i$, $\gamma_i(1) = b_i$ for $i = 1, \ldots, n$, and for any $t \in [0,1]$, $\gamma_1(t), \ldots, \gamma_n(t)$ satisfy the r-separation condition, and such that

$$\frac{1}{2}\sum_{i=1}^{n}\int_0^1 \|\dot{\gamma}_i(t)\|^2\, dt$$

is minimized.

The presence of the separation constraints makes the usual variational analysis infeasible. However, by "piling" the state together, we transform the problem into finding the curve $\gamma = (\gamma_1, \ldots, \gamma_n)$ in $M^{(n)} = M \times \ldots \times M$ connecting $a = (a_1, \ldots, a_n)$ to $b = (b_1, \ldots, b_n)$ with the least energy. The separation constraint implies that γ cannot enter the region W defined by

$$W = \{(p_1, \ldots, p_n) : d(p_i, p_j) < r \text{ for some } i \neq j\}$$

Evidently the optimal γ is a distance minimizing geodesic from a to b in $M^{(n)} \backslash W$, a manifold with complicated and nonsmooth boundary.

The case when $M = \mathbb{R}^2$ is studied indirectly in [7] using the notion of braids. It is found that the fundamental group of $M \setminus W$ in this case is isomorphic to the group of pure braids $\mathbf{PB}_n$. In fact, for each resolution maneuver $\gamma = (\gamma_1, \ldots, \gamma_n)$, one can build its braid representation in the following way: Let $\beta_i : [0,1] \to \mathbb{R}^2 \times [0,1]$ be the curve defined by $\beta_i(t) = (\gamma_i(t), t)$, $\forall t \in [0,1]$ for $i = 1, \ldots, n$. Then $\{\beta_i\}_{i=1}^n$ is a set of n disjoint strings in $\mathbb{R}^2 \times [0,1]$ connecting n points $\{(a_i, 0)\}_{i=1}^n$ at the bottom to n points $\{(b_i, 1)\}_{i=1}^n$ on the top. Such a set of strings is called a *pure braid*. Figure 4 shows the correspondence between a 3-aircraft resolution maneuver and its braid representation. Fixing the end points of all the strings, two pure braids are called *isotopic* if one can be deformed continuously to the other in such a way that the end points are fixed and no two strings intersect each other anywhere during the deformation. Then there is a one-to-one correspondence between the isotopy classes of pure braids (which can be made into the group $\mathbf{PB}_n$ by the operation of concatenation) and the homotopy classes of paths in $\mathbb{R}^{2n} \setminus W$ connecting a to b.

The method proposed in [7] to find the optimal homotopy class of resolution maneuvers is a randomized algorithm based on the model of Brownian motion, hence irrelevant to this survey. Suppose we fix the homotopy class, the problem of finding the shortest curve from a to b within this particular class is also studied in [7]. Denote with $\mathcal{G}$ the orientation-preserving isometry group of $\mathbb{R}^2$, which is a Lie group consisting of translations, rotations and their compositions. Let $g : [0, 1] \to \mathcal{G}$ be a piecewise smooth curve in $\mathcal{G}$ such that both $g(0)$ and $g(1)$ are the identity map. Then for each piecewise smooth curve $\gamma : [0, 1] \to \mathbb{R}^{2n} \setminus W$, the new curve $g(\gamma)$ defined as

$$g(\gamma)(t) = (g_t(\gamma_1(t)), \dots , g_t(\gamma_n(t))), \quad \forall t \in [0, 1]$$

is also a piecewise smooth curve in $\mathbb{R}^{2n} \setminus W$ connecting a to b and of the same homotopy type as γ. $g(\gamma)$ can be thought of as a local perturbation of γ if g is close to the identity. The shortest curve γ^* should satisfy the condition that the energy of $g(\gamma^*)$ is never smaller than the energy of γ^* for any such g. Using this observation, a set of necessary conditions for the optimality of γ^* are derived in [7]. These necessary conditions are sufficient in the sense that the geodesic equations on different smooth components of the boundary ∂W can be derived from them. Obviously this method can be generalized to the case when M is a Lie group.

Many problems remain unsolved even in this case. For example, we have the following conjecture.

Conjecture 1 *Suppose M is obtained by removing from $\mathbb{R}^n$ a finite number of convex cylinders, which may or may not intersect. Then there is a unique geodesic within each homotopy class of paths connecting two arbitrary points a and b of M.*

Loosely speaking, if one can define the “universal covering” $\tilde{M}$ of M, which is itself a manifold with boundary, then the above conjecture can be alternatively stated as: any two points in $\tilde{M}$ are connected by a unique geodesic. In the above example, the obstacle W is the union of $n(n-1)/2$ cylinders intersecting in a complicated way, hence a special case of Conjecture 1. Some preliminary results are reported in [8].

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