AIRCRAFT CONFLICT DETECTION IN PRESENCE OF SPATIALLY CORRELATED WIND PERTURBATIONS*

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We study the automated aircraft conflict detection problem. Specifically, we introduce a method for estimating the probability that two aircraft flying at the same altitude get closer than a prescribed safe distance thus causing a conflict. The correlation between the wind perturbations to the aircraft positions is taken into consideration when computing the probability of conflict. The procedure used to estimate the probability of conflict is based on the introduction of a Markov chain approximation of the stochastic processes modeling the aircraft motions.

Introduction

In this paper, we consider an encounter situation where two aircraft are flying at the same altitude in a certain region of the airspace, and each aircraft is trying to follow a prescribed flight plan. The encounter is said to generate a "conflict" if the two aircraft get closer than a minimum allowed distance, currently 5 nautical miles (nmi) in en-route airspace and 3 nmi in airspace close to the airports[1].

The procedure used to prevent the occurrence of a conflict typically consists of two phases, namely, aircraft conflict detection and aircraft conflict resolution. Automated tools are currently being studied to support the air traffic controllers (ATCs) in performing these tasks. A comprehensive overview of the methods proposed in the literature can be found in[2]. In automated conflict detection, models for predicting the aircraft future positions are introduced and the possibility that a conflict would happen within a certain time horizon is evaluated based on these models ([3–6]). If a conflict is predicted, then the aircraft flight plans are modified in the conflict resolution phase so as to avoid the actual occurrence of the predicted conflict. The cost of the resolution action in terms of, for example, delay, fuel consumption, deviation from originally planned itineraries, is usually taken into account when selecting the new flight plans ([7–14]).

In this paper we focus on the conflict detection issue and adopt a probabilistic viewpoint in addressing it. We describe the aircraft motion as the solution to a stochastic differential equation. We then quantify the criticality of a two-aircraft encounter by means of the probability that a conflict occurs within a certain time horizon: the higher the probability of conflict, the more critical the encounter situation. The methods proposed in the literature to compute the probability of conflict are generally based on the description of the aircraft future positions first proposed in[4]. In[4], each aircraft motion is described as a Gaussian random process whose variance grows in time. The processes modeling the motions of different aircraft are assumed to be uncorrelated. This assumption may be unrealistic in practice, and can cause erroneous evaluations of the probability of conflict, since wind is the main source of uncertainty and the correlation between the wind perturbations affecting the aircraft positions is stronger when two aircraft are closer to each other.

Inspired by[15], we introduce a model for a two-aircraft encounter, where the correlation between the wind perturbations affecting the aircraft positions is taken into account. We assume that each aircraft is trying to follow a flight plan, typically consisting of straight lines between given way points traveled at constant speed. The aircraft actual motion may deviate from its planned one because of different sources of uncertainty. In particular, we assume that the possible errors in tracking the flight plan are mainly due to the wind influence on the aircraft velocity. This leads to a model that tries to achieve a compromise between

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the two conflicting objectives of being realistic and at the same time still simple enough for the problem of evaluating the probability of conflict to be computationally tractable. The inspiring model in[15] is in fact interesting but too complicated to conceive a method for computing the probability of conflict based on it.

The computation of the probability of conflict based on the model proposed in this paper rests on the introduction of a Markov chain whose state space is obtained by discretizing the region of the airspace where the encounter takes place. By appropriately choosing its transition matrix, one can guarantee that the Markov chain converges weakly to the stochastic process modeling the aircraft motion as the grid size approaches zero. The probability of conflict can then be approximated by the corresponding quantity associated with the Markov chain, which is much easier to compute. A procedure for estimating the probability of conflict map, i.e., the probability of conflict as a function of the current positions of the two aircraft, is given. This procedure is based on the assumption that the wind correlation structure is homogeneous and isotropic in space. As expected, the computations show that the wind correlation effect cannot be neglected when estimating the probability of conflict.

The approximating Markov chain has a state space of dimension four. Under the additional assumption that the wind field is affine in space, the problem becomes simpler and can be solved by using an approximating Markov chain with a two-dimensional state space, which greatly reduces the computation load.

The rest of the paper is organized as follows. First of all, we describe the model of a two-aircraft encounter and precisely formalize the problem of computing the probability of conflict. We then propose a Markov chain approximation scheme to estimate the probability of conflict, starting from the case of an affine wind field and then extending the approach to the case of a general wind field. We present simulation examples for both cases. Finally, we draw some conclusions.

Model

In this paper we consider an encounter of two aircraft flying at the same altitude in a certain region of the airspace. We refer to the two aircraft as "aircraft 1" and "aircraft 2". Given that the aircraft are flying at the same altitude, the airspace can be identified with $\mathbb{R}^2$. Denote by $T = [0, t_f]$ the time interval of interest, where 0 is the current time instant, and t_f is a positive real number (possibly infinity) representing the look-ahead time horizon. We assume that during the time interval T each aircraft, say, aircraft $i \in \{1, 2\}$, tries to follow a flight plan specified by its velocity profile $u_i : T \to \mathbb{R}^2$, so that at time $t \in T$ aircraft i plans to fly at velocity $u_i(t)$. According to the common practice in the current Air Traffic Management (ATM) systems where aircraft are advised to follow piecewise linear motions specified by a series of timed way-points, the velocity profiles u_1 and u_2 are usually chosen to be piecewise constant functions.

We start by introducing a general model for aircraft motions, which will be simplified in later sections in order to compute the probability of conflict. Due to the presence of wind perturbations, the actual velocity of aircraft i at time $t \in T$ is the sum of $u_i(t)$ and an additional term representing the wind contribution. The latter can be further decomposed into two components: i) a deterministic term representing the nominal wind velocity, which may depend on the aircraft location and time t, and is assumed to be known to the ATC through measurements or forecast; and ii) a stochastic term representing the effect of air turbulence and errors in the wind speed measurements and forecast.

Denote by $X_i(t)$ the position of aircraft i at time $t \in T$. Then according to the above description, $X_1(\cdot)$ and $X_2(\cdot)$ are governed by the following stochastic differential equations over the time interval T:

$$dX_1(t) = u_1(t)dt + f(X_1, t)dt + g(X_1, t)dB(X_1, t), \quad (1)$$

$$dX_2(t) = u_2(t)dt + f(X_2, t)dt + g(X_2, t)dB(X_2, t), \quad (2)$$

where $f : \mathbb{R}^2 \times T \to \mathbb{R}^2$ is a time-varying vector field on $\mathbb{R}^2$ such that $f(x, t)$ represents the nominal wind velocity at position x and at time t. We call f the *wind field*. $B(\cdot, \cdot)$ is a time-varying random field on $\mathbb{R}^2$ modeling (the integral of) air turbulence perturbations to aircraft velocity as well as wind speed measurement errors, and has the following properties:

- for each fixed $x \in \mathbb{R}^2$, $B(x, \cdot)$ is a standard two-dimensional Brownian motion. Hence $dB(x, t)/dt$ can be thought of as a 2-D white noise;
- $B(\cdot, \cdot)$ is time increment independent. This implies, in particular, that the collections of random variables $\{B(x, t_2) - B(x, t_1)\}_{x \in \mathbb{R}^2}$ and $\{B(x, t_4) - B(x, t_3)\}_{x \in \mathbb{R}^2}$ are independent for any $t_1, t_2, t_3, t_4 \in T$, with $t_1 \le t_2 \le t_3 \le t_4$;
- for any $t_1 \le t_2$ in T, $\{B(x, t_2) - B(x, t_1)\}_{x \in \mathbb{R}^2}$ is an (uncountable) collection of Gaussian random variables with zero mean and covariance

$$E\big\{[B(x, t_2) - B(x, t_1)][B(y, t_2) - B(y, t_1)]^T\big\}$$
$$= h(\|x - y\|)(t_2 - t_1)I_2, \ \forall x, y \in \mathbb{R}^2,$$

  where I_2 is the 2-by-2 identity matrix, and $h : \mathbb{R}_+ \to \mathbb{R}$ is a continuous, decreasing function with $h(0) = 1$ and $h(\infty) = 0$. In addition, h has to be non-negative definite in the sense that the k-by-k matrix $[h(\|x_i - x_j\|)]_{i,j=1}^k$ is non-negative definite for arbitrary $x_1, \ldots, x_k \in \mathbb{R}^2$ and positive integer k. See[16] for more on this condition.

Remark 1 The choice of h reflects the fact that the closer the two aircraft, the more similar the wind influences on their positions. As the aircraft move farther

away from each other, the wind perturbations to their positions become nearly independent.

As a random field, $B(\cdot,\cdot)$ is Gaussian, stationary (its finite dimensional distributions remain unchanged when the origin of $\mathbb{R}^2 \times T$ is shifted), and isotropic in space (its finite dimensional distributions are not affected by a change of orthonormal coordinates in $\mathbb{R}^2$). Finally, $g : \mathbb{R}^2 \times T \to \mathbb{R}^{2\times 2}$ is the term introduced to modulate the variance of the random perturbations.

In this paper we shall assume that $g \equiv \sigma I_2$ for some constant $\sigma > 0$. This implies that the random contribution of the wind to the aircraft position after the modulation of g is isotropic, which is a reasonable assumption since in our model there is no apparent direction in $\mathbb{R}^2$ preferential to others. Equations (1) and (2) can then be rewritten as

$$dX_1(t) = u_1(t)dt + f(X_1,t)dt + \sigma dB(X_1,t), \quad (3)$$

$$dX_2(t) = u_2(t)dt + f(X_2,t)dt + \sigma dB(X_2,t), \quad (4)$$

with initial condition $X_1(0)$ and $X_2(0)$, respectively.

A *conflict* is defined to be the event that two aircraft get closer than a certain distance r. Our objective is to compute the probability that a conflict occurs within the time interval T, namely,

$$P_c \triangleq P\{\|X_1(t) - X_2(t)\| \le r \text{ for some } t \in T\}.$$

In the sequel we shall refer to P_c as the *probability of conflict*. Depending on $T = [0, t_f]$, this could be either a finite horizon ($t_f < \infty$) or infinite horizon ($t_f = \infty$) problem. Since P_c is difficult to compute analytically, in this paper we shall pursue the more realistic goal of obtaining a numerical estimate of P_c.

Affine Wind Field Case

In this section we consider the case when the wind field $f(x,t)$ is affine in x, i.e.,

$$f(x,t) = R(t)x + d(t), \quad \forall x \in \mathbb{R}^2, t \in T,$$

where $R : T \to \mathbb{R}^{2\times 2}$ and $d : T \to \mathbb{R}^2$ are continuous functions. We shall show that in this case there is a simple way to approximate the probability of conflict.

The relative position Y and the planned relative velocity v of aircraft 1 and aircraft 2 are given by

$$Y \triangleq X_2 - X_1, \quad v \triangleq u_2 - u_1,$$

respectively. Since the positions of the two aircraft, X_1 and X_2, are governed by equations (3) and (4), by subtracting (3) from (4), we have

$$dY(t) = v(t)dt + R(t)Y(t)dt + \sigma d[B(X_1,t) - B(X_2,t)]. \quad (5)$$

Let $Z(t) \triangleq B(X_2,t) - B(X_1,t)$. For each fixed X_1 and X_2, $Z(t)$ is Gaussian with zero mean and covariance

$$\begin{aligned} &E\{[Z(t_2) - Z(t_1)][Z(t_2) - Z(t_1)]^T\} \\ =&[2 - 2h(\|X_2 - X_1\|)](t_2 - t_1)I_2, \ \forall t_1 \le t_2. \end{aligned}$$

Note also that $Z(0) = 0$. Therefore, in terms of distribution we have for fixed X_1 and X_2

$$Z(t) \stackrel{d}{=} \sqrt{2[1 - h(\|X_2 - X_1\|)]}\, W(t), \quad (6)$$

where $W(t)$ is a standard 2-D Brownian motion. In general, since X_1, X_2 are time-varying stochastic processes whose outcomes depend on B, hence on Z, (6) is not valid. However, if the function h is flat at $\|X_1 - X_2\| \ge r$, then $h(\|X_2 - X_1\|)$ can be thought of as constant near each time epoch. So (6) still holds approximately, and in turn $dZ(t)$ can be approximated by $\sqrt{2[1 - h(\|Y\|)]}\, dW(t)$. We can thus approximate equation (5) weakly by

$$dY(t) = v(t)dt + R(t)Y(t)dt + \sigma\sqrt{2[1 - h(\|Y\|)]}\, dW(t). \quad (7)$$

To compute the probability of conflict numerically, we consider an open domain $U \in \mathbb{R}^2$ that contains the origin and has a compact support. U should be large enough so that the situation can be declared safe once the relative position Y of the two aircraft wanders outside U. With reference to the domain U, the probability of conflict can be expressed as

$$P_c^U \triangleq P\{Y \text{ hits } D \text{ before hitting } U^c \text{ within } T.\} \quad (8)$$

where $D \in \mathbb{R}^2$ is the closed disk of radius r centered at the origin modeling the protection zone surrounding each aircraft. Implicit in the above definition is that if Y hits neither D nor U^c during T, still no conflict occurs. For the purpose of computing (8), we can assume that in equation (7), Y is defined on the open domain $U \setminus D$ with initial condition $Y(0)$, and that it is stopped as soon as it hits the boundary $\partial U \cup \partial D$.

Markov Chain Approximation

We now describe an approach to approximate the solution $Y(\cdot)$ to equation (7) defined on $U \setminus D$. The idea is to discretize $U \setminus D$ into grid points and define on the grid points a Markov chain that converges weakly to $Y(\cdot)$ as the grid size approaches zero. A good estimate of P_c^U is then provided by the corresponding quantity of the Markov chain at a small grid size,.

We first introduce some notations. Fix a grid size $\delta > 0$. Denote by $\delta\mathbb{Z}^2$ the integer grids of $\mathbb{R}^2$ scaled by δ, namely, $\delta\mathbb{Z}^2 = \{(m\delta, n\delta) |\ m, n \in \mathbb{Z}\}$. Each grid point in $\delta\mathbb{Z}^2$ has four immediate neighbors (left, right, down, up positions). Define $S = (U \setminus D) \cap \delta\mathbb{Z}^2$, which consists of all those grid points in $\delta\mathbb{Z}^2$ that lie inside U but outside D. The interior of S, denoted by S^0, consists of all those points in S whose all four neighbors in $\delta\mathbb{Z}^2$ belong to S as well. The boundary of S is defined to be $\partial S = S \setminus S^0$, and is the union of two disjoint sets: $\partial S = \partial S_U \cup \partial S_D$, where points in ∂S_U have at least one neighbor outside U, and points in ∂S_D have at least one neighbor inside D.

We now define a time-inhomogeneous Markov chain $\{Q_k,\, k \ge 0\}$ on the state space S as follows:

1. each state in ∂S is an absorbing state, i.e., the state of the chain remains unchanged upon arriving at ∂S;
2. starting from a $q \in S^0$, the chain jumps to one of its four neighbors: $q_l = q + (-\delta, 0)$, $q_r = q + (\delta, 0)$, $q_d = q + (0, -\delta)$, $q_u = q + (0, \delta)$, or stays at q according to transition probabilities determined by its current location q and the current time step k as follows:

$$P\{Q_{k+1} = q' \,|\, Q_k = q\} = \begin{cases} p_l^k(q) = \dfrac{\exp(-\delta\xi_q^k)}{C_q^k}, & q' = q_l \\ p_r^k(q) = \dfrac{\exp(\delta\xi_q^k)}{C_q^k}, & q' = q_r \\ p_d^k(q) = \dfrac{\exp(-\delta\eta_q^k)}{C_q^k}, & q' = q_d \\ p_u^k(q) = \dfrac{\exp(\delta\eta_q^k)}{C_q^k}, & q' = q_u \\ p_o^k(q) = \dfrac{\chi_q^k}{C_q^k}, & q' = q. \end{cases} \tag{9}$$

Parameters in the above expression are given by

$$\begin{aligned} \xi_q^k &= \frac{[v(k\Delta t) + R(k\Delta t)q]_1}{2\sigma^2[1 - h(\|q\|)]}, \\ \eta_q^k &= \frac{[v(k\Delta t) + R(k\Delta t)q]_2}{2\sigma^2[1 - h(\|q\|)]}, \\ \chi_q^k &= \frac{1}{\lambda\sigma^2[1 - h(\|q\|)]} - 4, \\ C_q^k &= 2\,\mathrm{csh}(-\delta\xi_q^k) + 2\,\mathrm{csh}(-\delta\eta_q^k) + \chi_q^k, \end{aligned}$$

where for an arbitrary vector $z \in \mathbb{R}^2$, $[z]_i$ denotes its i-th component, $i = 1, 2$. $\lambda > 0$ is a constant that has to be chosen small enough such that χ_q^k defined above is positive for all q and all k. In particular, this is guaranteed if $0 < \lambda \le (4\sigma^2)^{-1}$. $\Delta t > 0$ is the amount of time elapsed between any two successive discrete time steps k and $k+1$, $k \ge 0$. Here, we set $\Delta t = \lambda\delta^2$.

For each time step k and each state $q \in S^0$, define

$$m_q^k = \frac{1}{\Delta t} E\{Q_{k+1} - Q_k \,|\, Q_k = q\},$$

$$V_q^k = \frac{1}{\Delta t} E\{(Q_{k+1} - Q_k)(Q_{k+1} - Q_k)^T \,|\, Q_k = q\}.$$

Direct computation shows that

$$m_q^k = \frac{2}{\lambda\delta C_q^k} \begin{bmatrix} \mathrm{sh}(\delta\xi_q^k) \\ \mathrm{sh}(\delta\eta_q^k) \end{bmatrix},$$

$$V_q^k = \frac{2}{\lambda C_q^k} \begin{bmatrix} \mathrm{csh}(\delta\xi_q^k) & 0 \\ 0 & \mathrm{csh}(\delta\eta_q^k) \end{bmatrix}.$$

If for each $\delta > 0$ we choose q to be a point in S^0 closest to a fixed point $y \in U \setminus D$, then it can be easily verified that as $\delta \to 0$, $m_q^k \to v(k\Delta t) + R(k\Delta t)y$, $V_q^k \to 2\sigma^2[1 - h(\|y\|)]I_2$. Suppose that the chain $\{Q_k, k \ge 0\}$ starts from a point $\bar{q} \in S$ closest to $Y(0)$. Then, by Theorem 8.7.1 in[17] (see also[18]), we conclude that

Proposition 1 *Fix $\delta > 0$ and consider the corresponding Markov chain $\{Q_k, k \ge 0\}$. Denote by $\{Q(t), t \ge 0\}$ the stochastic process that is equal to Q_k on the time interval $[k\Delta t, (k+1)\Delta t)$ for all k. Then as $\delta \to 0$, $\{Q(t), t \ge 0\}$ converges weakly to the solution $\{Y(t), t \ge 0\}$ to equation (7) defined on $U \setminus D$ with absorption on the boundary $\partial U \cup \partial D$.*

Let $k_f \triangleq \lfloor \frac{t_f}{\Delta t} \rfloor$ be the largest integer not exceeding $t_f/\Delta t$ ($k_f = \infty$ if $t_f = \infty$). Because of the weak convergence of $Q(t)$ to $Y(t)$, if we fix a small δ, then the probability P_c^U in (8) can be approximated by the corresponding probability

$$\begin{aligned} P_{c,\delta}^U \triangleq & P\{Q_{k_f} \in \partial S_D\} = P\{Q_k \text{ hits } \partial S_D \\ & \text{before hitting } \partial S_U \text{ within } 0 \le k \le k_f\}, \end{aligned} \tag{10}$$

with $\{Q_k, k \ge 0\}$ starting from a $\bar{q} \in S$ closest to $Y(0)$.

Remark 2 In a time interval of length $\Delta t = \lambda\delta^2$, the maximal distance that the Markov chain can travel is a unitary grid step δ. Thus given $v(t)$, $R(t)$, $t \in T$ and U, for $Y(t)$ to be approximated by the Markov chain, the quantity $\|v(t) + R(t)y\|$, $t \in T$, $y \in U$, has to be upper bounded roughly by $1/\lambda\delta$, which is $4\sigma^2/\delta$ if we choose λ to be $(4\sigma^2)^{-1}$. This condition translates into an upper bound on the admissible values for δ.

We next describe the procedures to compute $P_{c,\delta}^U$ in the finite and infinite horizon cases. In both cases, it is useful to refer to the following set of functions

$$P_{c,\delta}^{(k)}(q) \triangleq P\{Q_{k_f} \in \partial S_D \,|\, Q_k = q\}, \tag{11}$$

which are defined for $q \in S$ and indexed by $k = 0, 1, \ldots, k_f$. Since the chain $\{Q_k, k \ge 0\}$ starts at $\bar{q}$ at $k = 0$, the desired quantity $P_{c,\delta}^U$ defined in (10) is now given by $P_{c,\delta}^U = P_{c,\delta}^{(0)}(\bar{q})$.

To compute $P_{c,\delta}^{(0)}$, fix a k such that $0 \le k < k_f$. Then $P_{c,\delta}^{(k)} : S \to \mathbb{R}$ satisfies the following equation

$$P_{c,\delta}^{(k)}(q) = \begin{cases} p_o^k(q)P_{c,\delta}^{(k+1)}(q) + p_l^k(q)P_{c,\delta}^{(k+1)}(q_l) \\ +p_r^k(q)P_{c,\delta}^{(k+1)}(q_r) + p_d^k(q)P_{c,\delta}^{(k+1)}(q_d) \\ +p_u^k(q)P_{c,\delta}^{(k+1)}(q_u), & \text{if } q \in S^0 \\ 1, & \text{if } q \in \partial S_D \\ 0, & \text{if } q \in \partial S_U \end{cases} . \tag{12}$$

The procedures we propose below will determine the whole set of functions $P_{c,\delta}^{(k)} : S \to \mathbb{R}$ for $k = 0, 1, \ldots, k_f$. This has the advantage that at any future time $t \in [0, t_f]$, should the remaining flight plans be unchanged, an estimate of the probability of conflict over the new time horizon $[t, t_f]$ is readily available, eliminating the need for a re-computation.

Finite Horizon Case

In the finite horizon case ($t_f < \infty$), the probability $P^U_{c,\delta} = P^{(0)}_{c,\delta}(\bar{q})$ can be computed directly in a recursive way by iterating equation (12) backward k_f times starting from $k = k_f - 1$ and using the initialization $P^{(k_f)}_{c,\delta}(q) = \bar{P}(q)$, $q \in S$, where

$$\bar{P}(q) = \begin{cases} 1, & \text{if } q \in \partial S_D \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

We now summarize the procedure to compute an approximation of P^U_c in the finite horizon case.

Algorithm 1 Given $Y(0)$, $v(t) = u_2(t) - u_1(t)$ and $R(t)$, $t \in T$, then

1. Fix $\delta > 0$ and $\lambda \in (0, \frac{1}{4\sigma^2}]$. Set $\Delta t = \lambda\delta^2$. Define the Markov chain $\{Q_k, k \geq 0\}$ on $S = (U \backslash D) \cap \delta\mathbb{Z}^2$ with transition probabilities given by (9).

2. Set $\bar{k} = k_f$ and initialize $P^{(\bar{k})}_{c,\delta}$ with $\bar{P}$ in (13).

3. For $k = \bar{k} - 1, \ldots, 0$, compute $P^{(k)}_{c,\delta}$ from $P^{(k+1)}_{c,\delta}$ according to equation (12).

4. Output $P^U_{c,\delta} = P^{(0)}_{c,\delta}(\bar{q})$ for a $\bar{q} \in S$ closest to $Y(0)$.

Remark 3 For a fixed δ, the size of the state space S is $O(1/\delta^2)$. Since the number of iterations is given by $k_f \simeq t_f/\Delta t = O(1/\delta^2)$, the running time of Algorithm 1 grows as $O(1/\delta^4)$ as $\delta \to 0$. Accuracy in estimating P_c and computation time should be appropriately balanced when choosing δ.

Infinite Horizon Case

In the infinite horizon case $t_f = \infty$, so Algorithm 1 cannot be applied directly since it would take infinitely many iterations. In this section we consider a special case in which this difficulty can be easily overcome.

We start by rewriting the iteration law (12) in matrix form. Arrange the sequence $\{P^{(k)}_{c,\delta}(q), q \in S^0\}$ into a long column vector according to some fixed ordering of the points in S^0, and denote it by $\mathbf{P}^{(k)}_{c,\delta} \in \mathbb{R}^{|S^0|}$. Here $|S^0|$ is the cardinality of S^0. Rewrite (12) as

$$\mathbf{P}^{(k)}_{c,\delta} = \mathbf{A}^{(k)}\mathbf{P}^{(k+1)}_{c,\delta} + \mathbf{b}^{(k)} \tag{14}$$

for suitably chosen matrix $\mathbf{A}^{(k)} \in \mathbb{R}^{|S^0| \times |S^0|}$ and vector $\mathbf{b}^{(k)} \in \mathbb{R}^{|S^0|}$. Note that $\mathbf{A}^{(k)}$ is a sparse positive matrix with the property that the sum of its elements on each row is smaller than or equal to 1, where equality holds if and only if that row corresponds to a point in $(S^0)^0$, the interior of S^0 consisting of all those points in S^0 whose four neighbors all belong to S^0. On the other hand, $\mathbf{b}^{(k)}$ is a positive vector with nonzero elements on exactly those rows corresponding to points on the boundary $\partial(S^0) = S^0 \setminus (S^0)^0$ of S^0. Both $\mathbf{A}^{(k)}$ and $\mathbf{b}^{(k)}$ depend on the grid size δ. We do not write it explicitly to simplify the notation.

Suppose that from some time instant t_c on, both $v(t)$ and $R(t)$ remain constant. These are typical assumptions in the infinite horizon case since the ATC only has the aircraft flight plans and wind field information in the near future. Under these assumptions, we have that $\mathbf{A}^{(k)} \equiv \mathbf{A}$ and $\mathbf{b}^{(k)} \equiv \mathbf{b}$ for $k > k_c \triangleq \lfloor \frac{t_c}{\Delta t} \rfloor$. Hence, for $k > k_c$ equation (14) becomes

$$\mathbf{P}^{(k)}_{c,\delta} = \mathbf{A}\mathbf{P}^{(k+1)}_{c,\delta} + \mathbf{b}. \tag{15}$$

We next address the problem of computing $\mathbf{P}^{(k_c+1)}_{c,\delta}$. Once we have determined $\mathbf{P}^{(k_c+1)}_{c,\delta}$, we can execute Algorithm 1 with step 2 replaced by

2'. Set $\bar{k} = k_c + 1$ and initialize $P^{(\bar{k})}_{c,\delta}$ with $P^{(k_c+1)}_{c,\delta}$.

to determine the approximation $P^{(0)}_{c,\delta}(\bar{q})$ of P^U_c.

The procedure to compute $\mathbf{P}^{(k_c+1)}_{c,\delta}$ rests on the following two lemmas, whose proofs are omitted here.

Lemma 1 *The eigenvalues of* $\mathbf{A}$ *are all in the interior of the unit disk of the complex plane.*

Lemma 2 *Consider equation*

$$\mathbf{P}^{(k)} = \mathbf{A}\mathbf{P}^{(k+1)} + \mathbf{b}. \tag{16}$$

i) There is a unique $\mathbf{P} \in \mathbb{R}^{|S^0|}$ *satisfying*

$$\mathbf{P} = \mathbf{A}\mathbf{P} + \mathbf{b}. \tag{17}$$

ii) Starting from any initial value $\mathbf{P}^{(k_0)}$ *at some* k_0 *and iterating equation* (16) *backward in time,* $\mathbf{P}^{(k)}$ *converges to the fix point* $\mathbf{P}$ *as* $k \to -\infty$. *Moreover, if* $\mathbf{P}^{(k_0)} \geq \mathbf{P}$, *then* $\mathbf{P}^{(k)} \geq \mathbf{P}$ *for all* $k \leq k_0$. *Conversely, if* $\mathbf{P}^{(k_0)} \leq \mathbf{P}$, *then* $\mathbf{P}^{(k)} \leq \mathbf{P}$ *for all* $k \leq k_0$. *Note that here the symbols* $\geq$ *and* $\leq$ *denote component-wise comparison between vectors.*

Lemma 2 shows that equation (15) has a fixed point $\mathbf{P}$ to which $\mathbf{P}^{(k)}$ obtained by iterating from any initial condition converges as $k \to -\infty$. Such a fixed point is the desired quantity $\mathbf{P}^{(k_c+1)}_{c,\delta}$. Thus one way to compute $\mathbf{P}^{(k_c+1)}_{c,\delta}$ is to solve the linear equation $(I - \mathbf{A})\mathbf{P}^{(k_c+1)}_{c,\delta} = \mathbf{b}$ directly, using sparse matrix computation tools if possible. In our simulations, we determined $\mathbf{P}^{(k_c+1)}_{c,\delta}$ by iterating equation (16) starting at some k_0 from two initial conditions $\mathbf{P}^{(k_0)}_l$ and $\mathbf{P}^{(k_0)}_u$ that are respectively a lower bound and an upper bound of $\mathbf{P}$ (for example, one can choose $\mathbf{P}^{(k_0)}_l$ to be identically 0 on S^0 and $\mathbf{P}^{(k_0)}_u$ to be identically 1 on S^0). By Lemma 2, the iterated results at every $k \leq k_0$ for the two initial conditions will provides a lower bound and an upper bound of $\mathbf{P}^{(k_c+1)}_{c,\delta}$, respectively, which

also converge toward each other (hence to $\mathbf{P}_{c,\delta}^{(k_c+1)}$ as well) as $k \to -\infty$. By running the iterations for the upper and lower bounds in parallel we can determine an approximation of $\mathbf{P}_{c,\delta}^{(k_c+1)}$ within any accuracy.

Remark 4 As $\delta \to 0$, the size of matrix $\mathbf{A}$ becomes larger. Moreover, the ratio $|(S^0)^0|/|S^0| \to 1$. Hence $\mathbf{A}$ will have an eigenvalue close to 1 whose corresponding eigenvector is close to $(1, \ldots, 1)$. This causes slower convergence for the iteration (16) and numerical problems for the solution to the fixed point equation (17).

Extension to the case when the aircraft current positions are uncertain

The procedure for estimating P_c^U can be extended to the case when the aircraft current positions $X_1(0)$ and $X_2(0)$ are not known precisely, and they are described as random variables with a certain joint distribution. First, define the probability of conflict map as $p_c^U(y) \triangleq P\{Y \text{ hits } D \text{ before hitting } U^c \text{within } T | Y(0) = y\}$, which assigns to each $y \in U \setminus D$ the probability of conflict over the time horizon T when the two aircraft initial relative position is $Y(0) = y$. In the case addressed here, $Y(0)$ is a random variable. If we denote by $\mu_Y(y)$, $y \in U \setminus D$, the distribution of $Y(0)$, the probability of conflict can be expressed in terms of the probability of conflict map as $P_c^U = \int_{U \setminus D} p_c^U(y) d\mu_Y(y)$. This integral reduces to a finite summation when approximating p_c^U with $P_{c,\delta}^{(0)}$. Based on the preceding analysis, for each $y \in U \setminus D$, $p_c^U(y)$ can in fact be approximated by $P_{c,\delta}^{(0)}(q_y)$ defined in (11) for small δ, q_y being a closest point to y in the grid space $S = (U \setminus D) \cap \delta\mathbb{Z}^2$.

Examples

We now present a few examples to illustrate Algorithm 1. Unless otherwise stated, in all of the examples we use the parameters listed below. The safe distance is $r = 3$. The parameter σ and the function h appearing in equation (7) are respectively $\sigma = 1$ and $h(w) = \exp(-\beta w)$, $w \geq 0$, where β is a positive constant. The time interval of interest is $T = [0, 40]$. The relative velocity of the two aircraft during the time horizon T is given by

$$v(t) = \begin{cases} (2,0), & 0 \leq t < 10; \\ (0,1), & 10 \leq t < 20; \\ (2,0), & 20 \leq t \leq 40. \end{cases}$$

Based on the choices of T and $v(t)$, $t \in T$, the domain U is chosen to be $(-80, 10) \times (-40, 10)$. Finally, λ is set equal to $\lambda = (4\sigma^2)^{-1} = 0.25$, and $\delta = 1$.

Example 1 Suppose first that the wind field is identically zero: $R(t) \equiv 0$, $d(t) \equiv 0$, for $t \in T$. Let $\beta = 1/5$, so that $h(w) = \exp(-w/5)$ for $w \geq 0$. By executing Algorithm 1, we compute the set of functions $P_{c,\delta}^{(k)} : S \to [0,1]$, $k = 0, 1, \ldots, k_f$. In Fig. 1

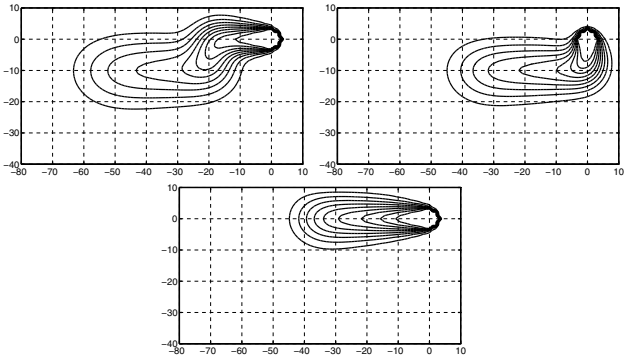

Fig. 1 Level curves of the estimated probability of conflict over $T = [t, 40]$ as a function of the aircraft relative position at time t. Top left: $t = 0$; Top right: $t = 10$; Bottom: $t = 20$ ($\beta = 1/5$).

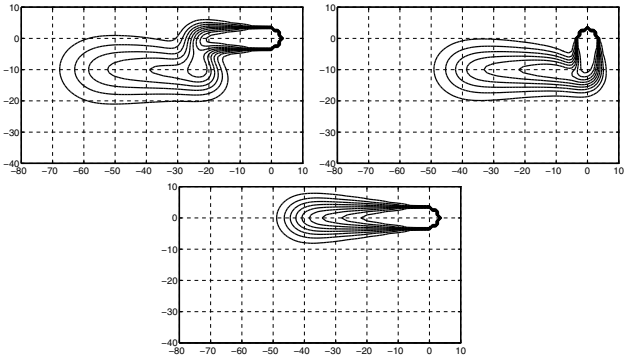

Fig. 2 Level curves of the estimated probability of conflict over the time horizon $[t, 40]$ as a function of the aircraft relative position at time t. Top left: $t = 0$; Top right: $t = 10$; Bottom: $t = 20$ ($\beta = 1/20$).

we plot the level curves of $P_{c,\delta}^{(k)}$, for $k = \lfloor t/\Delta t \rfloor$, at $t = 0, 10, 20$. For each t, $P_{c,\delta}^{(\lfloor t/\Delta t \rfloor)}$ represents an estimate of the probability of conflict over the time horizon $[t, t_f]$ as a function of the aircraft relative position at time t. As one can expect, the probability of conflict over $[t, t_f]$ takes higher values along the nominal path, which is the path traced by a point that starts from the origin at time $t_f = 40$ and moves backward in time according to the nominal relative velocity $v(\cdot)$ until time t. Furthermore, as the relative positions between the aircraft at time t moves farther way from that path, the probability of conflict decreases. Experiments (not reported here) show that the smaller the variance parameter σ, the faster this decrease.

Example 2 This example differs from the previous one only in the value of β, which is now set equal to $\beta = 1/20$. Then $h(w) = \exp(-w/20)$ for $w \geq 0$, which decreases much more slowly than in the previous case as w increases. Since h characterizes the strength of spatial correlation in the isotropic random field $B(\cdot, \cdot)$, this means that the random components of the wind contributions to the two aircraft velocities tend to be more correlated to each other than

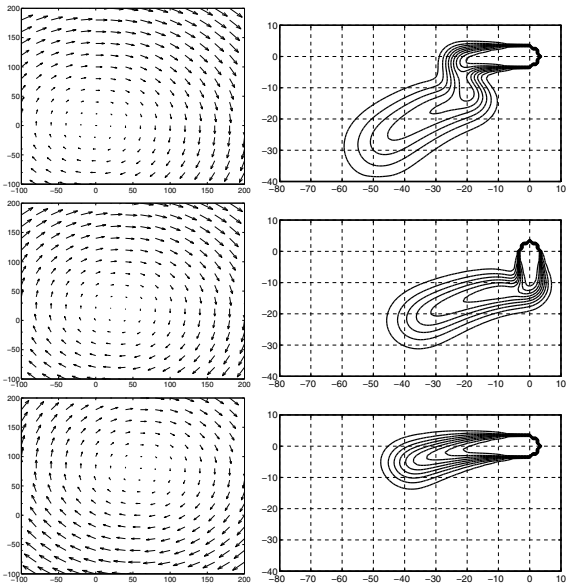

Fig. 3 Left: wind field at time t. Right: level curves of the estimated probability of conflict over the time horizon $[t, 40]$ as a function of the aircraft relative position at time t. Top: $t = 0$; Middle: $t = 10$; Bottom: $t = 20$ ($\beta = 1/20$).

in Example 1. In Fig. 2, we plot the level curves of $P_{c,\delta}^{(k)}$ computed by Algorithm 1, for $k = \lfloor t/\Delta t \rfloor$, in the cases $t = 0, 10, 20$. Compared to the plots in Fig. 1, the regions with higher probability of conflict in Fig. 2 are more concentrated along the nominal path, which is especially evident near the origin. In a sense, this implies that the current approaches to estimating the probability of conflict, based on the assumption of independent wind perturbations to the aircraft velocities, could be pessimistic. The intuitive explanation of this is that random wind perturbations to the aircraft velocities with larger correlations are more likely to cancel each other, resulting in more predictable behaviors and hence smaller probability of conflict.

Example 3 In this example, we choose $\beta = 1/20$ as in Example 2. However, we assume that there is a nontrivial affine wind field f defined by

$$f(x,t) = R(t)[x - z(t)], \quad x \in \mathbb{R}^2,\ t \in [0, 40],$$
$$R(t) \equiv \frac{1}{50}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad z(t) = \begin{bmatrix} 3t \\ t^2/5 \end{bmatrix}.$$

The wind field f can be viewed as a windstorm swirling clockwise, whose center $z(t)$ accelerates along a hyperbolic curve during T. In fact, the choice of $z(t)$ will have no effect on the probability of conflict since it does not affect the aircraft relative position. On the left of Fig. 3, we plot the wind field f on the region $[-100, 200] \times [-100, 200]$ at the time instants $t = 0, 10, 20$. On the right we plot the level curves of $P_{c,\delta}^{(k)}$, where $k = \lfloor t/\Delta t \rfloor$, at $t = 0, 10, 20$. One can

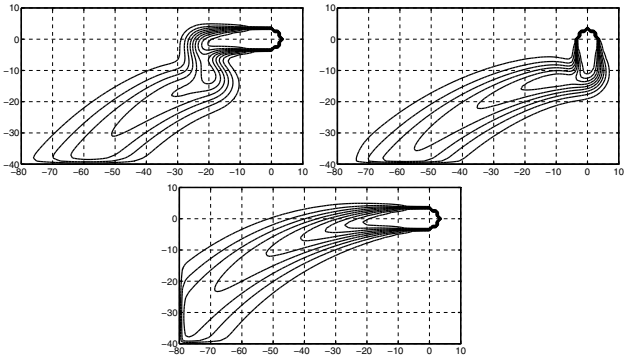

Fig. 4 Level curves of the estimated probability of conflict over the time horizon $[t, \infty]$ as a function of the aircraft relative position at time t. Top left: $t = 0$; Top right: $t = 10$; Bottom: $t = 20$ ($\beta = 1/20$).

see that, compared to the results in Fig. 2, the regions with high probability of conflict are "bent" counterclockwise, and the farther way from the origin, the more the bending. This is because the net effect of the wind field f on the relative velocity v of the two aircraft is RY, which points clockwise when the relative position Y is in the third quarter of the plane.

Example 4 Suppose now that in Example 3 we change the ending epoch t_f from 40 to infinity, and assume that the relative velocity v remain constant and equal to $(2, 0)^T$ from time 20 on. For this infinite horizon problem, we can evoke the procedure described in this section, and obtain an estimate of the probability of conflict at time $t = 0, 10, 20$ as drawn in Fig. 4. Note that, unlike in the previous examples, the regions with high probability of conflict extend outside the domain U and are truncated. This is the price we pay to evaluate numerically the probability of conflict.

General Wind Field Case

In the previous section, we have described a numerical approach to compute the probability of conflict for a two-aircraft encounter in presence of a wind field that is affine in space. In this section, we extend this approach to the general case when the wind field is not necessarily affine. The idea is still to use a Markov chain approximation method. However, in the general case the dimension of the state space of the introduced Markov chain is four instead of two, because one has to keep track of each aircraft position and not simply of their relative position.

Consider the model of the two-aircraft encounter where the motions $X_1(\cdot)$ and $X_2(\cdot)$ of the aircraft are governed by equations (3) and (4), respectively. Since no assumption is made on the wind field $f(x, t)$, it no longer suffices to consider only the relative position of the two aircraft as in the previous section. We define

$$\hat{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathbb{R}^4.$$

Then equations (3) and (4) can be written in terms of $\hat{X}$ as a single equation:

$$d\hat{X}(t) = \hat{u}(t)dt + \hat{f}(\hat{X},t)dt + \sigma d\hat{B}(\hat{X},t), \qquad (18)$$

where we set

$$\hat{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \qquad \hat{f}(\hat{X},t) = \begin{bmatrix} f(X_1,t) \\ f(X_2,t) \end{bmatrix},$$
$$\hat{B}(\hat{X},t) = \begin{bmatrix} B(X_1,t) \\ B(X_2,t) \end{bmatrix}, \qquad t \in T.$$

Given $\hat{X} = (X_1\, X_2)$, $\{\hat{B}(\hat{X},t), t \geq 0\}$ is a Gaussian process with zero mean and covariance

$$E[\hat{B}(\hat{X},t)\hat{B}(\hat{X},t)^T] = \begin{bmatrix} t\, I_2 & h(\|X_1 - X_2\|)\, t\, I_2 \\ h(\|X_1 - X_2\|)\, t\, I_2 & t\, I_2 \end{bmatrix}.$$

Analogously to the analysis in the previous section, we can think of $h(\|X_1 - X_2\|)$ as locally constant near each time instant. So in terms of distribution,

$$d\hat{B}(\hat{X},t) \overset{d}{\sim} \Sigma(\hat{X})d\hat{W}(t),$$

where $\hat{W}(t)$ is a standard Brownian motion in $\mathbb{R}^4$, and

$$\Sigma(\hat{X}) \triangleq \begin{bmatrix} I_2 & \hat{h}(\hat{X})\, I_2 \\ \hat{h}(\hat{X})\, I_2 & I_2 \end{bmatrix}^{1/2} \in \mathbb{R}^{4\times 4},$$

with $\hat{h}(\hat{X}) \triangleq h(\|X_1 - X_2\|)$. As a result, (18) becomes

$$d\hat{X}(t) = \hat{u}(t)dt + \hat{f}(\hat{X},t)dt + \sigma\Sigma(\hat{X})d\hat{W}(t). \qquad (19)$$

Define $F = \{(x_1,x_2) \in \mathbb{R}^4 : x_2 - x_1 \in D\}$. Then a conflict occurs if and only if $\hat{X}$ enters the unsafe set F within the time horizon T. To evaluate numerically the probability that a conflict occurs, we choose bounded open regions V_1 and V_2 of $\mathbb{R}^2$ large enough so that whenever the state $\hat{X}$ wanders outside $V \triangleq V_1 \times V_2$ the situation can be declared to be safe. The quantity of interest then becomes

$$P_c^V \triangleq P\{\hat{X} \text{ hits } F \text{ before hitting } V^c \text{ within the time interval } T\}, \qquad (20)$$

where $\hat{X}$ is the solution to the diffusion equation (19) defined on the open domain $V \setminus F$ with initial condition $\hat{X}(0) = (X_1(0), X_2(0)) \in V \setminus F$ and absorption on the boundary $\partial V \cup \partial F$.

Markov Chain Approximation

We now proceed as in the affine wind field case. For each $\delta > 0$, let $\delta\mathbb{Z}^4$ be the integer grid in $\mathbb{R}^4$ scaled by δ, i.e., $\delta\mathbb{Z}^4 = \{(i\delta, j\delta, m\delta, n\delta) | \, i,j,m,n \in \mathbb{Z}\}$. For each point q in $\delta\mathbb{Z}^4$, we consider the following eight neighbors:

$$q_{ll} = q + (-\delta, 0, -\delta, 0), \qquad q_{rr} = q + (\delta, 0, \delta, 0),$$
$$q_{lr} = q + (-\delta, 0, \delta, 0), \qquad q_{rl} = q + (\delta, 0, -\delta, 0)$$
$$q_{dd} = q + (0, -\delta, 0, -\delta), \qquad q_{uu} = q + (0, \delta, 0, \delta),$$
$$q_{du} = q + (0, -\delta, 0, \delta), \qquad q_{ud} = q + (0, \delta, 0, -\delta).$$

Define a Markov chain $\{Q_k, k \geq 0\}$ with state space $S \triangleq (V \setminus F) \cap \delta\mathbb{Z}^4$. Denote by S^0 the interior of S consisting of those points in S whose eight neighbors in $\delta\mathbb{Z}^4$ all belong to S. Let ∂S_V be the set of points in S with at least one neighbor outside V, and let ∂S_F be the set of points in S with at least one neighbor inside F. If a point satisfies both the conditions, then we assign it only to ∂S_F. This will eventually lead to an overestimation of the probability of conflict. However, if V_1 and V_2 are chosen to be large enough, this effect on the probability of conflict to be computed is negligible. The union $\partial S \triangleq \partial S_V \cup \partial S_F = S \setminus S^0$ of the two disjoint sets ∂S_V and ∂S_F is the boundary of S.

Suppose that each state in ∂S is an absorbing state for Q_k, and starting from an arbitrary state q in S^0 at time step $k \geq 0$, the chain stays at the same state or jumps to one of the eight neighbors listed above according to the following transition probabilities:

$$P\{Q_{k+1} = q' | \, Q_k = q\} = \begin{cases} p_{oo}^k(q) = \dfrac{\chi}{\chi+4}, & q' = q \\ p_{ll}^k(q) = \dfrac{(1+\alpha_q^k)\exp(-\delta\xi_q^k)}{[1+\exp(-\delta\xi_q^k)](\chi+4)}, & q' = q_{ll} \\ p_{rr}^k(q) = \dfrac{1+\alpha_q^k}{[1+\exp(-\delta\xi_q^k)](\chi+4)}, & q' = q_{rr} \\ p_{lr}^k(q) = \dfrac{1-\alpha_q^k}{[1+\exp(-\delta\eta_q^k)](\chi+4)}, & q' = q_{lr} \\ p_{rl}^k(q) = \dfrac{(1-\alpha_q^k)\exp(-\delta\eta_q^k)}{[1+\exp(-\delta\eta_q^k)](\chi+4)}, & q' = q_{rl} \\ p_{dd}^k(q) = \dfrac{(1+\alpha_q^k)\exp(-\delta\mu_q^k)}{[1+\exp(-\delta\mu_q^k)](\chi+4)}, & q' = q_{dd} \\ p_{uu}^k(q) = \dfrac{1+\alpha_q^k}{[1+\exp(-\delta\mu_q^k)](\chi+4)}, & q' = q_{uu} \\ p_{du}^k(q) = \dfrac{1-\alpha_q^k}{[1+\exp(-\delta\nu_q^k)](\chi+4)}, & q' = q_{du} \\ p_{ud}^k(q) = \dfrac{(1-\alpha_q^k)\exp(-\delta\nu_q^k)}{[1+\exp(-\delta\nu_q^k)](\chi+4)}, & q' = q_{ud}, \end{cases} \qquad (21)$$

whose parameters are specified by

$$\chi = \frac{2}{\lambda\sigma^2} - 4,$$
$$\alpha_q^k = \hat{h}(q),$$
$$\xi_q^k = \frac{2([\hat{F}(q,k\Delta t)]_3 + [\hat{F}(q,k\Delta t)]_1}{\sigma^2(1+\alpha_q^k)},$$
$$\eta_q^k = \frac{2([\hat{F}(q,k\Delta t)]_3 - [\hat{F}(q,k\Delta t)]_1)}{\sigma^2(1-\alpha_q^k)},$$
$$\mu_q^{k\Delta t} = \frac{2([\hat{F}(q,k\Delta t)]_4 + [\hat{F}(q,k\Delta t)]_2)}{\sigma^2(1+\alpha_q^k)},$$
$$\nu_q^{k\Delta t} = \frac{2([\hat{F}(q,k\Delta t)]_4 - [\hat{F}(q,k\Delta t)]_2)}{\sigma^2(1-\alpha_q^k)}.$$

Here

$$\hat{F}(q, k\Delta t) \triangleq \hat{f}(q, k\Delta t) + \hat{u}(k\Delta t),$$

and $[z]_i$ denotes the i-th component of a vector $z \in \mathbb{R}^4$, $i = 1, 2, 3, 4$. λ satisfies $0 < \lambda \leq (2\sigma^2)^{-1}$. Δt is the time elapsed between successive jumps and is set equal to $\Delta t = \lambda\delta^2$.

For each $q \in S^0$ and each $k \geq 0$, define

$$m_q^k = \frac{1}{\Delta t} E\{Q_{k+1} - Q_k | \, Q_k = q\},$$
$$V_q^k = \frac{1}{\Delta t} E\{(Q_{k+1} - Q_k)(Q_{k+1} - Q_k)^T | \, Q_k = q\}.$$

Then it can be verified that

$$m_q^k = \frac{1}{\lambda\delta(\chi+4)} \begin{bmatrix} (1+\alpha_q^k)\frac{1-\exp(-\delta\xi_q^k)}{1+\exp(-\delta\xi_q^k)} - (1-\alpha_q^k)\frac{1-\exp(-\delta\eta_q^k)}{1+\exp(-\delta\eta_q^k)} \\ (1+\alpha_q^k)\frac{1-\exp(-\delta\mu_q^k)}{1+\exp(-\delta\mu_q^k)} - (1-\alpha_q^k)\frac{1-\exp(-\delta\nu_q^k)}{1+\exp(-\delta\nu_q^k)} \\ (1+\alpha_q^k)\frac{1-\exp(-\delta\xi_q^k)}{1+\exp(-\delta\xi_q^k)} + (1-\alpha_q^k)\frac{1-\exp(-\delta\eta_q^k)}{1+\exp(-\delta\eta_q^k)} \\ (1+\alpha_q^k)\frac{1-\exp(-\delta\mu_q^k)}{1+\exp(-\delta\mu_q^k)} + (1-\alpha_q^k)\frac{1-\exp(-\delta\nu_q^k)}{1+\exp(-\delta\nu_q^k)} \end{bmatrix},$$
$$V_q^k = \begin{bmatrix} \sigma^2 I_2 & \sigma^2\alpha_q^k I_2 \\ \sigma^2\alpha_q^k I_2 & \sigma^2 I_2 \end{bmatrix}.$$

So if $\delta \to 0$ and we always choose q to be a point in S^0 closest to a fixed $\hat{x} \in V \setminus F$, then

$$m_q^k \to \hat{u}(k\Delta t) + \hat{f}(\hat{x}, k\Delta t),$$
$$V_q^k \to \sigma^2 \begin{bmatrix} I_2 & \hat{h}(\hat{x})I_2 \\ \hat{h}(\hat{x})I_2 & I_2 \end{bmatrix} = \sigma^2\Sigma^2(\hat{x}).$$

Therefore, we conclude that

Proposition 2 *Fix $\delta > 0$ and consider the corresponding Markov chain $\{Q_k, k \geq 0\}$. Denote by $\{Q(t), t \geq 0\}$ the stochastic process that is equal to Q_k on the time interval $[k\Delta t, (k+1)\Delta t)$ for $k \geq 0$. Then as $\delta \to 0$, $\{Q(t), t \geq 0\}$ converges weakly to the solution $\{\hat{X}(t), t > 0\}$ to equation (19) defined on $V \setminus F$ with absorption on the boundary $\partial V \cup \partial F$.*

As a result, a good approximation to the probability of conflict (20) is given by

$$P_{c,\delta}^V \triangleq P\{Q_k \text{ hits } \partial S_F \text{ before hitting } \partial S_V \text{ within } 0 \leq k \leq k_f\}, \tag{22}$$

where $k_f = \lfloor \frac{t_f}{\Delta t} \rfloor$, for δ sufficiently small. Here the Markov chain $\{Q_k, k \geq 0\}$ is assumed to start from a point $\bar{q}$ in S closest to $\hat{X}(0)$,

Remark 5 In a time interval of length $\Delta t = \lambda\delta^2$, the maximal distance that the Markov chain can travel in any of the four dimensions is δ. Thus, given $u_1(t)$, $u_2(t)$, $t \in T$, and V_1, V_2, for $\hat{X}(t)$ to be approximated by the Markov chain, the quantities $\|u_i(t) + f(x_i, t)\|$, $i = 1, 2$, have to be upper bounded on $V_i \times T$, $i = 1, 2$, respectively, by $1/\lambda\delta$, which is $2\sigma^2/\delta$ if we choose λ to be equal to $(2\sigma^2)^{-1}$. These two conditions translate into an upper bound on the admissible values for δ.

If we introduce the set of functions

$$P_{c,\delta}^{(k)}(q) \triangleq P\{Q_{k_f} \in \partial S_F | \, Q_k = q\}, \quad q \in S,$$

$k = 0, \ldots, k_f$, and consider the finite horizon case $t_f < \infty$, then $P_{c,\delta}^V$ defined in (22) can be computed recursively by Algorithm 2 described below.

Algorithm 2 Given $X_1(0), X_2(0)$, $u_1(t), u_2(t)$, $t \in T$, and the wind field $f : \mathbb{R}^2 \times T \to \mathbb{R}^2$.

1. Fix $\delta > 0$ and $\lambda \in (0, \frac{1}{2\sigma^2}]$. Set $\Delta t = \lambda\delta^2$. Define the Markov chain $\{Q_k, k \geq 0\}$ with state space $S = (V \setminus F) \cap \delta\mathbb{Z}^4$ and transition probabilities given by (21).

2. Set $\bar{k} = k_f$ and initialize $P_{c,\delta}^{(\bar{k})}$ according to

$$P_{c,\delta}^{(\bar{k})}(q) = \begin{cases} 1, & \text{if } q \in \partial S_F, \\ 0, & \text{otherwise.} \end{cases}$$

3. For $k = \bar{k} - 1, \ldots, 0$, compute $P_{c,\delta}^{(k)}$ from $P_{c,\delta}^{(k+1)}$ in the following way. For each $q \in S$,

$$P_{c,\delta}^{(k)}(q) = \begin{cases} p_{oo}^k(q)P_{c,\delta}^{(k+1)}(q) + p_{ll}^k(q)P_{c,\delta}^{(k+1)}(q_{ll}) \\ +p_{rr}^k(q)P_{c,\delta}^{(k+1)}(q_{rr}) + p_{lr}^k(q)P_{c,\delta}^{(k+1)}(q_{lr}) \\ +p_{rl}^k(q)P_{c,\delta}^{(k+1)}(q_{rl}) + p_{dd}^k(q)P_{c,\delta}^{(k+1)}(q_{dd}) \\ +p_{uu}^k(q)P_{c,\delta}^{(k+1)}(q_{uu}) + p_{du}^k(q)P_{c,\delta}^{(k+1)}(q_{du}) \\ +p_{ud}^k(q)P_{c,\delta}^{(k+1)}(q_{ud}), & \text{if } q \in S^0, \\ 1, & \text{if } q \in \partial S_F, \\ 0, & \text{if } q \in \partial S_V. \end{cases}$$

4. Choose a point $\bar{q}$ in S closest to $\hat{X}(0) = (X_1(0), X_2(0))$, and output $P_{c,\delta}^V = P_{c,\delta}^{(0)}(\bar{q})$.

By Proposition 2, $P_{c,\delta}^V$ converges to P_c^V in (20) as $\delta \to 0$, thus providing a good approximation of the probability of conflict when δ is sufficiently small.

Remark 6 The size of the state space S is of the order of $1/\delta^4$. Since the number of iterations is $k_f \simeq t_f/\Delta t = O(1/\delta^2)$, the running time of Algorithm 2 grows as $O(1/\delta^6)$ as $\delta \to 0$. Due to the increase in the state space dimension, the running time of Algorithm 2 is significantly larger than that of Algorithm 1.

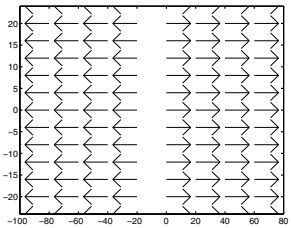

Fig. 5 Non-affine wind field

Algorithm 2 only applies in the finite horizon case. Under the assumption that both aircraft travel at constant velocity and that the wind field f is time-invariant from some time t_c on, one can adopt a two-step procedure to compute $P^V_{c,\delta}$: first $P^{(k_c+1)}_{c,\delta}$ with $k_c = \lfloor t_c/\Delta t \rfloor$ is estimated, and then $P^V_{c,\delta}$ is computed through Algorithm 2 with step 2 replaced by

2'. Set $\bar{k} = k_c + 1$ and initialize $P^{(\bar{k})}_{c,\delta}$ with $P^{(k_c+1)}_{c,\delta}$.

Since the procedure to determine $P^{(k_c+1)}_{c,\delta}$ is entirely similar to the one described in the affine wind field case, we omit the details here.

One can also extend the approach to the case when the aircraft initial positions are uncertain in a way completely similar to the previous section.

Remark 7 There are many ways of defining the transition probabilities for $\{Q_k, k \geq 0\}$ on S such that Proposition 2 holds. Our choice in (21) is "sparse" in the sense that starting from a point in S^0 the chain can only jump to eight of its neighboring grid points. The advantage of such a choice is that each iteration step in Algorithm 2 takes less time. The disadvantage is that if the chain starts from a certain point, it can only reach approximately one half of the grid points in S, i.e., those points in S with the same parity as the starting point. Hence the level curves of $P^{(0)}_{c,\delta}$ computed by Algorithm 2 tend to be saw-like. One solution can be to smooth $P^{(0)}_{c,\delta}$ by passing it through a low pass filter. This is equivalent to saying that there is some uncertainty in the aircraft initial positions.

Examples

If the wind field is affine, Algorithm 2 returns similar results to those of Algorithm 1. So only a non-affine wind field example is reported here.

Example 5 Suppose the safe distance is $r = 3$. The parameter σ and the function $\hat{h}$ in equation (18) are respectively $\sigma = 2$ and $\hat{h}((x_1, x_2)) = \exp(-\|x_2 - x_1\|)$, $x_1, x_2 \in \mathbb{R}^2$. The time interval of interest is $T = [0, 20]$. The velocities of the two aircraft during T are supposed to be constant: $u_1(t) \equiv (4, 0)$, $u_2(t) \equiv (2, 0)$. The wind field is assumed to depend only on the spatial coordinate $x \in \mathbb{R}^2$ as $f(x,t) = (\frac{\exp[([x]_1+20)/2]-1}{\exp[([x]_1+20)/2]+1}, 0)$. See Fig. 5 for a plot of f.

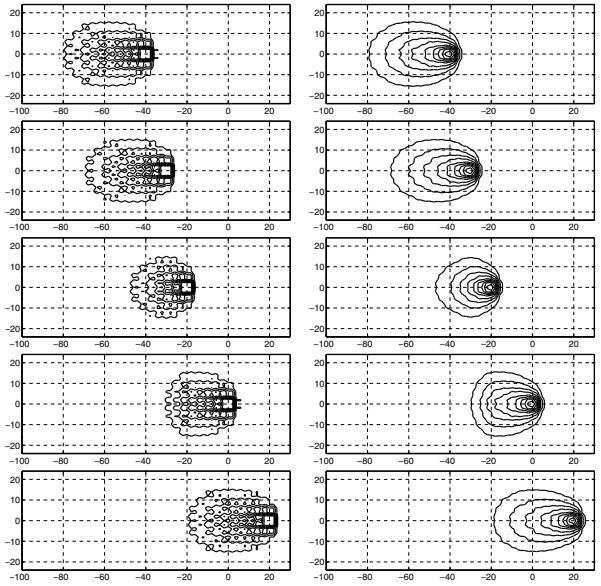

Fig. 6 Left: Level curves of the estimated probability of conflict over the time horizon $[0, 20]$ as a function of the initial position of aircraft 1 for fixed initial position of aircraft 2 (from top to bottom: $(-40, 0)$, $(-30, 0)$, $(-20, 0)$, $(0, 0)$, and $(20, 0)$). Right: Level curve of a smooth version of the corresponding quantity on the left. (Non-affine wind field.)

Based on the values taken by T, and $u_1(t), u_2(t)$, $t \in T$, the domains V_1 and V_2 are chosen to be the open rectangles $V_1 = (-100, 30) \times (-24, 24)$ and $V_2 = (-60, 80) \times (-16, 16)$. Finally, we set $\lambda = (2\sigma^2)^{-1} = 0.125$ and $\delta = 1.5$, so that $\Delta t = \lambda\delta^2 = 9/32$.

In Fig. 6, we plot the level curves of the probability of conflict (22) as a function of the initial position of aircraft 1, for five different initial positions of aircraft 2: $(-40, 0)$, $(-30, 0)$, $(-20, 0)$, $(0, 0)$, and $(20, 0)$, moving from top to bottom in the figure. On each row, the figure on the left side corresponds to the probability of conflict as computed by Algorithm 2. Since we use a relative coarse grid $\delta = 1.5$, the level curves are not smooth. For better visualization, we plot on the right side the level curves of a smoothed version of the corresponding probability of conflict map on the left side, whose value at each grid point $w \in V_1 \cap \delta\mathbb{Z}^2$ is the average value of the probability of conflict at w and its four immediate neighboring points w_l, w_r, w_d, w_u. In effect, this is equivalent to passing the original probability of conflict map through a low pass filter $\frac{1}{5}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. This amounts to assuming that there is uncertainty in the initial position of aircraft 1, such that aircraft 1 occupies its nominal position and the four neighboring grid points with equal probability.

Unlike the affine wind field case, in this example the probability of conflict in general depends on the initial positions of both aircraft, not just on their initial rel-

ative position. This is more eminent at those places where there is a large acceleration (or deceleration) in wind components, i.e., at those places with higher degree of nonlinearity in the wind field. If the nonlinearity of the wind field is relatively small, Algorithm 1 can be used instead of Algorithm 2, significantly reducing the computation time.

Probability of Intrusion

Our approach to estimate the probability of conflict for a two-aircraft encounter can be generalized to estimate the probability of intrusion of an aircraft into a protected area of the airspace with an arbitrary shape.

Suppose that the position X of the aircraft is governed by the stochastic differential equation:

$$dX(t) = u(t)dt + f(X,t)dt + \sigma dB(X,t), \quad t \in T, \ (23)$$

with initial condition $X(0)$, where $u(t)$, $t \in T$, is the aircraft velocity profile, and all of the other terms appearing in the equation are defined as in the section where we introduced the model for a two-aircraft encounter. Denote by D the arbitrarily shaped closed region of $\mathbb{R}^2$ that the aircraft should avoid, and by U an open subset of $\mathbb{R}^2$ containing D and large enough such that whenever the aircraft wanders outside U the situation can be declared to be safe. Then, the probability of intrusion of the aircraft into the protected area D over the time horizon $T = [0, t_f]$ can be defined as the probability that X hits D before it (ever) hits U^c within the time interval T. By adopting a procedure analogous to the one introduced for computing the probability of conflict in the affine wind field case, we can construct a Markov chain $\{Q_k,\, k \geq 0\}$ on the grid space $(U \setminus D) \cap \delta\mathbb{Z}^2$ so that as the discretization size $\delta \to 0$, Q_k converges weakly to $X(t)$. Then the probability of intrusion for $X(t)$ can be approximated by the corresponding probability for Q_k for δ small enough. We omit the details here.

Conclusions

Correlation between the wind perturbations to the aircraft positions is largely ignored in the literature on aircraft conflict detection. In this paper, a model of a two-aircraft encounter is introduced to address this issue. Based on this model, one can effectively estimate the probability of conflict by using a Markov chain approximation scheme. Simulation results show that the correlation between wind perturbations does affect the values of the probability of conflict.

One distinctive advantage of the approach proposed in this paper is that, upon completion, one has the probability of conflict not only at the current time, but also at all future time instants, which eliminates the need for re-computation if the flight plans remain unchanged. The approach can also be extended to address some more general cases such as, for example, computing the probability of conflict when the current aircraft positions are uncertain, or estimating the probability of intrusion into a protected area of the airspace with an arbitrary shape.

Given the abundance of information that our approach returns, it will be interesting to see if and how one can use this information to assist in the decision process at the conflict resolution stage. This is one of our current directions of research.

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