

Stabilizing Switched Linear Systems With Unstabilizable Subsystems

Wei Zhang, Alessandro Abate and Jianghai Hu

Abstract

This paper studies the exponential stabilization problem for discrete-time switched linear systems based on a control-Lyapunov function approach. A number of versions of converse control-Lyapunov function theorems are proved and their connections to the switched LQR problem are derived. It is shown that the system is exponentially stabilizable if and only if there exists a finite integer N such that the N -horizon value function of the switched LQR problem is a control-Lyapunov function. An efficient algorithm is also proposed which is guaranteed to yield a control-Lyapunov function and a stabilizing strategy whenever the system is exponentially stabilizable.

I. INTRODUCTION

One of the basic problems for switched systems is to design a switched-control feedback strategy that ensures the stability of the closed-loop system [1]. The stabilization problem for switched systems, especially autonomous switched linear systems, has been extensively studied in recent years [2]. Most of the previous results are based on the existence of a switching strategy and a Lyapunov or Lyapunov-like function with decreasing values along the closed-loop system trajectory [3], [4]. These existence results have also led to some constructive ways to find the stabilizing switching strategy [5], [6]. The main idea is to parameterize the switching strategy and the Lyapunov function in terms of some matrices and then translate the Lyapunov theorem to some matrix inequalities. The solution of these matrix inequalities, when existing, will define a stabilizing switching strategy. However, these matrix inequalities are usually NP-hard to solve and relaxations and heuristic methods are often required. A similar idea is used to study the stabilization problem of nonautonomous switched linear systems [7], [8]. By assuming a linear state-feedback form for the continuous control of each mode, the problem is also formulated as a matrix inequality problem, where the feedback-gain matrices are part of the design variables. Although some sufficient and necessary conditions are derived for quadratic stabilizability [4], [9], [10], most of the previous stabilization results are far from necessary in the sense that the system may be asymptotically or exponentially stabilizable without satisfying the proposed conditions or the derived matrix inequalities.

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W. Zhang and J. Hu are with the School of Electrical and Computer Engineering, Purdue University, West Lafayette, IN 47906. Email:{zhang70,jianghai@purdue.edu}.

A. Abate is with the Department of Aeronautics and Astronautics, Stanford University, CA 94305. Email:{aabate@stanford.edu}.

In this paper, we study the exponential stabilization problem for discrete-time switched linear systems. Our goal is to develop a computationally appealing way to construct both a switching strategy and a continuous control strategy to exponentially stabilize the system when none of the subsystems is stabilizable but the switched system is exponentially stabilizable. Unlike most previous methods, we propose a controller synthesis framework based on the control-Lyapunov function which embeds the controller design in the design of the Lyapunov function. The control-Lyapunov function approach has been widely used for studying the stabilization problem of general nonlinear systems [11], [12]. However, its application in switched linear systems has not been adequately investigated. Another novelty of this paper is the derivation of some nice connections between the stabilization problem and the switched LQR problem. In particular, we show that the switched linear system is exponentially stabilizable if and only if there exists a finite integer N such that the N -horizon value function of the switched LQR problem is a control-Lyapunov function. This result not only serves as a converse control-Lyapunov function theorem, but also transforms the stabilization problem into the switched LQR problem. Motivated by the results of the switched LQR problem recently developed in [13], [14], [15], an efficient algorithm is proposed which is guaranteed to yield a control-Lyapunov function and a stabilizing strategy whenever the system is exponentially stabilizable. A numerical example is also carried out to demonstrate the effectiveness of the proposed algorithm.

II. PROBLEM FORMULATION

We consider the discrete-time switched linear systems described by:

$$x(t+1) = A_{v(t)}x(t) + B_{v(t)}u(t), \quad t \in \mathbb{Z}^+, \quad (1)$$

where \mathbb{Z}^+ denotes the set of nonnegative integers, $x(t) \in \mathbb{R}^n$ is the continuous state, $v(t) \in \mathbb{M} \triangleq \{1, \dots, M\}$ is the switching control¹, and $u(t) \in \mathbb{R}^p$ is the continuous control. The integers n , M and p are all finite and the control u is unconstrained. The sequence of pairs $\{(u(t), v(t))\}_{t=0}^\infty$ is called the *hybrid control sequence*. For each $i \in \mathbb{M}$, A_i and B_i are constant matrices of appropriate dimensions and the pair (A_i, B_i) is called a subsystem. This switched linear system is time invariant in the sense that the set of available subsystems $\{(A_i, B_i)\}_{i=1}^M$ is independent of time t . We assume that there is no internal forced switchings, i.e., the system can stay at or switch to any mode at any time instant. At each time $t \in \mathbb{Z}^+$, denote by $\xi_t \triangleq (\mu_t, \nu_t) : \mathbb{R}^n \rightarrow \mathbb{R}^p \times \mathbb{M}$ the *hybrid control law* of system (1), where $\mu_t : \mathbb{R}^n \rightarrow \mathbb{R}^p$ is called the *continuous control law* and $\nu_t : \mathbb{R}^n \rightarrow \mathbb{M}$ is called the *switching control law*. A sequence of hybrid control laws constitutes an *infinite-horizon feedback policy*: $\pi \triangleq \{\xi_0, \xi_1, \dots, \dots\}$. Denote by Π the set of all admissible policies, i.e., the set of all sequences of functions $\pi = \{\xi_0, \xi_1, \dots\}$ with $\xi_t : \mathbb{R}^n \rightarrow \mathbb{R}^p \times \mathbb{M}$ for $t \in \mathbb{Z}^+$. If system (1) is driven by a feedback policy π , then the closed-loop dynamics is governed by

$$x(t+1) = A_{\nu_t(x(t))}x(t) + B_{\nu_t(x(t))}\mu_t(x(t)), \quad t \in \mathbb{Z}^+. \quad (2)$$

¹In this paper, $v(t)$ is an external control rather than an internal state and is thus called the switching control instead of the discrete mode.

In this paper, the policy π is allowed to be time-varying and the feedback law $\xi_t = (\mu_t, \nu_t)$ at each time step can be an arbitrary function of the state. The special policy $\pi = \{\xi, \xi, \dots\}$ with the same feedback law $\xi_t = \xi$ at each time t is called a *stationary policy*.

Definition 1: The origin of system (2) is exponentially stable if there exist constants $a > 0$ and $0 < c < 1$ such that the system trajectory starting from any initial state x_0 satisfies:

$$\|x(t)\|^2 \leq ac^t \|x_0\|^2,$$

where $\|\cdot\|$ denotes the standard Euclidean norm in \mathbb{R}^n .

Definition 2: The system (1) is called *exponentially stabilizable* if there exists a feedback policy $\pi = \{(\mu_t, \nu_t)\}_{t \geq 0}$ under which the closed-loop system (2) is exponentially stable.

Clearly, system (1) is exponentially stabilizable if one of the subsystems is stabilizable. A nontrivial problem is to stabilize the system when none of the subsystems are stabilizable. The main purpose of this paper is to develop an efficient and constructive way to solve the following stabilization problem.

Problem 1 (Stabilization Problem): Suppose that the pair (A_i, B_i) is not stabilizable for any $i \in \mathbb{M}$. Find, if possible, a feedback policy π under which the closed-loop system (2) is exponentially stable.

Most stabilization problems studied in the literature [4], [7], [16] assume that the hybrid-feedback law is time invariant, i.e., $(\mu_t, \nu_t) = (\mu, \nu)$, and the continuous-feedback law is a linear function of the state for each mode, i.e., $\mu(x) = F_{\nu(x)}x$, for some $\{F_i\}_{i=1}^M$. Compared with these problems, Problem 1 is more general as it allows the hybrid-feedback law to be an arbitrary time-varying function of the state. It will be shown in Section V that if the system is exponentially stabilizable, then the stabilizing policy can always be made stationary; however, the continuous-feedback law may not be a simple linear function of the state for each mode. See Remark 3 for details.

III. A CONTROL-LYAPUNOV FUNCTION FRAMEWORK

We first recall a version of the Lyapunov theorem for exponential stability.

Theorem 1 ([17]): Suppose that there exist a policy π and a nonnegative function $V : \mathbb{R}^n \rightarrow \mathbb{R}^+$ satisfying:

- (i) $\kappa_1 \|z\|^2 \leq V(z) \leq \kappa_2 \|z\|^2$ for some finite positive constants κ_1 and κ_2 ;
- (ii) $V(x(t)) - V(x(t+1)) \geq \kappa_3 \|x(t)\|^2$ for some constant $\kappa_3 > 0$, where $x(t)$ is the closed-loop trajectory of system (2) under policy π .

Then system (2) is exponentially stable under π .

To solve the stabilization problem, one usually needs to first propose a valid policy and then construct a Lyapunov function that satisfies the conditions in the above theorem. A more convenient way is to combine these two steps together, resulting in the control-Lyapunov function approach.

Definition 3 (ECLF): The nonnegative function $V : \mathbb{R}^n \rightarrow \mathbb{R}^+$ is called an *exponentially stabilizing control Lyapunov function* (ECLF) of system (1) if

- (i) $\kappa_1 \|z\|^2 \leq V(z) \leq \kappa_2 \|z\|^2$ for some finite positive constants κ_1 and κ_2 ;
- (ii) $V(z) - \inf_{\{v \in \mathbb{M}, u \in \mathbb{R}^p\}} V(A_v z + B_v u) \geq \kappa_3 \|z\|^2$ for some constant $\kappa_3 > 0$.

The ECLF, if exists, represents certain abstract energy of the system. The second condition of Definition 3 guarantees that by choosing proper hybrid controls, the abstract energy decreases by a constant factor at each step. This together with the first condition implies the exponential stabilizability of system (1).

Theorem 2: If system (1) has an ECLF, then it is exponentially stabilizable by a stationary policy.

Proof: Follows directly from Theorem 1 and Definition 3. ■

If $V(z)$ is an ECLF, then one can always find a feedback law ξ that satisfies the conditions of Theorem 1. Such a feedback law is exponentially stabilizing, but may result in a large control action. A systematic way to stabilize the system with a reasonable control effort is to choose the hybrid control (u, v) that minimizes the abstract energy at the next step $V(A_v z + B_v u)$ plus certain kind of control energy expense. Toward this purpose, we introduce the following feedback law:

$$\xi_V(z) = (\mu_V(z), \nu_V(z)) = \arg \inf_{u \in \mathbb{R}^p, v \in \mathbb{M}} [V(A_v z + B_v u) + u^T R_v u], \quad (3)$$

where for each $v \in \mathbb{M}$, $R_v = R_v^T \succ 0$ characterizes the penalizing metric for the continuous control u . Since the quantity inside the bracket is bounded from below and grows to infinity as $\|u\| \rightarrow \infty$, the minimizer of (3) always exists in $\mathbb{R}^p \times \mathbb{M}$.

Lemma 1: Let $V : \mathbb{R}^n \rightarrow \mathbb{R}^+$ be a nonnegative function satisfying the first condition of Definition 3. Let $\xi_V = (\mu_V, \nu_V)$ be defined by V through (3). If

$$V(z) - V(A_{\nu_V(z)} z + B_{\nu_V(z)} \mu_V(z)) \geq \kappa_3 \|z\|^2, \quad (4)$$

for some constant $\kappa_3 > 0$, then system (1) is exponentially stabilizable by the stationary policy $\{\xi_V, \xi_V, \dots\}$.

If there exists a function satisfying the conditions in Lemma 1, we can use (3) to construct a stabilizing feedback policy with a reasonable control effort. Clearly, the challenge is how to find such a function. In the rest of this paper, we will focus on a particular class of piecewise quadratic functions as candidates for the ECLFs of system (1). Each of these functions can be written as a pointwise minimum of a finite number of quadratic functions as follows:

$$V_{\mathcal{H}}(z) = \min_{P \in \mathcal{H}} z^T P z, \quad (5)$$

where \mathcal{H} is a finite set of positive definite matrices, hereby referred to as the *FPD set*. The main reason that we focus on functions of the form (5) is that this form is sufficiently rich in terms of characterizing the ECLFs of system (1). It will be shown in Section V that the system is exponentially stabilizable if and only if there exists an ECLF of the form (5) satisfying (4).

With the particular structure of the candidate ECLFs as defined in (5), the feedback law defined in (3) can be derived in closed form. Its expression is closely related to the Riccati equation and the Kalman gain of the classical LQR problem. To derive this expression, we first define a few notations. Let \mathcal{A} be the *positive semidefinite cone*, namely, the set of all symmetric positive semidefinite (p.s.d.) matrices. For each subsystem $i \in \mathbb{M}$, define a mapping $\rho_i^0 : \mathcal{A} \rightarrow \mathcal{A}$ as:

$$\rho_i^0(P) = A_i^T P A_i - A_i^T P B_i (R_i + B_i^T P B_i)^{-1} B_i^T P A_i. \quad (6)$$

It will become clear in Section IV that the mapping ρ_i^0 is the difference Riccati equation of subsystem i with a zero state-weighting matrix. For each subsystem $i \in \mathbb{M}$ and each p.s.d. matrix P , the Kalman gain is defined as

$$K_i(P) \triangleq (R_i + B_i^T P B_i)^{-1} B_i^T P A_i. \quad (7)$$

Lemma 2: Let \mathcal{H} be an arbitrary FPD set. Let $V_{\mathcal{H}} : \mathbb{R}^n \rightarrow \mathbb{R}^+$ be defined by \mathcal{H} through (5). Then the feedback law defined in (3) is given by

$$\xi_{V_{\mathcal{H}}}(z) = (-K_{i_{\mathcal{H}}(z)}(P_{\mathcal{H}}(z))z, i_{\mathcal{H}}(z)), \quad (8)$$

where $K_i(\cdot)$ is the Kalman gain defined in (7) and

$$(P_{\mathcal{H}}(z), i_{\mathcal{H}}(z)) = \arg \min_{P \in \mathcal{H}, i \in \mathbb{M}} z^T \rho_i^0(P) z. \quad (9)$$

Proof: By (3), to find ξ_V , we need to solve the following optimization problem:

$$\begin{aligned} f(z) &\triangleq \inf_{u \in \mathbb{R}^p, i \in \mathbb{M}} \left[\min_{P \in \mathcal{H}} (A_i z + B_i u)^T P (A_i z + B_i u) + u^T R_i u \right] \\ &= \min_{i \in \mathbb{M}, P \in \mathcal{H}} \left\{ \inf_{u \in \mathbb{R}^p} [(A_i z + B_i u)^T P (A_i z + B_i u) + u^T R_i u] \right\}. \end{aligned} \quad (10)$$

For each $i \in \mathbb{M}$ and $P \in \mathcal{H}$, the quantity inside the square bracket is quadratic in u . Thus, the optimal value of u can be easily computed as $u^* = -K_i(P)z$, where $K_i(P)$ is the Kalman gain defined in (7). Substituting u^* into (10) and simplifying the resulting expression yield $f(z) = z^T \rho_{i_{\mathcal{H}}(z)}^0(P_{\mathcal{H}}(z))z$, where $P_{\mathcal{H}}(z)$ and $i_{\mathcal{H}}(z)$ are defined in (9). \blacksquare

Remark 1: It is worth mentioning that the minimizer $(P_{\mathcal{H}}(z), i_{\mathcal{H}}(z))$ of (9) is radially invariant, indicating that under the feedback law $\xi_{V_{\mathcal{H}}}$, the states along the same radial direction have the same hybrid-control action. It can also be easily verified that the feedback law $\xi_{V_{\mathcal{H}}}$ partitions the state space into at most $|\mathcal{H}|$ conic decision regions, each of which corresponds to a different pair of feedback gain and switching control.

To check whether a function $V_{\mathcal{H}}$ defined by a FPD set \mathcal{H} is an ECLF, it is convenient to introduce another FPD set $\mathcal{F}_{\mathcal{H}}$ defined as:

$$\mathcal{F}_{\mathcal{H}} = \{\rho_i^0(P) : i \in \mathbb{M} \text{ and } P \in \mathcal{H}\}. \quad (11)$$

In other words, $\mathcal{F}_{\mathcal{H}}$ contains all the possible images of the mapping $\rho_i^0(P)$ as i ranges over \mathbb{M} and P ranges over \mathcal{H} .

Theorem 3: Let \mathcal{H} be an arbitrary FPD set. Let $V_{\mathcal{H}} : \mathbb{R}^n \rightarrow \mathbb{R}^+$ and $V_{\mathcal{F}_{\mathcal{H}}} : \mathbb{R}^n \rightarrow \mathbb{R}^+$ be defined by \mathcal{H} and $\mathcal{F}_{\mathcal{H}}$, respectively, by (5). Then the stationary policy $\pi_{V_{\mathcal{H}}} = \{\xi_{V_{\mathcal{H}}}, \xi_{V_{\mathcal{H}}}, \dots\}$ is exponentially stabilizing if

$$V_{\mathcal{H}}(z) - V_{\mathcal{F}_{\mathcal{H}}}(z) \geq \kappa_3 \|z\|^2, \quad (12)$$

for all $z \in \mathbb{R}^n$ and some constant $\kappa_3 > 0$.

Proof: Obviously, $V_{\mathcal{H}}$ satisfies the first condition of Definition 3. For simplicity, let $z \in \mathbb{R}^n$ be arbitrary but fixed. Denote by (\hat{P}, \hat{i}) the minimizer of (9) for this fixed z . Let $\hat{u} = -K_{\hat{i}}(\hat{P})z$ and $\hat{x}(1) = A_{\hat{i}}z + B_{\hat{i}}\hat{u}$ be the

continuous control at time 0 and the state at time 1, respectively, provided that the system starts from z at time 0 driven by the feedback law $\xi_{V_{\mathcal{H}}}$ as defined in (8). Using equations (7) and (6) into \hat{u} , we have

$$\begin{aligned} V_{\mathcal{H}}(\hat{x}(1)) &= \min_{P \in \mathcal{H}} [\hat{x}^T(1)P\hat{x}(1)] \leq \hat{x}^T(1)\hat{P}\hat{x}(1) \\ &\leq \hat{x}^T(1)\hat{P}\hat{x}(1) + \hat{u}^T R_i \hat{u} = z^T \rho_i^0(\hat{P})z \end{aligned}$$

Considering (9) and (11), we have

$$V_{\mathcal{H}}(\hat{x}(1)) \leq z^T \rho_i^0(\hat{P})z = V_{F_{\mathcal{H}}}(z),$$

which implies

$$V_{\mathcal{H}}(z) - V_{\mathcal{H}}(\hat{x}(1)) \geq V_{\mathcal{H}}(z) - V_{F_{\mathcal{H}}}(z) \geq \kappa_3 \|z\|^2.$$

Therefore, $V_{\mathcal{H}}$ also satisfies (4) and the desired result follows from Lemma 1. \blacksquare

For a given function $V_{\mathcal{H}}$ of the form (5), to see whether it is an ECLF, we should check condition (12). Since both $V_{\mathcal{H}}$ and $V_{F_{\mathcal{H}}}$ are homogeneous, we only need to consider the points on the unit sphere in \mathbb{R}^n to verify (12). In \mathbb{R}^2 , a practical way of checking (12) is to plot the functions $V_{\mathcal{H}}(z)$ and $V_{F_{\mathcal{H}}}(z)$ along the unit circle to see whether $V_{\mathcal{H}}(z)$ is uniformly above $V_{F_{\mathcal{H}}}(z)$. In higher dimensional state spaces, there is no general way to efficiently verify this condition. Nevertheless, a sufficient convex condition can be obtained using the S -procedure.

Corollary 1 (Convex Test): With the same notations as in Theorem 3, the stationary policy

$$\pi_{V_{\mathcal{H}}} = \{\xi_{V_{\mathcal{H}}}, \xi_{V_{\mathcal{H}}}, \dots\}$$

is exponentially stabilizing if for each $P \in \mathcal{H}$, there exists nonnegative constants α_j , $j = 1, \dots, k$, such that

$$\sum_{j=1}^k \alpha_j = 1, \text{ and } P \succ \sum_{j=1}^k \alpha_j \hat{P}^{(j)}, \quad (13)$$

where $k = |\mathcal{F}_{\mathcal{H}}|$ and $\{\hat{P}^{(j)}\}_{j=1}^k$ is an enumeration of $\mathcal{F}_{\mathcal{H}}$.

Proof: Let $\{P^{(i)}\}_{i=1}^{|\mathcal{H}|}$ be an enumeration of \mathcal{H} . Let $z \in \mathbb{R}^n$ be arbitrary. If $z = 0$, then (12) is trivially satisfied. Suppose that $z \neq 0$. By (13), for each $i = 1, \dots, |\mathcal{H}|$, we have

$$z^T P^{(i)} z > \sum_{j=1}^k \alpha_j \left(z^T \hat{P}^{(j)} z \right) > z^T \hat{P}^{(j_i)} z$$

for some $\hat{P}^{(j_i)} \in \mathcal{F}_{\mathcal{H}}$. Thus,

$$V_{\mathcal{H}}(z) = \min_{i \leq |\mathcal{H}|} z^T P^{(i)} z > \min_{i \leq |\mathcal{H}|} z^T \hat{P}^{(j_i)} z \geq V_{F_{\mathcal{H}}}(z).$$

Since $z \neq 0$, this implies

$$V_{\mathcal{H}}(z) - V_{F_{\mathcal{H}}}(z) \geq \kappa_3 \|z\|^2,$$

for some constant $\kappa > 0$. Therefore, inequality (12) is always satisfied and the desired result follows from Theorem 3. \blacksquare

IV. A CONVERSE ECLF THEOREM USING DYNAMIC PROGRAMMING

By focusing on the ECLFs of the form (5) and the feedback laws of the form (3), the stabilization problem becomes a quadratic optimal control problem. The main purpose of this section is to prove that system (1) is exponentially stabilizable if and only if there exists an ECLF that satisfies (4). Our approach is based on the theory of the switched LQR problem recently developed in [13], [15].

A. The Switched LQR Problem

Let $Q_i = Q_i^T \succ 0$ and $R_i = R_i^T \succ 0$ be the weighting matrices for the state and the control, respectively, for subsystem $i \in \mathbb{M}$. Define the running cost as

$$L(x, u, v) = x^T Q_v x + u^T R_v u, \quad (14)$$

for $x \in \mathbb{R}^n, u \in \mathbb{R}^p, v \in \mathbb{M}$. Denote by $J_\pi(z)$ the total cost, possibly infinite, starting from $x(0) = z$ under policy π , i.e.,

$$J_\pi(z) = \sum_{t=0}^{\infty} L(x(t), \mu_t(x(t)), \nu_t(x(t))). \quad (15)$$

Define $V^*(z) = \inf_{\pi \in \Pi} J_\pi(z)$. Since the running cost is always nonnegative, the infimum always exists. The function $V^*(z)$ is usually called the *infinite-horizon value function*. It will be infinite if $J_\pi(z)$ is infinite for all the policies $\pi \in \Pi$. As a natural extension of the classical LQR problem, the *Discrete-time Switched LQR problem* (DSLQR) is defined as follows.

Problem 2 (DSLQR problem): For a given initial state $z \in \mathbb{R}^n$, find the infinite-horizon policy $\pi \in \Pi$ that minimizes $J_\pi(z)$ subject to equation (2).

B. The Value Functions of the DSLQR Problem

Dynamic programming solves the DSLQR problem by introducing a sequence of value functions. Define the N -horizon value function $V_N : \mathbb{R}^n \rightarrow \mathbb{R}$ as:

$$V_N(z) = \inf_{\substack{u(t) \in \mathbb{R}^p, v(t) \in \mathbb{M} \\ 0 \leq t \leq N-1}} \left\{ \sum_{t=0}^{N-1} L(x(t), u(t), v(t)) \mid \text{subject to (1) with } x(0)=z \right\}. \quad (16)$$

For any function $V : \mathbb{R}^n \rightarrow \mathbb{R}^+$ and any feedback law $\xi = (\mu, \nu) : \mathbb{R}^n \rightarrow \mathbb{R}^p \times \mathbb{M}$, denote by \mathcal{T}_ξ the operator that maps V to another function $\mathcal{T}_\xi[V]$ defined as:

$$\mathcal{T}_\xi[V](z) = L(z, \mu(z), \nu(z)) + V(A_{\nu(z)}z + B_{\nu(z)}\mu(z)), \forall z \in \mathbb{R}^n. \quad (17)$$

Similarly, for any function $V : \mathbb{R}^n \rightarrow \mathbb{R}^+$, define the operator \mathcal{T} by

$$\mathcal{T}[V](z) = \inf_{u \in \mathbb{R}^p, v \in \mathbb{M}} \{ L(z, u, v) + V(A_v z + B_v u) \}, \forall z \in \mathbb{R}^n. \quad (18)$$

The equation defined above is called the *one-stage value iteration* of the DSLQR problem. We denote by \mathcal{T}^k the composition of the mapping \mathcal{T} with itself k times, i.e., $\mathcal{T}^k[V](z) = \mathcal{T}[\mathcal{T}^{k-1}[V]](z)$ for all $k \in \mathbb{Z}^+$ and $z \in \mathbb{R}^n$. Some standard results of Dynamic Programming are summarized in the following lemma.

Lemma 3 ([18]): Let $V_0(z) = 0$ for all $z \in \mathbb{R}^n$. Then

- (i) $V_N(z) = \mathcal{T}^N[V_0](z)$ for all $N \in \mathbb{Z}^+$ and $z \in \mathbb{R}^n$;
- (ii) $V_N(z) \rightarrow V^*(z)$ pointwise in \mathbb{R}^n as $N \rightarrow \infty$.
- (iii) The infinite-horizon value function satisfies the Bellman equation, i.e., $\mathcal{T}[V^*](z) = V^*(z)$ for all $z \in \mathbb{R}^n$.

To derive the value function of the DSLQR problem, we introduce a few definitions. Denote by $\rho_i : \mathcal{A} \rightarrow \mathcal{A}$ the *Riccati Mapping* of subsystem $i \in \mathbb{M}$, i.e.,

$$\rho_i(P) = Q_i + A_i^T P A_i - A_i^T P B_i (R_i + B_i^T P B_i)^{-1} B_i^T P A_i. \quad (19)$$

Definition 4: Let $2^{\mathcal{A}}$ be the power set of \mathcal{A} . The mapping $\rho_{\mathbb{M}} : 2^{\mathcal{A}} \rightarrow 2^{\mathcal{A}}$ defined by: $\rho_{\mathbb{M}}(\mathcal{H}) = \{\rho_i(P) : i \in \mathbb{M} \text{ and } P \in \mathcal{H}\}$ is called the *Switched Riccati Mapping* associated with Problem 2.

Definition 5: The sequence of sets $\{\mathcal{H}_k\}_{k=0}^N$ generated iteratively by $\mathcal{H}_{k+1} = \rho_{\mathbb{M}}(\mathcal{H}_k)$ with initial condition $\mathcal{H}_0 = \{0\}$ is called the *Switched Riccati Sets* of Problem 2.

The switched Riccati sets always start from a singleton set $\{0\}$ and evolve according to the switched Riccati mapping. For any finite N , the set \mathcal{H}_N consists of up to M^N p.s.d. matrices. An important fact about the DSLQR problem is that its value functions are completely characterized by the switched Riccati sets.

Theorem 4 ([13]): The N -horizon value function for the DSLQR problem is given by

$$V_N(z) = \min_{P \in \mathcal{H}_N} z^T P z. \quad (20)$$

C. A Converse ECLF Theorem

The main purpose of this subsection is to show that if system (1) is exponentially stabilizable, then an ECLF must exist and can be chosen to be the infinite-horizon value function V^* of the DSLQR problem. Denote by $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ the smallest and the largest eigenvalue of a p.s.d. matrix, respectively. Define

$$\begin{aligned} \lambda_Q^- &= \min_{i \in \mathbb{M}} \{\lambda_{\min}(Q_i)\}, \quad \lambda_Q^+ = \max_{i \in \mathbb{M}} \{\lambda_{\max}(Q_i)\}, \\ \lambda_R^- &= \min_{i \in \mathbb{M}} \{\lambda_{\min}(R_i)\}, \quad \lambda_R^+ = \max_{i \in \mathbb{M}} \{\lambda_{\max}(R_i)\}, \\ \sigma_A^+ &= \max_{i \in \mathbb{M}} \left\{ \sqrt{\lambda_{\max}(A_i^T A_i)} \right\}. \end{aligned}$$

Denote by $I_+^B \subset \mathbb{M}$ be set of indices of nonzero B matrices, i.e., $I_+^B \triangleq \{i \in \mathbb{M} : B_i \neq 0\}$. Let $\sigma_{\min}^+(\cdot)$ be the smallest positive singular value of a nonzero matrix. If $I_+^B \neq \emptyset$, define $\sigma_{B+}^- = \min_{i \in I_+^B} \{\sigma_{\min}^+(B_i)\}$. We now present a technical lemma whose proof can be found in the Appendix.

Lemma 4: Let $B \in \mathbb{R}^{n \times p}$ be arbitrary but not identically zero. Let $u \in \mathbb{R}^p$ be in the column space of B^T , i.e., $u \in \text{col}(B^T)$. Then $\|u\| \leq \|Bu\| / \sigma_{\min}^+(B)$.

We now prove some important properties of V^* .

Lemma 5: If $R_v \succ 0$ for all $v \in \mathbb{M}$, then there exists a feedback law ξ^* such that $\mathcal{T}_{\xi^*}[V^*](z) = V^*(z)$, $\forall z \in \mathbb{R}^n$.

Proof: By Lemma 3, $V^*(z)$ satisfies the Bellman equation, i.e., $\forall z \in \mathbb{R}^n$,

$$V^*(z) = \inf_{u \in \mathbb{R}^p, v \in \mathbb{M}} \{L(z, u, v) + V^*(A_v z + B_v u)\}. \quad (21)$$

Let z be arbitrary and fixed. If $V^*(z)$ is infinite, then an arbitrary $\xi^*(z) \in \mathbb{R}^p \times \mathbb{M}$ achieves the infimum of (21) which is infinite. Now suppose $V^*(z)$ is finite. Then there exists a hybrid control (u, v) under which the quantity inside the bracket of (21) is finite. Denote by \hat{V} this finite number. Since $R_v \succ 0$ for all $v \in \mathbb{M}$, there must exist a compact set \mathcal{U} such that $L(z, u, v) \geq \hat{V}$ as long as $u \notin \mathcal{U}$. This implies that

$$V^*(z) = \inf_{u \in \mathcal{U}, v \in \mathbb{M}} \{L(z, u, v) + V^*(A_v z + B_v u)\}.$$

Since \mathcal{U} is compact, there always exists a hybrid control that achieves the infimum of (21). Therefore, in any case, there must exist a feedback law $\xi^*(z) = (\mu^*(z), \nu^*(z))$ such that $\mathcal{T}_{\xi^*}[V^*](z) = V^*(z)$ for each $z \in \mathbb{R}^n$. ■

Lemma 6: Suppose that system (1) is exponentially stabilizable. Let $a < \infty$ and $c \in (0, 1)$ be the constants such that the closed-loop trajectory satisfies $\|x(t)\|^2 \leq ac^t \|x_0\|^2$. Then there exists a positive constant $\beta < \infty$ such that $\lambda_Q^- \|z\|^2 \leq V^*(z) \leq \beta \|z\|^2$, for all $z \in \mathbb{R}^n$. Furthermore, one possible choice of the bound β is given by

$$\beta = \begin{cases} \frac{a\lambda_Q^+}{1-c}, & \text{if } I_+^B = \emptyset \\ \max\{\lambda_Q^+, \lambda_R^+\} \frac{a(\sigma_A^+ + 1) + c - 1}{\sigma_{B^+}^- (1-c)}, & \text{otherwise.} \end{cases} \quad (22)$$

Proof: Let $z \in \mathbb{R}^n$ be arbitrary and fixed. Obviously, $V^*(z)$ must be no smaller than the one-step state cost, which implies $V^*(z) \geq \lambda_Q^- \|z\|^2$. To prove the second inequality, let $\pi = \{(\mu_t, \nu_t)\}_{t=0}^\infty$ be an exponentially stabilizing policy. By Definition 2, we know that the closed-loop trajectory $x(t)$ with initial condition $x(0) = z$ satisfies $\|x(t)\|^2 \leq ac^t \|z\|^2$, for some $a < \infty$ and $c \in (0, 1)$. Thus, $\sum_{t=0}^\infty \|x(t)\|^2 \leq \frac{a}{1-c} \|z\|^2$. Denote by $(u(t), v(t))$ the hybrid-control sequence generated by π , i.e., $u(t) = \mu_t(x(t))$ and $v(t) = \nu_t(x(t))$. If $I_+^B = \emptyset$, then $u(t)$ can be chosen to be zero for each $t \geq 0$. Thus,

$$V^*(z) = \sum_{t=0}^\infty x^T(t) Q_{v(t)} x(t) \leq \frac{a\lambda_Q^+}{1-c} \|z\|^2,$$

which is the desired result with $\beta = \frac{a\lambda_Q^+}{1-c}$. We now suppose that $I_+^B \neq \emptyset$, which assures that $\sigma_{B^+}^- > 0$. Define a new control sequence

$$\tilde{u}(t) = \begin{cases} 0, & \text{if } B_{v(t)} = 0, \\ [u(t)]_{B_{v(t)}^T}, & \text{otherwise,} \end{cases}$$

where $[\cdot]_{B_{v(t)}^T}$ denotes the projection of a given vector onto the column space of $B_{v(t)}^T$. Thus, $u(t) - \tilde{u}(t)$ is in the null space of $B_{v(t)}$, implying that $B_{v(t)}\tilde{u}(t) = B_{v(t)}u(t)$. By Lemma 4, we have

$$\begin{aligned} \sum_{t=0}^{\infty} \|\tilde{u}(t)\|^2 &\leq \frac{1}{\sigma_{B^+}^-} \sum_{t=0}^{\infty} \|B_{v(t)}\tilde{u}(t)\|^2 \\ &= \frac{1}{\sigma_{B^+}^-} \sum_{t=0}^{\infty} \|B_{v(t)}u(t)\|^2 \\ &\leq \frac{1}{\sigma_{B^+}^-} \sum_{t=0}^{\infty} \|x(t+1) - A_{v(t)}x(t)\|^2 \\ &\leq \frac{1}{\sigma_{B^+}^-} \left[\frac{a}{1-c} - 1 + \sigma_A^+ \frac{a}{1-c} \right] \|z\|^2 \\ &\leq \frac{a(\sigma_A^+ + 1) + c - 1}{\sigma_{B^+}^-(1-c)} \|z\|^2. \end{aligned}$$

Let $\tilde{x}(t)$ be the closed-loop trajectory starting from z driven by the hybrid-control sequence $(\tilde{u}(t), v(t))$. Then $\tilde{x}(t) = x(t)$ for all $t \in \mathbb{Z}^+$. Since (\tilde{u}, v) is just one choice of the hybrid-control sequences, we have

$$V^*(z) \leq \sum_{t=0}^{\infty} L(\tilde{x}(t), \tilde{u}(t), v(t)) \leq \max\{\lambda_Q^+, \lambda_R^+\} \frac{a(\sigma_A^+ + 1) + c - 1}{\sigma_{B^+}^-(1-c)} \|z\|^2.$$

The desired result is proved. ■

We now prove the main theorem of this section, which relates the exponential stabilizability with the infinite-horizon value function V^* .

Theorem 5 (Converse ECLF Theorem I): System (1) is exponentially stabilizable if and only if $V^*(z)$ is an ECLF of system (1) that satisfies condition (4).

Proof: We only need to show the “only if” part. Suppose system (1) is exponentially stabilizable. By Lemma 6, $V^*(z)$ satisfies the first condition of Definition 3. Furthermore, by Lemma 5, there exists a feedback law $\xi^* = (\mu^*, \nu^*)$ such that $V^*(z) = \mathcal{T}_{\xi^*}[V^*](z)$. This implies that

$$V^*(z) - V^*(A_{\nu^*(z)}z + B_{\nu^*(z)}\mu^*(z)) - [\mu^*(z)]^T R_{\nu^*(z)}[\mu^*(z)] \geq \lambda_Q^- \|z\|^2.$$

Let $\xi_{V^*} = (\hat{\mu}, \hat{\nu})$ be defined as in (3) with V replaced by V^* . Then we have

$$\begin{aligned} &V^*(z) - V^*(A_{\hat{\nu}(z)}z + B_{\hat{\nu}(z)}\hat{\mu}(z)) \\ &\geq V^*(z) - V^*(A_{\hat{\nu}(z)}z + B_{\hat{\nu}(z)}\hat{\mu}(z)) - [\hat{\mu}(z)]^T R_{\hat{\nu}(z)}[\hat{\mu}(z)] \\ &\geq V^*(z) - V^*(A_{\nu^*(z)}z + B_{\nu^*(z)}\mu^*(z)) - [\mu^*(z)]^T R_{\nu^*(z)}[\mu^*(z)] \geq \lambda_Q^- \|z\|^2, \end{aligned}$$

where the second inequality follows from the definition of ξ_{V^*} in (3). Thus, V^* also satisfies condition (4). Hence, V^* is an ECLF satisfying (4). ■

By this theorem, whenever system (1) is exponentially stabilizable, $V^*(z)$ can be used as an ECLF to construct an exponentially stabilizing feedback law ξ_{V^*} . However, from a design view point, such an existence result is not very useful as V^* can seldom be obtained exactly. In the next section, we will develop an efficient algorithm to compute an approximation of V^* which is also guaranteed to be an ECLF of system (1).

V. EFFICIENT COMPUTATION OF ECLFs

A. Approximation of V^* as an ECLF

Although the infinite-horizon value function V^* can not be obtained exactly, it may be approximated by some simple function which can be efficiently computed. If the approximating function is uniformly close to V^* with sufficient accuracy, then it will also be an ECLF of system (1). By part (ii) of Lemma 3, the finite-horizon value function V_N converges pointwise to V^* as $N \rightarrow \infty$. This motivates us to use V_N to approximate V^* for large N . To guarantee that V_N will eventually become an ECLF, we shall first ensure that the convergence of V_N to V^* can be made uniform within a compact set, say the unit ball.

Theorem 6 ([14]): If $V^*(z) \leq \beta \|z\|^2$ for some $\beta < \infty$, then

$$|V_{N_1}(z) - V_N(z)| \leq \alpha_\beta \gamma_\beta^N \|z\|^2, \quad (23)$$

for any $N_1 \geq N \geq 1$, where

$$\gamma_\beta = \frac{1}{1 + \lambda_Q^- / \beta} < 1 \quad \text{and} \quad \alpha_\beta = \max\{1, \frac{\sigma_A^+}{\gamma_\beta}\}. \quad (24)$$

Remark 2: A distinctive feature of the above theorem is the analytical characterization of the convergence rate in terms subsystem matrices and the bound β . Thus, for a given bound β , the number of value iterations required for achieving a certain numerical accuracy can be easily computed.

By this theorem, the N -horizon value function V_N approaches V^* exponentially fast as $N \rightarrow \infty$. Therefore, as we increase N , V_N will quickly become an ECLF of system (1).

Theorem 7 (Converse ECLF Theorem II): If system (1) is exponentially stabilizable, then there exists a constant $\beta < \infty$ such that $V_N(z)$ is an ECLF satisfying condition (4) for all $N \geq N_\beta$, where

$$N_\beta = \frac{\ln \lambda_Q^- / \alpha_\beta}{\ln \gamma_\beta} + 1 < \infty, \quad (25)$$

with γ_β and α_β be defined as in (24).

Proof: Define

$$\xi_N^*(z) = (\mu_N^*, \nu_N^*) \triangleq \arg \inf_{u \in \mathbb{R}^p, v \in \mathbb{M}} \{L(z, u, v) + V_N(A_v z + B_v u)\}. \quad (26)$$

By Lemma 3 and equation (26), we know that

$$V_{N+1}(z) = \mathcal{T}[V_N](z) = \mathcal{T}_{\xi_N^*}(z)[V_N](z), \forall z \in \mathbb{R}^n.$$

We now fix an arbitrary $z \in \mathbb{R}^n$ and let $u^* = \mu_N^*(z)$, $v^* = \nu_N^*(z)$ and $x^*(1) = A_{v^*} z + B_{v^*} u^*$. Therefore,

$$V_{N+1}(z) - V_N(x^*(1)) - (u^*)^T R_{v^*}(u^*) \geq \lambda_Q^- \|z\|^2. \quad (27)$$

By Lemma 6, the exponential stabilizability implies the existence of a positive constant $\beta < \infty$ such that $V^*(z) \leq \beta \|z\|^2$, $\forall z \in \mathbb{R}^n$. Let γ and α be defined in terms of β as in (24). By Theorem 6, $V_{N+1}(z) \leq V_N(z) + \alpha \gamma^N \|z\|^2$.

Substituting this inequality to (27) yields

$$V_N(z) - V_N(x^*(1)) - (u^*)^T R_{v^*}(u^*) \geq (\lambda_Q^- - \alpha \gamma^N) \|z\|^2.$$

Let N_β be defined as in (25). Then,

$$\kappa \triangleq (\lambda_Q^- - \alpha_\beta \gamma_\beta^N) > 0, \quad \forall N \geq N_\beta.$$

Fix an arbitrary $N \geq N_\beta$. Define $\xi_{V_N} = (\hat{\mu}, \hat{\nu})$ as in (3) with V replaced by V_N . Then we have

$$\begin{aligned} & V_N(z) - V_N(A_{\hat{\nu}(z)}z + B_{\hat{\nu}(z)}\hat{\mu}(z)) \\ & \geq V_N(z) - V_N(A_{\hat{\nu}(z)}z + B_{\hat{\nu}(z)}\hat{\mu}(z)) - [\hat{\mu}(z)]^T R_{\hat{\nu}(z)} [\hat{\mu}(z)] \\ & \geq V_N(z) - V_N(x^*(1)) - (u^*)^T R_{v^*} (u^*) \\ & \geq (\lambda_Q^- - \alpha_\beta \gamma_\beta^N) \|z\|^2 = \kappa \|z\|^2, \end{aligned}$$

where the second inequality follows from the definition of ξ_{V_N} in (3). Hence, V_N is an ECLF satisfying (4) for all $N \geq N_\beta$. \blacksquare

Theorem 7 implies that when the system is exponentially stabilizable, the ECLF not only exists but also can be chosen to be a piecewise quadratic function of the form (5). Furthermore, as N increases, the N -horizon value function V_N will eventually become an ECLF. In this case, by our analysis in Section ??, the system is exponentially stabilizable by the feedback law ξ_{V_N} , where ξ_{V_N} is defined by (8) with \mathcal{H} replaced by \mathcal{H}_N .

Remark 3: As discussed in Remark 1, the feedback law ξ_{V_N} divides the state space into several conic decision regions. It is important to notice that the number of the decision regions may not equal to the number of subsystems and different decision regions may have the same switching control but with different feedback gains. These properties make ξ_{V_N} more general than many other feedback laws studied in the literature [4], [7], [16]. A more important advantage of ξ_{V_N} is that if the system is exponentially stabilizable by a general feedback policy, then it is guaranteed to be stabilizable by ξ_{V_N} for some large N .

B. Numerical Relaxation

By Theorem 7, to solve the stabilization problem, we only need to compute the switched Riccati set \mathcal{H}_N . However, this method may not be computationally feasible as the size of \mathcal{H}_N grows exponentially fast as N increases. Fortunately, if we allow a small numerical relaxation, an approximation of V_N can be efficiently computed [15].

Definition 6 (Numerical Redundancy): A matrix $\hat{P} \in \mathcal{H}_N$ is called (numerically) ϵ -redundant with respect to \mathcal{H}_N if

$$\min_{P \in \mathcal{H}_N \setminus \hat{P}} z^T P z \leq \min_{P \in \mathcal{H}_N} z^T (P + \epsilon I_n) z, \text{ for any } z \in \mathbb{R}^n.$$

Remark 4: Numerical redundancy can also be defined in terms of the completeness concept for a set of matrices []. It is easy to see that \hat{P} is ϵ -redundant in \mathcal{H}_N if and only if the set of matrices $\{P^{(j)} - P + \epsilon I_n\}_{j=1}^{|\mathcal{H}_N|-1}$ is complete, where $\{P^{(j)}\}_{j=1}^{|\mathcal{H}_N|-1}$ is an enumeration of $\mathcal{H}_N \setminus \hat{P}$. In this paper, we provide our direct definition to emphasize its role in simplifying the computations of the ECLFs.

Definition 7 (ϵ -ES): The set \mathcal{H}_N^ϵ is called an ϵ -Equivalent-Subset (ϵ -ES) of \mathcal{H}_N if $\mathcal{H}_N^\epsilon \subset \mathcal{H}_N$ and for all $z \in \mathbb{R}^n$,

$$\min_{P \in \mathcal{H}_N} z^T P z \leq \min_{P \in \mathcal{H}_N^\epsilon} z^T P z \leq \min_{P \in \mathcal{H}_N} z^T (P + \epsilon I_n) z.$$

Algorithm 1

- (i) Denote by $P^{(i)}$ the i^{th} matrix in \mathcal{H}_N . Specify a tolerance ϵ and set $\mathcal{H}_N^{(1)} = \{P^{(1)}\}$.
 - (ii) For each $i = 2, \dots, |\mathcal{H}_N|$, if $P^{(i)}$ satisfies the condition in Lemma 7 with respect to \mathcal{H}_N , then $\mathcal{H}_N^{(i)} = \mathcal{H}_N^{(i-1)}$; otherwise $\mathcal{H}_N^{(i)} = \mathcal{H}_N^{(i-1)} \cup \{P^{(i)}\}$.
 - (iii) Return $\mathcal{H}_N^{(|\mathcal{H}_N|)}$.
-

Removing the ϵ -redundant matrices may introduce some error for the value function; but the error is no larger than ϵ for $\|z\| \leq 1$. To simplify the computation, for a given tolerance ϵ , we want to prune out as many ϵ -redundant matrices as possible. The following lemma provides a sufficient condition for testing the ϵ -redundancy for a given matrix.

Lemma 7 (Redundancy Test): \hat{P} is ϵ -redundant in \mathcal{H}_N if there exist nonnegative constants $\{\alpha_i\}_{i=1}^{k-1}$ such that $\sum_{i=1}^k \alpha_i = 1$ and $\hat{P} + \epsilon I_n \succeq \sum_{i=1}^k \alpha_i P^{(i)}$, where $k = |\mathcal{H}_N|$ and $\{P^{(i)}\}_{i=1}^{k-1}$ is an enumeration of $\mathcal{H}_N \setminus \{\hat{P}\}$.

The condition in Lemma 7 can be easily verified using various existing convex optimization algorithms [19]. To compute an ϵ -ES of \mathcal{H}_N , we only need to remove the matrices in \mathcal{H}_N that satisfy the condition in Lemma 7. The detailed procedure is summarized in Algorithm 1. Denote by $Algo_\epsilon(\mathcal{H}_N)$ the ϵ -ES of \mathcal{H}_N returned by the algorithm. To further reduce the complexity, we can remove the ϵ -redundant matrices after every switched Riccati mapping. To this end, we define the *relaxed switched Riccati sets* $\{\mathcal{H}_k^\epsilon\}_{k=0}^N$ iteratively as:

$$\mathcal{H}_0^\epsilon = \mathcal{H}_0 \quad \text{and} \quad \mathcal{H}_{k+1}^\epsilon = Algo_\epsilon(\rho_{\mathbb{M}}(\mathcal{H}_k^\epsilon)), \quad \text{for } k \leq N-1. \quad (28)$$

The function defined based on \mathcal{H}_N^ϵ is very close to V_N but much easier to compute as \mathcal{H}_N^ϵ usually contains much fewer matrices than \mathcal{H}_N . We now use the following example to demonstrate the simplicity of computing the set \mathcal{H}_N^ϵ .

$$\begin{aligned} A_1 &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 1.5 & 1 \\ 0 & 1.5 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \\ B_2 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, Q_i = I_2, R_i = 1, i = 1, 2. \end{aligned} \quad (29)$$

Clearly, neither subsystem is stabilizable. As shown in Fig. 1, a direct computation of $\{\mathcal{H}_k\}_{k=0}^N$ results in a combinatorial complexity of the order 10^9 for $N = 30$. However, if we use the relaxed iteration (28) with $\epsilon = 10^{-3}$, eventually \mathcal{H}_N^ϵ contains only 16 matrices. This example shows that the numerical relaxation can dramatically simplify the computation of \mathcal{H}_N . Our next task is to show that this relaxation does not change the value function too much. Define $V_N^\epsilon(z) = \min_{P \in \mathcal{H}_N^\epsilon} z^T P z$. It is proved in [15] that the total error between $V_N^\epsilon(z)$ and $V_N(z)$ can be bounded uniformly with respect to N .

Lemma 8 ([15]): If $V^*(z) \leq \beta \|z\|^2$ for some $\beta < \infty$, then

$$V_N(z) \leq V_N^\epsilon(z) \leq V_N(z) + \epsilon \eta_\beta \|z\|^2, \quad (30)$$

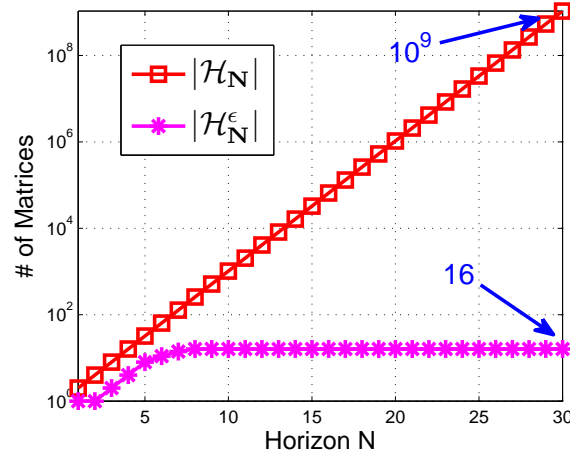


Fig. 1. Evolution of $|\mathcal{H}_N^\epsilon|$ with $\epsilon = 10^{-3}$.

where

$$\eta_\beta = \frac{1 + (\beta/\lambda_Q^- - 1)\gamma_\beta}{1 - \gamma_\beta}, \quad (31)$$

with γ_β defined in (24).

The above lemma indicates that by choosing ϵ small enough, V_N^ϵ can approximate V_N uniformly within the unit ball with arbitrary accuracy. This warrants V_N^ϵ as an ECLF for large N and small ϵ .

Theorem 8 (Converse ECLF Theorem III): If system (1) is exponentially stabilizable, then there exists a positive constant $\beta < \infty$ such that $V_N^\epsilon(z)$ is an ECLF of system (2) satisfying condition (4) for all $N \geq \tilde{N}_\beta$ and all $\epsilon \leq \epsilon_\beta$, where

$$\tilde{N}_\beta = \frac{\ln(\lambda_Q^-/(2\alpha_\beta))}{\ln \gamma_\beta} + 1, \text{ and } \epsilon_\beta = \frac{\lambda_Q^-}{2\eta_\beta}, \quad (32)$$

with γ_β and α_β defined in (24) and η_β defined in (31).

Proof: Fix an arbitrary $z \in \mathbb{R}^n$. Define

$$\xi_N^\epsilon(z) = (\mu_N^\epsilon, \nu_N^\epsilon) \triangleq \arg \inf_{u \in \mathbb{R}^p, v \in \mathbb{M}} \{L(z, u, v) + V_N^\epsilon(A_v z + B_v u)\}. \quad (33)$$

Let $\tilde{V}_{N+1}^\epsilon(z) = \mathcal{T}_{\xi_N^\epsilon}[V_N^\epsilon](z)$, i.e.,

$$\tilde{V}_{N+1}^\epsilon(z) = \inf_{u \in \mathbb{R}^p, v \in \mathbb{M}} \{L(z, u, v) + V_N^\epsilon(A_v z + B_v u)\} = \min_{v \in \mathbb{M}, P \in \mathcal{H}_N^\epsilon} \{z^T \rho_v(P) z\} = \min_{P \in \rho_{\mathbb{M}}(\mathcal{H}_N^\epsilon)} z^T P z. \quad (34)$$

By (28), we know that $\mathcal{H}_{N+1}^\epsilon = \text{Algo}_\epsilon(\rho_{\mathbb{M}}(\mathcal{H}_N^\epsilon))$. Then it follows directly from Definition 7 that

$$\tilde{V}_{N+1}^\epsilon(z) \leq V_{N+1}^\epsilon(z) = \min_{P \in \mathcal{H}_{N+1}^\epsilon} z^T P z.$$

Let $u^\epsilon = \mu_N^\epsilon(z)$, $v^\epsilon = \nu_N^\epsilon(z)$ and $x^\epsilon(1) = A_{v^\epsilon} z + B_{v^\epsilon} u^\epsilon$. According to (34), we have

$$\tilde{V}_{N+1}^\epsilon(z) - V_N^\epsilon(x^\epsilon(1)) - (u^\epsilon)^T R_{v^\epsilon}(u^\epsilon) \geq \lambda_Q^- \|z\|^2. \quad (35)$$

By the exponential stabilizability, there exists a constant $\beta < \infty$, such that $V^*(z) \leq \beta \|z\|^2$, $\forall z \in \mathbb{R}^n$. Then by Theorem 6 and Lemma 8, we have

$$\begin{aligned}\tilde{V}_{N+1}^\epsilon(z) &\leq V_{N+1}^\epsilon(z) \leq V_{N+1}(z) + \epsilon \eta_\beta \|z\|^2 \\ &\leq V_N(z) + (\alpha_\beta \gamma_\beta^N + \epsilon \eta_\beta) \|z\|^2 \\ &\leq V_N^\epsilon(z) + (\alpha_\beta \gamma_\beta^N + \epsilon \eta_\beta) \|z\|^2\end{aligned}$$

Combining this with inequality (35) yields

$$\begin{aligned}&V_N^\epsilon(z) - V_N^\epsilon(x^\epsilon(1)) - (u^\epsilon)^T R_{v^\epsilon}(u^\epsilon) \\ &\geq \tilde{V}_{N+1}^\epsilon(z) - V_N^\epsilon(x^\epsilon(1)) - (u^\epsilon)^T R_{v^\epsilon}(u^\epsilon) - (\alpha_\beta \gamma_\beta^N + \epsilon \eta_\beta) \|z\|^2 \\ &\geq (\lambda_Q^- - \alpha_\beta \gamma_\beta^N - \epsilon \eta_\beta) \|z\|^2.\end{aligned}$$

Let \tilde{N}_β and ϵ_β be defined as in (32). It can be easily seen that $\lambda_Q^- - \alpha_\beta \gamma_\beta^N - \epsilon \eta_\beta > 0$ for all $N \geq \tilde{N}_\beta$ and $\epsilon \leq \epsilon_\beta$. Then, by a similar argument as in the proof of Theorem 7, we can conclude that V_N^ϵ is an ECLF satisfying (4) for all $N \geq \tilde{N}_\beta$ and $\epsilon \leq \epsilon_\beta$. ■

C. Overall Algorithm

In summary, if the system is exponentially stabilizable, we can always find an ECLF of the form (5) defined by \mathcal{H}_N^ϵ . To compute such an ECLF, we can start from a reasonable guess of ϵ and perform the relaxed switched Riccati mapping (28). After each iteration, we need to check whether the condition of Corollary 1 are met. If so, an ECLF is found; otherwise we should continue iteration (28). If the maximum iteration number N_{max} is reached, we should reduce ϵ and restart iteration (28). Since V_N^ϵ converges exponentially fast, N_{max} can usually be chosen rather small. The above procedure of constructing an ECLF is summarized in Algorithm 2. This algorithm is computationally efficient and guarantees to yield an ECLF provided that ϵ_{min} is sufficiently small and N_{max} is sufficiently large.

VI. NUMERICAL EXAMPLES

A. Example 1

Consider the same two-mode switched system as defined in (29). Neither of the subsystems is stabilizable by itself. However, this switched system is stabilizable through a proper hybrid control. The stabilization problem can be easily solved using Algorithm 2. If we start from $\epsilon = 1$, then the algorithm terminates after 5 steps which results in an ECLF V_6^1 defined by the relaxed switched Riccati set \mathcal{H}_6^1 . We have also tried a smaller relaxation $\epsilon = 0.1$. In this case, the algorithm stops after 4 steps resulting in an ECLF $V_5^{0.1}$ defined by the relaxed switched Riccati set $\mathcal{H}_5^{0.1}$. It is worth mentioning that \mathcal{H}_6^1 contains only two matrices and $\mathcal{H}_5^{0.1}$ contains 3 matrices. With these matrices, starting from any initial position x_0 , the feedback laws corresponding to \mathcal{H}_6^1 and $\mathcal{H}_5^{0.1}$ can be easily computed using equation (8) and (9). The closed-loop trajectories generated by these two feedback laws starting from the same initial position $x_0 = [0, 1]^T$ are plotted on the left of Fig. 2. On the right of the same figure, the continuous control

Algorithm 2 (Computation of ECLF)

```

Specify proper values for  $\epsilon$ ,  $\epsilon_{min}$  and  $N_{max}$ .
while  $\epsilon > \epsilon_{min}$  do
  for  $N = 0$  to  $N_{max}$  do
     $\mathcal{H}_{N+1} = \text{Algo}_\epsilon(\rho_{\mathbb{M}}(\mathcal{H}_N))$ 
    if  $\mathcal{H}_{N+1}^\epsilon$  satisfies the condition of Corollary 1 then
      stop and return  $V_N^\epsilon$  as an ECLF
    end if
  end for
   $\epsilon = \epsilon/2$ 
end while

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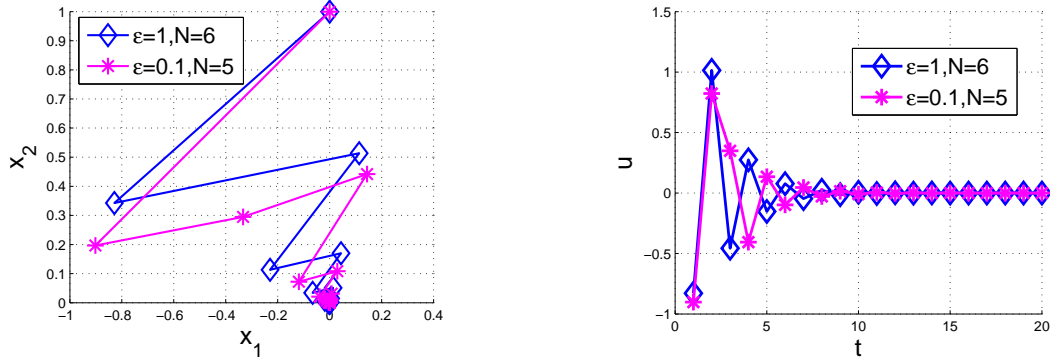


Fig. 2. Simulation Results. Left figure: phase-plane trajectories generated by the ECLFs V_6^1 and $V_5^{0.1}$ starting from the same initial condition $x_0 = [0, 1]^T$. Right figure: the corresponding continuous controls

signals associated with the two trajectories are plotted. In both cases, the switching signals jump to the other mode at every time step and are not shown in the figure. It can also be seen that the ECLF $V_5^{0.1}$ stabilizes the system with a faster convergence speed and a smaller control energy than V_6^1 . This is because it has a smaller relaxation ϵ which makes the resulting trajectory closer to the optimal trajectory of the DSLQR problem.

B. Example II

Consider another two-mode switched system, where

$$A_1 = \begin{bmatrix} 0.3 & 1 \\ 0 & 1.3 \end{bmatrix}, A_2 = \begin{bmatrix} 1.2 & 1 \\ 0 & 0.3 \end{bmatrix}, B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and $Q_i = I_2, R_i = 1, i = 1, 2$. It can be easily seen that this switched system can be exponentially stabilized by alternating between the two subsystems at each time step. Such a switching strategy is nonstationary and does not depend the system state. By our analysis, the system must also be stabilizable by a stationary feedback policy,

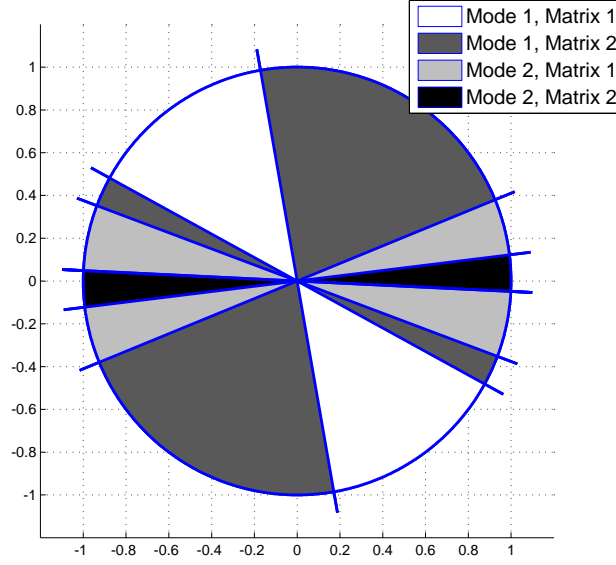


Fig. 3. Decision regions of Example II

which is nontrivial to obtain without the result of this paper. To find the stationary policy, we apply Algorithm 2 with $\epsilon = 1$. The algorithm terminates after 4 steps, resulting in an ECLF V_5^1 . The switched Riccati set \mathcal{H}_5^1 contains only 2 matrices. Then according to Theorem 8 and Lemma 2, the stationary policy $\{\xi_{V_5^1}, \xi_{V_5^1}, \dots\}$ is exponentially stabilizing and it divides the state space into 4 (possibly nonconnected) decision regions, depending on which pair of (i, P_j) achieves the minimum of (9), where $i, j = 1, 2$ and $\mathcal{H}_5^1 = \{P_1, P_2\}$. Figure 3 illustrates the decision regions and the corresponding minimizing pairs (i, P_j) . The figure indicates that the stationary feedback policy may be complex even when the system can be trivially stabilized by a nonstationary policy.

C. Example III

We now consider a multi-dimensional example with four subsystems:

$$A_1 = \begin{bmatrix} \frac{1}{2} & -1 & 2 & 3 \\ 0 & \frac{-1}{2} & 2 & 4 \\ 0 & -1 & \frac{5}{2} & 2 \\ 0 & 0 & 0 & \frac{3}{2} \end{bmatrix}, A_2 = \begin{bmatrix} \frac{-1}{2} & -1 & 2 & 1 \\ 0 & \frac{3}{2} & -2 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ -2 & -1 & 2 & \frac{5}{2} \end{bmatrix}, A_3 = \begin{bmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 1 & 1 & \frac{1}{2} & \frac{-1}{2} \\ 0 & \frac{1}{2} & 1 & \frac{-1}{2} \\ 1 & 0 & 0 & \frac{1}{2} \end{bmatrix}, A_4 = \begin{bmatrix} \frac{1}{2} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & -2 & \frac{1}{2} \end{bmatrix},$$

$B_1 = B_3 = [1, 2, 3, 4]^T$, $B_2 = B_4 = [4, 3, 2, 1]^T$, and $Q_i = I_4$, $R_i = 1$, for $i = 1, \dots, 4$. It can be verified none of the subsystems is stabilizable. Algorithm 2 is used to solve the stabilization problem with $\epsilon = 1$. The algorithm terminates after 6 steps, resulting in an ECLF V_7^1 defined by the relaxed switched Riccati set \mathcal{H}_7^1 which consists of 13 matrices. Compared with the previous two examples, this example requires more matrices to characterize the

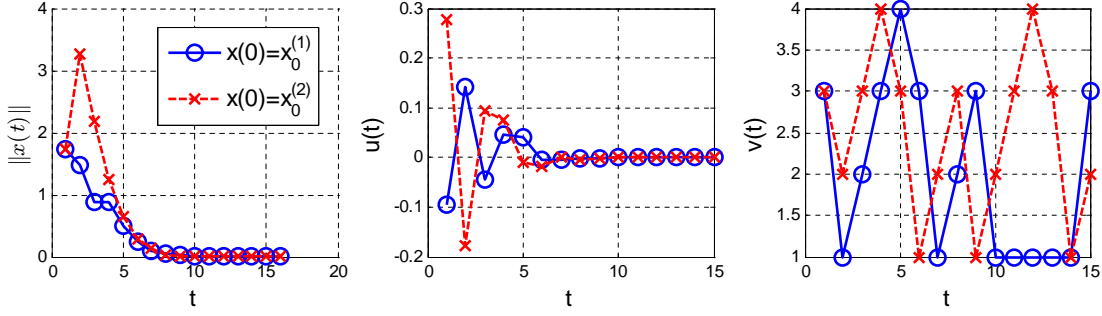


Fig. 4. Simulation Results for Example III with two different initial conditions: $x_0^{(1)} = [1, 1, 1, 0]^T$ and $x_0^{(2)} = [1, 0, 1, 1]^T$. Left figure: norms of the closed-loop trajectories associated with the two initial conditions; middle figure: the corresponding continuous control sequences; right figure: the corresponding mode sequences.

stabilizing policy due to the increase of the state dimension and the number of subsystems. To test the controller performance, the feedback hybrid-control sequences are computed using \mathcal{H}_7^1 based on (8) and (9) for two different initial conditions $x(0) = x_0^{(1)} = [1, 1, 0, -1]^T$ and $x(0) = x_0^{(2)} = [1, 0, -1, 1]^T$. The hybrid-control sequences and the norms of the closed-loop trajectories are plotted in Fig. 4. It can be seen that, for both initial conditions, the system utilizes multiple modes to maintain the stability of switched system.

VII. CONCLUSION

This paper studies the exponential stabilization problem for the discrete-time switched linear system. It has been proved that if the system is exponentially stabilizable, then there must exist a piecewise quadratic ECLF. More importantly, this ECLF can be chosen to be a finite-horizon value function of the switched LQR problem. An efficient algorithm has been developed to compute such an ECLF and the corresponding stabilizing policy whenever the system is exponentially stabilizable. Indicated by some numerical examples, the ECLF and the stabilizing policy can usually be characterized by only a few p.s.d. matrices which can be easily computed using the relaxed switched Riccati mapping. Future research will focus on extending the algorithm to solve the robust stabilization problem for uncertain switched linear systems.

APPENDIX

Proof: The result follows immediately when B is full column rank. Suppose that B is not full column rank and let $\sigma_{B^+} = \sigma_{min}^+(B)$. By the theory of singular value decomposition, there exists unitary matrices $U = [U_1, U_2]$ and $V = [V_1, V_2]$ such that

$$B = [U_1, U_2] \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$$

Since the column space $\text{col}(B^T)$ is the orthogonal complement of the null space of B , we have $V_2^T u = 0$. Thus, $\|u\| = \|V^T u\| = \|V_1^T u\|$. Therefore,

$$\|Bu\|^2 = u^T V_1 \Sigma V_1^T u \geq \sigma_{B+}^2 \|V_1^T u\|^2 \geq \sigma_{B+}^2 \|u\|^2.$$

The desired result follows by taking the square root of the above inequality. ■

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