Resilient Stabilization of Switched Linear Control Systems against Adversarial Switching

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***Abstract*—This paper studies the problem of stabilizing discrete-time switched linear control systems (SLCSs) using continuous input by a user against adversarial switching by an adversary. It is assumed that at each time the adversary knows the user's decision on the continuous input but not vice versa. A quantitative metric of stabilizability is proposed. Systems at the margin of stabilizability are further classified and studied via the notions of defectiveness and reducibility. Analytical bounds on the stabilizability metric are derived using (semi)norms, with tight bounds provided by extremal norms. Numerical algorithms are also developed for computing this metric. An application example in networked control systems is presented.**

***Index Terms*—Switched control systems, stabilization, robustness, resilient systems.**

I. Introduction

Switched control systems are hybrid systems controlled by a continuous input signal and a switching signal (or mode sequence in the discrete-time case). Stabilization of switched control systems is the problem of designing control laws for the controllable input signals to achieve a stable closed-loop system [1]–[6]. The existing approaches are roughly classified into two categories. In the first category (e.g., [1], [3], [4], [6], [7]), both the continuous input and the switching signal are utilized for stabilization. In the second category, the continuous input is used as a control, whereas the switching signal is treated as a disturbance subject to certain constraints (e.g., switching frequency and dwell time constraints). A common assumption in prior work of this category (e.g., [8]–[15]) is that the continuous controller knows exactly the current mode for at least some duration of time following each switching, and hence can take the form of a collection of mode-dependent state feedback controllers. Additional assumptions (e.g., controllability of individual subsystems [9], [13], and minimal dwell time [8], [15]) are often imposed to ensure the stabilizability of the switched control systems.

The resilient stabilization problem studied in this paper belongs to the second category but assumes a different information structure: at each time, the user decides the continuous input without any knowledge of the current mode, whereas the adversary is aware of the current continuous input. This disadvantage for the user makes the resilient stabilization a very challenging task. For example, even if each subsystem is stabilizable to the origin in one time step, the switched control system may not be stabilizable (see Example II.1). Applications of the resilient stabilization problem include robust networked control systems with uncertain network delay [16] and the (stability, safety) control of survivable cyber-physical systems under malicious cyber attacks and sabotages [17].

The resilient stabilization problem has been addressed in different contexts before. It can be formulated as the robust stabilization of linear control systems with polytopic uncertainty. However, to our knowledge, the existing work either assumes uncertain but constant system matrices [18] or only considers special cases such as quadratic stabilizability [16], [19], [20] and linear control policies [21]. Other relevant results include simultaneous stabilization of multiple linear systems [22] and stabilization of switched systems under delayed switching observability [13]. These results provide conservative sufficient conditions for resilient stabilization.

The contributions of this paper are four folds: (i) Sufficient and necessary conditions as well as a quantitative metric of resilient stabilizability are developed; (ii) SLCSs at the margin of resilient stabilizability are characterized; (iii) Theoretical results and numerical algorithms are developed that can produce more accurate bounds on the stabilizability metric than the existing approaches; (iv) We show that the resiliently stabilizing controllers are nonlinear in general (cf. Example V.2). The results in this paper extend those on the stability of autonomous switched linear systems (SLSs) [23]–[27] to the stabilization of SLCS using continuous control input.

This paper is organized as follows. The σ-resilient stabilization problem is formulated in Section II. The concepts of nondefective and irreducible systems are introduced in Section III. In Sections IV and V, theoretical and practical bounds on the σ-resilient stabilizing rate are established. Section VI presents an application in network control systems. Finally, concluding remarks are given in Section VII.

II. Resilient Stabilizability

Consider the discrete-time switched linear control system (SLCS) on $\mathbb{R}^n$ with the state $x(\cdot) \in \mathbb{R}^n$:

$$x(t+1) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad t \in \mathbb{Z}_+. \tag{1}$$

Here, $u(\cdot) \in \mathbb{R}^p$ and $\sigma(\cdot) \in \mathcal{M} := \{1, \ldots, m\}$ are the (continuous) control input and switching sequence, respectively; the sets $\mathbb{Z}_+ := \{0, 1, \ldots\}$ and $\mathbb{N} := \{1, 2, \ldots\}$. For brevity, the SLCS is denoted by $\{(A_i, B_i)\}_{i\in\mathcal{M}}$, where $A_i \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times p}$ specify the dynamics of subsystem i.

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The following assumption is made throughout this paper.

Assumption II.1 (*Admissible Control and Switching Policies*)**.** Denote by $\mathcal{F}_t := (x_{0:t}, u_{0:t-1}, \sigma_{0:t-1})$ the causal information available at time $t \in \mathbb{Z}_+$, where $x_{0:t}$ denotes $\{x(0), \ldots, x(t)\}$ and similarly for $u_{0:t-1}$ and $\sigma_{0:t-1}$, with the understanding that $\mathcal{F}_0 := \{x(0)\}$. An *admissible control policy* $\mathbf{u} := \{\mathbf{u}_0, \mathbf{u}_1, \ldots\}$ consists of a sequence of feedback control laws $\mathbf{u}_t : \mathbb{R}^{n(t+1)} \times \mathbb{R}^{pt} \times \mathcal{M}^t \to \mathbb{R}^p$ so that $u(t) = \mathbf{u}_t(\mathcal{F}_t)$, $\forall t$. Denote by $\mathcal{U}$ the set of all admissible control policies. An *admissible switching policy* $\sigma := (\sigma_0, \sigma_1, \ldots)$ consists of a sequence of feedback switching laws $\sigma_t : \mathbb{R}^{n(t+1)} \times \mathbb{R}^{pt} \times \mathcal{M}^t \times \mathbb{R}^p \to \mathcal{M}$ so that the switching sequence at time t is specified by the adversary as $\sigma(t) = \sigma_t(\mathcal{F}_t, u(t))$. The set of all admissible switching policies is denoted by $\mathcal{S}$.

Thus, the user and the adversary are playing a dynamic game: at each time t, the user decides $u(t)$ first and then the adversary decides $\sigma(t)$ with the full knowledge of $u(t)$.

Denote by $x(\cdot; \sigma, \mathbf{u}, z)$ the solution to the SLCS from the initial state z under the control policy $\mathbf{u} \in \mathcal{U}$ and switching policy $\sigma \in \mathcal{S}$. Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$.

Definition II.1. The SLCS is called *σ-resiliently exponentially stabilizable* if there exist an admissible control policy $\mathbf{u} \in \mathcal{U}$ and constants $\kappa \in [0, \infty)$, $\rho \in [0, 1)$ such that

$$\|x(t; \sigma, \mathbf{u}, z)\| \leq \kappa \rho^t \|z\|, \quad \forall\, t \in \mathbb{Z}_+, \forall\, z \in \mathbb{R}^n, \forall\, \sigma \in \mathcal{S}. \quad (2)$$

Definition II.2. For the SLCS (1), the infimum of all $\rho \geq 0$ for which (2) holds for some $\kappa \geq 0$ and $\mathbf{u} \in \mathcal{U}$ is called the *σ-resilient stabilizing rate* and denoted by ρ^*. An *optimal control* (if exists) is a $\mathbf{u} \in \mathcal{U}$ such that (2) holds for $\rho = \rho^*$.

Note that $\rho^* \in [0, \infty)$ provides a quantitative metric of the σ-resilient exponential stabilizability and its value is independent of the choice of the norm $\|\cdot\|$. The SLCS is σ-resiliently exponentially stabilizable if and only if $\rho^* < 1$.

When studying the σ-resilient stabilizability, the set $\mathcal{S}$ of adversarial switching policies can be equivalently replaced with the smaller set $\mathcal{M}^\infty$ of all *open-loop* switching policies, namely, switching sequences that are determined at time $t = 0$. This is because whenever a switching policy $\sigma \in \mathcal{S}$ destabilizes the SLCS, so does at least one switching sequence in $\mathcal{M}^\infty$, namely, the one actually produced by the policy σ. Hence, in the rest of this paper, we assume $\mathcal{S} = \mathcal{M}^\infty$ and think of $\sigma \in \mathcal{S}$ as switching sequences.

As a related notion, the *σ-resiliently asymptotical stabilizability* of the SLCS (1) is defined as the existence of $\mathbf{u} \in \mathcal{U}$ such that $x(t; \sigma, \mathbf{u}, z) \to 0$ as $t \to \infty$ for all $z \in \mathbb{R}^n$ and $\sigma \in \mathcal{S}$. The following result is proved in Appendix A.

Theorem II.1. *The SLCS (1) is σ-resiliently asymptotically stabilizable if and only if it is σ-resiliently exponentially stabilizable.*

The rest of the paper will focus on the σ-resilient (exponential) stabilizability and the stabilizing rate ρ^* for the SLCS (1). We first establish a homogeneous property of ρ^*.

Lemma II.1. *Let ρ^* be the σ-resilient stabilizing rate of the SLCS $\{(A_i, B_i)\}_{i\in\mathcal{M}}$. For any $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$, the SLCS $\{(\alpha A_i, \beta B_i)\}_{i\in\mathcal{M}}$ has the σ-resilient stabilizing rate $|\alpha| \cdot \rho^*$.*

Proof. This result is trivial if $\alpha = 0$. When $\alpha \neq 0$, the conclusion follows directly from the observation that the scaled SLCS $\{(\widetilde{A}_i = \alpha A_i, \widetilde{B}_i = \beta B_i)\}_{i\in\mathcal{M}}$ has the solution $\tilde{x}(t; \sigma, \tilde{\mathbf{u}}, z) = \alpha^t \cdot x(t; \sigma, \mathbf{u}, z)$ where $\tilde{\mathbf{u}}_t = \beta^{-1}\alpha^{t+1}\mathbf{u}_t, \forall\, t, z, \sigma$. □

When all $B_i = 0$, the σ-resilient stabilizability is reduced to the stability of the resulting autonomous SLS defined by $\{A_i\}_{i\in\mathcal{M}}$ under arbitrary switching, and ρ^* becomes the joint spectral radius (JSR) [28] of the matrix set $\{A_i\}_{i\in\mathcal{M}}$. More generally, under a static linear state feedback control policy $u(t) = Kx(t)$, the ρ^* of the closed-loop system is the JSR of $\{A_i + B_i K\}_{i\in\mathcal{M}}$. Note that the smallest possible JSR of $\{A_i + B_i K\}_{i\in\mathcal{M}}$ achieved by all gain matrices K is a conservative estimate of the ρ^* of the SLCS (1), since the optimal control policies are nonlinear in general (see Example V.2).

Example II.1. Consider a one-dimensional (1D) SLCS on $\mathbb{R}$ with two subsystems, where $A_1 = a_1$, $B_1 = b_1$, $A_2 = a_2$, and $B_2 = b_2$ are real numbers with $b_1^2 + b_2^2 \neq 0$. At any time t, given the state $x(t) \in \mathbb{R}$, the optimal control $u^*(t)$ can be shown to achieve the following infimum:

$$\inf_{u(t)\in\mathbb{R}} \max\Big\{|a_1 x(t) + b_1 u(t)|,\ |a_2 x(t) + b_2 u(t)|\Big\} = \frac{|a_1 b_2 - a_2 b_1|}{|b_1| + |b_2|}\,|x(t)| := \rho^* \cdot |x(t)|, \quad (3)$$

which also specifies the σ-resilient stabilizing rate ρ^*. Indeed, the optimal control $u^*(t)$ is $-[(a_1 - a_2)/(b_1 - b_2)]x(t)$ if $b_1 b_2 < 0$ and $-[(a_1 + a_2)/(b_1 + b_2)]x(t)$ if $b_1 b_2 > 0$. If $b_1 = 0$, then $u^*(t)$ can be any value between $(a_1 - a_2)x(t)/b_2$ and $-(a_1 + a_2)x(t)/b_2$. A similar result holds when $b_2 = 0$.

If $a_1 a_2 b_1 b_2 \leq 0$, then ρ^* in (3) is between the stabilizing rates $|a_1|$ and $|a_2|$ of the two autonomous subsystems. If $a_1 a_2 b_1 b_2 > 0$, then ρ^* can be smaller than both $|a_1|$ and $|a_2|$. For instance, $\rho^* = 0$ if $a_1/b_1 = a_2/b_2$, i.e., the two subsystems are scaled versions of each other. Indeed, from any $x(0)$, the control $u^*(0) = -(a_1/b_1)x(0) = -(a_2/b_2)x(0)$ ensures that $x(1) = 0$ regardless of $\sigma(0)$. Finally, since $b_1 b_2 \neq 0$, both the subsystems are controllable hence stabilizable; however, the SLCS may not be σ-resiliently exponentially stabilizable. □

Remark II.1. In the above example, the optimal user control policy $\mathbf{u}$ is of the static state feedback form $\mathbf{u}_t(\mathcal{F}_t) = g(x(t))$. That $\mathbf{u}_t$ depends solely on $x(t)$ is not surprising since the stabilizability property is entirely based on the behavior (i.e., convergence) of the future state solution, which depends on the past u, σ, and x only through the current state $x(t)$. The adversary will not gain any further advantage by knowing the user's optimal feedback control policy in advance. This observation remains valid for all the subsequent examples in this paper. On the other hand, if the user adopts an *open-loop* control policy (i.e., a control sequence), then the adversary by knowing such a sequence in advance will have a much greater advantage. In fact, it would be impossible to stabilize the SLCS in Example II.1 in the latter setting.

III. Defectiveness and Reducibility

In this section, we study those SLCSs whose σ-resilient stabilizing rates ρ^* can be exactly achieved.

Definition III.1 (Defectiveness)**.** The SLCS is called *nondefective* if there exist a control policy $\mathbf{u} \in \mathcal{U}$ and a constant $\kappa \geq 0$ such that $\|x(t;\sigma,\mathbf{u},z)\| \leq \kappa(\rho^*)^t\|z\|$, $\forall\, t \in \mathbb{Z}_+$, $\forall\, z \in \mathbb{R}^n$, $\forall\, \sigma \in \mathcal{S}$. Otherwise, it is called *defective*.

The notion of defectiveness helps to further distinguish the σ-resilient stabilizability of those SLCSs at the margin (i.e., with $\rho^* = 1$). The SLCS is called *σ-resiliently Lyapunov stabilizable* if there exist $\mathbf{u} \in \mathcal{U}$ and $\kappa \in [0,\infty)$ such that $\|x(t;\sigma,\mathbf{u},z)\| \leq \kappa\|z\|, \forall t \in \mathbb{Z}_+$, $\forall z \in \mathbb{R}^n$, $\forall \sigma \in \mathcal{S}$. By Definition III.1, the σ-resilient Lyapunov stabilizability is equivalent to either of the following two cases: (i) $\rho^* < 1$; (ii) $\rho^* = 1$ and the SLCS is nondefective.

An SLCS with $\rho^* = 0$ is nondefective if and only if it is resiliently controllable to the origin in one time step: for any $z \in \mathbb{R}^n$, there exists $v \in \mathbb{R}^p$ such that $A_i z + B_i v = 0, \forall i \in \mathcal{M}$; or equivalently, $A_i = B_i K, \forall i \in \mathcal{M}$, for some matrix K.

As an example, consider the LTI system (A,B), where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. As B is zero, $\rho^* = 1$ is the spectral radius of A. Since $x(t) = A^t x(0)$ is unbounded for some $x(0)$, the SLCS is defective. For another example, consider the LTI system (A,B) with $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which is controllable to the origin in two (but not one) steps. Thus, the system has $\rho^* = 0$ and is defective.

Tests for defectiveness are difficult to obtain. We establish easily verified conditions for (non-)defectiveness as follows.

Definition III.2. A subset $\mathcal{V} \subset \mathbb{R}^n$ is called *a control σ-invariant set* if for any $z \in \mathcal{V}$, there exists $v \in \mathbb{R}^p$ such that $A_i z + B_i v \in \mathcal{V}$ for all $i \in \mathcal{M}$. If $\mathcal{V}$ is further a subspace of $\mathbb{R}^n$, then it is called a *control σ-invariant subspace*.

Two trivial control σ-invariant subspaces are $\{0\}$ and $\mathbb{R}^n$.

Definition III.3 (Reducibility)**.** The SLCS (1) is called *irreducible* if it does not have any control σ-invariant subspaces other than $\{0\}$ and $\mathbb{R}^n$. Otherwise, it is called *reducible*.

If the SLCS is reducible, then there exists a proper nontrivial control σ-invariant subspace $\mathcal{V} \subsetneq \mathbb{R}^n$. After a common coordinate change $x = T\tilde{x} = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \tilde{x}$ where the range of T_1 is $\mathcal{V}$, the subsystem dynamics matrices of the SLCS (still denoted by A_i and B_i for simplicity) will be of the form

$$A_i = \begin{bmatrix} A_{i,11} & * \\ B_{i,2}K & A_{i,22} \end{bmatrix}, \quad B_i = \begin{bmatrix} B_{i,1} \\ B_{i,2} \end{bmatrix}, \quad \forall\, i \in \mathcal{M}, \tag{4}$$

where $*$ indicates a matrix of proper size and the matrix K is independent of i. Repeating this process if possible, the subsystem dynamics matrices will eventually have the form

$$A_i = \begin{bmatrix} A_{i,11} & * & \cdots & * \\ B_{i,2}K_1 & A_{i,22} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ B_{i,r}K_1 & B_{i,r}K_2 & \cdots & A_{i,rr} \end{bmatrix}, \quad B_i = \begin{bmatrix} B_{i,1} \\ B_{i,2} \\ \vdots \\ B_{i,r} \end{bmatrix} \tag{5}$$

for some matrices $K_1, \ldots, K_{r-1}$, where each of the SLCSs $\{(A_{i,jj}, B_{i,j})\}_{i\in\mathcal{M}}$ is irreducible for $j = 1,\ldots,r$. By the change of variables $u = \tilde{u} - \begin{bmatrix} K_1 & \cdots & K_{r-1} & 0 \end{bmatrix} x$, we derive the following standard form of reducible SLCSs:

$$\widetilde{A}_i = \begin{bmatrix} \widetilde{A}_{i,11} & * & \cdots & * \\ 0 & \widetilde{A}_{i,22} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \widetilde{A}_{i,rr} \end{bmatrix}, \quad \widetilde{B}_i = \begin{bmatrix} B_{i,1} \\ B_{i,2} \\ \vdots \\ B_{i,r} \end{bmatrix}, \tag{6}$$

where for each $j = 1,\ldots,r$, the SLCS $\{(\widetilde{A}_{i,jj}, B_{i,j}\}_{i\in\mathcal{M}}$ is irreducible. Clearly, the SLCSs (5) and (6) have the same σ-resilient stabilizing rate ρ^* as that of (1).

If an SLCS has $\rho^* = 0$ and is nondefective, then any subspace of $\mathbb{R}^n$ will be control σ-invariant; thus the system is reducible if its state dimension is greater than one.

Assume $\rho^* > 0$. Define an extended real valued function $\zeta : \mathbb{R}^n \to \mathbb{R}_+ \cup \{\infty\}$ as

$$\zeta(z) := \inf_{\mathbf{u}\in\mathcal{U}} \sup_{\sigma\in\mathcal{S}} \sup_{t\in\mathbb{Z}_+} \frac{\|x(t;\sigma,\mathbf{u},z)\|}{(\rho^*)^t}, \quad \forall\, z \in \mathbb{R}^n, \tag{7}$$

which is positively homogeneous of degree one: $\zeta(\alpha z) = \alpha\zeta(z)$ for all $\alpha \geq 0$. Since $\sup_\sigma \sup_t \|x(t;\sigma,\mathbf{u},z)\|/(\rho^*)^t$ is jointly convex in $\mathbf{u}$ and z and $\mathcal{U}$ is a vector space hence convex, by [29, pp. 87], ζ is convex on $\mathbb{R}^n$. Thus, the set

$$\mathcal{W} := \{z \in \mathbb{R}^n \,|\, \zeta(z) < \infty\} \tag{8}$$

is a subspace of $\mathbb{R}^n$. In Appendix B we will show that $\mathcal{W}$ is control σ-invariant and that the following result holds.

Theorem III.1. *An irreducible SLCS with $\rho^* > 0$ is nondefective.*

The converse of Theorem III.1 may not hold. A counterexample is given by the LTI system (A,B) with $B = 0$ and $A \in \mathbb{R}^{3\times 3}$ having two real eigenvalues $0 < \lambda_1 < \lambda_2$ with a Jordan block of order two for the eigenvalue λ_1.

Remark III.1. The concepts of defectiveness and reducibility have been proposed in the study of the JSR and the stability of autonomous SLS [23], [30]. They are extended to the SLCS in this paper. In particular, the proof of Theorem III.1 is an extension of that of [25, Theorem 2.1].

IV. Bounds on σ-Resilient Stabilizing Rate

In this section, a systematic approach for deriving bounds of the σ-resilient stabilizing rate ρ^* is developed.

A. Motivating Example

We first discuss a motivating example of the SLCSs.

Example IV.1. Consider the following SLCS on $\mathbb{R}^2$:

$$A_1 = \begin{bmatrix} a_1 & 0 \\ 0 & f_1 \end{bmatrix}, B_1 = \begin{bmatrix} b_1 \\ g_1 \end{bmatrix}; \quad A_2 = \begin{bmatrix} a_2 & 0 \\ 0 & f_2 \end{bmatrix}, B_2 = \begin{bmatrix} b_2 \\ g_2 \end{bmatrix},$$

which is obtained from two 1D SLCSs $\{(a_i,b_i)\}_{i=1,2}$ and $\{(f_i,g_i)\}_{i=1,2}$ that share a common control input and a common switching sequence. Denote by ρ^*, ρ_1^*, and ρ_2^* the σ-resilient stabilizing rate of the 2D SLCS and the two 1D SLCSs, respectively. Obviously, $\rho^* \geq \max\{\rho_1^*, \rho_2^*\}$.

In what follows, we assume that $a_1 \neq f_1$ or $a_2 \neq f_2$, and that B_1 and B_2 are not collinear, i.e., $b_1 g_2 \neq b_2 g_1$. Hence the following two constants cannot be both zero: $\alpha := (a_1 - f_1)g_2 - (a_2 - f_2)g_1$, and $\beta := (a_1 - f_1)b_2 - (a_2 - f_2)b_1$. Define two nonnegative functions $V, W : \mathbb{R}^2 \to \mathbb{R}_+$ by

$$V(z) := |\alpha z_1 - \beta z_2|, \quad W(z) := |\beta z_1 + \alpha z_2|, \ \forall z \in \mathbb{R}^2,$$

where $z = (z_1, z_2)$. Their null sets $\mathcal{N}_V := \{z \in \mathbb{R}^2 \,|\, V(z) = 0\}$ and $\mathcal{N}_W := \{z \in \mathbb{R}^2 \,|\, W(z) = 0\}$ are orthogonal 1D subspaces. It follows from (3) and $\alpha b_i - \beta g_i = (b_1 g_2 - b_2 g_1)(a_i - f_i)$, $\forall i = 1, 2$, that for any time t and $x(t) = z$,

$$\begin{aligned}
&\inf_{u(t)\in\mathbb{R}} \max_{i=1,2} V(A_i x(t) + B_i u(t)) \\
&= \inf_{u(t)\in\mathbb{R}} \max_{i=1,2} \left|(\alpha b_i - \beta g_i)u(t) + (\alpha a_i z_1 - \beta f_i z_2)\right| \\
&= \Bigg| \frac{(\alpha b_1 - \beta g_1)(\alpha a_2 z_1 - \beta f_2 z_2)}{\sum_{i=1}^2 |\alpha b_i - \beta g_i|} \\
&\qquad - \frac{(\alpha b_2 - \beta g_2)(\alpha a_1 z_1 - \beta f_1 z_2)}{\sum_{i=1}^2 |\alpha b_i - \beta g_i|} \Bigg| \\
&= \frac{|a_1 f_2 - a_2 f_1|}{|a_1 - f_1| + |a_2 - f_2|} \cdot V(x(t)) \, := \, \rho_0 \cdot V(x(t)). \quad (9)
\end{aligned}$$

The optimal $u^*(t)$ achieving the above infimum is given by

$$u^*(t) := -\frac{(\alpha a_1 z_1 - \beta f_1 z_2) \pm (\alpha a_2 z_1 - \beta f_2 z_2)}{(\alpha b_1 - \beta g_1) \pm (\alpha b_2 - \beta g_2)}, \qquad (10)$$

which is a linear state feedback controller with the sign "$\pm$" being "+" if $(a_1 - f_1)(a_2 - f_2) \geq 0$ and "−" otherwise.

The result in (9) implies that $\mathcal{N}_V$ is a control σ-invariant subspace. Also, if at each time t, the adversary chooses $\sigma(t) = \arg\max_i V(A_i x(t) + B_i u(t))$, then $V(x(t+1)) \geq \rho_0 V(x(t))$ regardless of $u(t)$. As $V(\cdot)$ is positively homogeneous of degree one, we conclude that $x(t)$ cannot decay at an exponential rate faster than ρ_0 from $x(0)$ satisfying $V(x(0)) > 0$, i.e.,

$$\rho^* \geq \rho_0 \, = \, \frac{|a_1 f_2 - a_2 f_1|}{|a_1 - f_1| + |a_2 - f_2|}. \qquad (11)$$

If the user adopts the feedback control strategy in (10), then

$$\begin{aligned}
V(x(t+1)) &= V(A_{\sigma(t)} x(t) + B_{\sigma(t)} u^*(t)) \\
&\leq \rho_0 \cdot V(x(t)), \quad \forall \sigma(t) \in \{1, 2\}, \forall x(t). \quad (12)
\end{aligned}$$

When $\rho_0 < 1$, $x(t) \to \mathcal{N}_V$ as $t \to \infty$ for any $\sigma \in \mathcal{S}$. To ensure that $x(t) \to 0$, one needs in addition that $x(t)$ will not diverge along $\mathcal{N}_V$. Pick any $x(t) = (z_1, z_2) \in \mathcal{N}_V$, i.e., $\alpha z_1 = \beta z_2$. It can be verified that, with the sign in (10) being either "+" or "−", we have $W(A_i x(t) + B_i u^*(t)) = \rho_i \cdot W(x(t))$, where

$$\rho_i := \frac{|g_i(a_1 b_2 - a_2 b_1) - b_i(f_1 g_2 - f_2 g_1)|}{|b_1 g_2 - b_2 g_1|}, \quad i = 1, 2. \quad (13)$$

Thus $W(x(t+1)) \leq \max\{\rho_1, \rho_2\} \cdot W(x(t))$ regardless of $\sigma(t)$. This, together with (12), implies that if $\max\{\rho_0, \rho_1, \rho_2\} < 1$, then the system is σ-resiliently stabilized by u^*. In view of Theorem II.1, we have $\rho^* < 1$ if $\max\{\rho_0, \rho_1, \rho_2\} < 1$. As $\max\{\rho_0, \rho_1, \rho_2\}$ has the exact same scaling properties as ρ^* in Lemma II.1, we obtain via a scaling argument that

$$\rho^* \leq \max\{\rho_0, \rho_1, \rho_2\}. \qquad (14)$$

In the case $\rho_0 \geq \max\{\rho_1, \rho_2\}$, $\rho^* = \rho_0$ by (11). For example, assume $a_1 b_2 = a_2 b_1$ and $f_1 g_2 = f_2 g_1$. By Example II.1, $\rho_1^* = \rho_2^* = 0$, while $\rho^* = \rho_0 > 0$ provided $a_1 f_2 - a_2 f_1 \neq 0$. □

B. Bounds via Seminorms

We now formalize the technique employed in Example IV.1. Recall that a *seminorm* on $\mathbb{R}^n$ is defined as a nonnegative function $\xi : \mathbb{R}^n \to \mathbb{R}_+$ that is convex (hence continuous) and positively homogeneous of degree one [31]. A seminorm is a norm if it is positive definite, i.e., $\xi(z) > 0$ whenever $z \neq 0$.

Lemma IV.1. *For an arbitrary seminorm ξ on $\mathbb{R}^n$, let the mapping $\mathcal{T} : \xi \mapsto \xi_\sharp$ be defined by, $\forall z \in \mathbb{R}^n$,*

$$\xi_\sharp(z) = \mathcal{T}[\xi](z) := \inf_{v\in\mathbb{R}^p} \max_{i\in\mathcal{M}} \xi(A_i z + B_i v). \qquad (15)$$

More generally, for any $h \in \mathbb{N}$, define the mapping $\mathcal{T}^{(h)}$ by

$$\mathcal{T}^{(h)}[\xi](z) := \inf_{v(0)\in\mathbb{R}^p} \max_{i(0)\in\mathcal{M}} \cdots \inf_{v(h-1)\in\mathbb{R}^p} \max_{i(h-1)\in\mathcal{M}}$$
$$\xi\left(A_{i(h-1)} \cdots A_{i(0)} z + \sum_{j=0}^{h-1} A_{i(h-1)} \cdots A_{i(j+1)} B_{i(j)} v(j)\right).$$

Then $\mathcal{T}(\xi)$ and $\mathcal{T}^{(h)}(\xi)$ are also seminorms on $\mathbb{R}^n$, i.e., $\mathcal{T}$ and $\mathcal{T}^{(h)}$ are self maps of seminorms on $\mathbb{R}^n$.

Proof. That $\xi_\sharp(\cdot)$ is pointwise finite and nonnegative is obvious. It is convex since $\max_{i\in\mathcal{M}} \xi(A_i z + B_i v)$ is convex in (z, v) [29, pp. 88]. To show the homogeneity, let $\alpha \neq 0$ be arbitrary. By setting $v' := v/\alpha$, we have $\xi_\sharp(\alpha z) = \inf_{v\in\mathbb{R}^p} \max_{i\in\mathcal{M}} \xi(\alpha A_i z + B_i v) = \inf_{v'\in\mathbb{R}^p} \max_{i\in\mathcal{M}} |\alpha| \cdot \xi(A_i z + B_i v') = |\alpha| \cdot \xi_\sharp(z)$. When $\alpha = 0$, it is obvious from (15) that $\xi_\sharp(0) = 0$. This shows that $\xi_\sharp$ is a seminorm on $\mathbb{R}^n$. The proof for $\mathcal{T}^{(h)}$ is similar hence omitted. □

If $\xi(\cdot) = \|\cdot\|$ is a norm on $\mathbb{R}^n$, then $\xi_\sharp(\cdot)$, which we denote as $\|\cdot\|_\sharp$, is a seminorm but not necessarily a norm on $\mathbb{R}^n$. For instance, if the two 1D subsystem dynamics in Example II.1 are scaled version of each other, i.e., $a_1/b_1 = a_2/b_2$, then $|\cdot|_\sharp \equiv 0$, which is not a norm on $\mathbb{R}$. Also, $\mathcal{T}^{(h)}[\xi](z)$ defined above is the solution to the h-horizon problem $\inf_{\mathbf{u}} \max_\sigma \xi(x(h; \sigma, \mathbf{u}, z))$ and $\mathcal{T}^{(h)} = \mathcal{T}$ if $h = 1$.

Lemma IV.2. *Suppose ξ is a seminorm on $\mathbb{R}^n$. Then for any given $z \in \mathbb{R}^n$, the function $f(v) := \max_{i\in\mathcal{M}} \xi(A_i z + B_i v)$ attains a (possibly non-unique) minimizer in $\mathbb{R}^p$.*

Proof. Let $\mathcal{V}$ be the subspace $\{v \in \mathbb{R}^p \,|\, \xi(B_i v) = 0, \forall i \in \mathcal{M}\}$ and $\mathcal{V}^\perp$ be its orthogonal complement. Any $v \in \mathbb{R}^p$ can be decomposed uniquely as $v = v_1 + v_2$, where $v_1 \in \mathcal{V}$ and $v_2 \in \mathcal{V}^\perp$. Since $f(v) = f(v_2)$, v_1 can be set to zero without loss of generality. Define $g(v_2) := \max_{i\in\mathcal{M}} \xi(B_i v_2)$, which is a norm on $\mathcal{V}^\perp$. Then, $g(v_2) \leq \max_{i\in\mathcal{M}}[\xi(A_i z + B_i v_2) + \xi(-A_i z)] \leq f(v_2) + \max_{i\in\mathcal{M}} \xi(A_i z)$. This implies that any nonempty sub-level set of $f(\cdot)$ (which is closed as f is continuous) restricted to $\mathcal{V}^\perp$ is contained in a sub-level set of $g(\cdot)$ and thus bounded and compact. Since f is continuous, a minimizer exists. □

Lemma IV.3. *The mapping $\mathcal{T} : \xi \mapsto \xi_\sharp$ defined in (15) has the following properties.*

- **(Monotonicity)**: *For two extended real-valued seminorms* ξ *and* ξ' *with* $\xi \le \xi'$, $\mathcal{T}(\xi) \le \mathcal{T}(\xi')$.
- **(Monotone Continuity)**: *Let* $(\xi_k)_{k\in\mathbb{N}}$ *be a nonincreasing sequence of seminorms whose (pointwise) limit is denoted by* $\xi_\infty = \lim_{k\to\infty} \xi_k$. *Then* $\lim_{k\to\infty} \mathcal{T}(\xi_k) = \mathcal{T}(\xi_\infty)$.

Proof. The first property is trivial. To show the second property, suppose $\xi_k(z) \downarrow \xi_\infty(z)$, $\forall\, z \in \mathbb{R}^n$. Being the pointwise limit of the seminorms ξ_k, ξ_∞ is also a seminorm on $\mathbb{R}^n$. Let $\eta_k := \mathcal{T}(\xi_k)$. By Lemma IV.1 and the first property, (η_k) is a nonincreasing sequence of seminorms, whose pointwise limit $\eta_\infty := \lim_{k\to\infty} \eta_k$ is also a seminorm. As $\xi_k \ge \xi_\infty$, $\eta_k \ge \mathcal{T}(\xi_\infty), \forall\, k$, which implies $\eta_\infty \ge \mathcal{T}(\xi_\infty)$.

To prove the other direction, fix an arbitrary $z \in \mathbb{R}^n$. Since ξ_∞ is a seminorm, Lemma IV.2 implies that the minimizer $v^* = \arg\min_v [\max_{i\in\mathcal{M}} \xi_\infty(A_i z + B_i v)]$ exists. Since $\max_{i\in\mathcal{M}} \xi_\infty(\cdot)$ is also a pointwise limit of $\max_{i\in\mathcal{M}} \xi_k(\cdot)$ as $k \to \infty$, we deduce that for any $\varepsilon > 0$, there exists N large enough such that $\max_{i\in\mathcal{M}} \xi_k(A_i z + B_i v^*) \le \max_{i\in\mathcal{M}} \xi_\infty(A_i z + B_i v^*) + \varepsilon$ for all $k \ge N$. Thus,

$$\begin{aligned}\eta_k(z) \;&=\; \inf_v \max_{i\in\mathcal{M}} \xi_k(A_i z + B_i v) \;\le\; \max_{i\in\mathcal{M}} \xi_k(A_i z + B_i v^*) \\ &\le\; \max_{i\in\mathcal{M}} \xi_\infty(A_i z + B_i v^*) + \varepsilon \;=\; \mathcal{T}[\xi_\infty](z) + \varepsilon,\end{aligned}$$

for all $k \ge N$. Letting $k \to \infty$ and noting that $\varepsilon > 0$ is arbitrary, we have $\eta_\infty(z) \ge \mathcal{T}[\xi_\infty](z)$ for any $z \in \mathbb{R}^n$. □

Proposition IV.1. *Let* ξ *be a non-zero seminorm on* $\mathbb{R}^n$ *and* $\alpha \ge 0$ *be a constant such that* $\xi_\sharp(\cdot) \ge \alpha\, \xi(\cdot)$. *Then* $\rho^* \ge \alpha$.

Proof. Assume the adversary adopts the switching policy $\sigma(t) = \arg\max_i \xi(A_i x(t) + B_i u(t))$ at each t for any given $x(t), u(t)$. Hence $\sigma \in \mathcal{S}$. Then from $x(0)$ with $\xi(x(0)) > 0$, we have, for any $u(t)$,

$$\begin{aligned}\xi(x(t+1)) \;&=\; \xi(A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t)) \\ &=\; \max_{i\in\mathcal{M}} \xi(A_i x(t) + B_i u(t)) \;\ge\; \xi_\sharp(x(t)) \;\ge\; \alpha\, \xi(x(t)), \ \forall\, t.\end{aligned}$$

This shows that $c\|x(t)\| \ge \xi(x(t)) \ge \alpha^t \cdot \xi(x(0))$, $\forall\, t \in \mathbb{Z}_+$, where $c := \sup_{\|z\|=1} \xi(z) > 0$. Hence there exist no $\rho < \alpha$ and a constant $\kappa > 0$ so that $\|x(t)\| \le \kappa \rho^t \|x(0)\|$, $\forall\, t$. □

Proposition IV.1 has been applied in Example IV.1 with $\xi(\cdot) = V(\cdot)$ and $\alpha = \rho_0$ in equation (9).

Proposition IV.2. *Let* $\|\cdot\|$ *be a norm on* $\mathbb{R}^n$ *such that* $\|\cdot\|_\sharp \le \beta\, \|\cdot\|$ *for some* $\beta \ge 0$. *Then,* $\rho^* \le \beta$.

Proof. Suppose the user adopts the control policy $u^*(t) = \mathbf{u}_t(x(t)) := \arg\min_v \max_{i\in\mathcal{M}} \|A_i x(t) + B_i v\|$, $\forall\, t$, which exists by Lemma IV.2. Then for any $\sigma \in \mathcal{S}$ and any $t \in \mathbb{Z}_+$,

$$\begin{aligned}\|x(t+1)\| \;&=\; \left\|A_{\sigma(t)} x(t) + B_{\sigma(t)} u^*(t)\right\| \\ &\le\; \max_{i\in\mathcal{M}} \left\|A_i x(t) + B_i u^*(t)\right\| \;=\; \|x(t)\|_\sharp \le \beta \|x(t)\|.\end{aligned}$$

This implies that $\|x(t)\| \le \beta^t \|x(0)\|$, $\forall\, t$, i.e., $\rho^* \le \beta$. □

The next result follows from Propositions IV.1 and IV.2.

Corollary IV.1. *If* $\alpha\|\cdot\| \le \|\cdot\|_\sharp \le \beta\|\cdot\|$ *for some norm* $\|\cdot\|$ *on* $\mathbb{R}^n$, *then* $\alpha \le \rho^* \le \beta$.

By using the operator $\mathcal{T}^{(h)}$ instead of $\mathcal{T}$ and considering per-h-step growth of the state solutions, we obtain the following result whose proof is similar hence omitted.

Proposition IV.3. *Let* $h \in \mathbb{N}$. *If* ξ *is a nonzero seminorm on* $\mathbb{R}^n$ *and* $\mathcal{T}^{(h)}(\xi) \ge \alpha\xi$ *for some constant* $\alpha \ge 0$, *then* $\rho^* \ge \sqrt[h]{\alpha}$. *Further, if* ξ *is a norm on* $\mathbb{R}^n$ *and* $\alpha\xi \le \mathcal{T}^{(h)}(\xi) \le \beta\xi$ *for some constants* $\alpha, \beta \ge 0$, *then* $\sqrt[h]{\alpha} \le \rho^* \le \sqrt[h]{\beta}$.

C. Extremal Norms

By Corollary IV.1, associated with each norm $\|\cdot\|$ are the following lower and upper bounds of ρ^*:

$$\begin{aligned}\alpha^* &:= \sup\left\{\alpha \in \mathbb{R}_+ \mid \alpha\|\cdot\| \le \|\cdot\|_\sharp\right\}, \\ \beta^* &:= \inf\left\{\beta \in \mathbb{R}_+ \mid \|\cdot\|_\sharp \le \beta\|\cdot\|\right\}.\end{aligned}$$

A natural question arises: can such bounds be tight?

Definition IV.1. *A norm* $\|\cdot\|$ *on* $\mathbb{R}^n$ *is called an (upper) extremal norm of the SLCS (1) if* $\|\cdot\|_\sharp \le \rho^*\|\cdot\|$.

Suppose an extremal norm $\|\cdot\|$ exists. Then the property $\|\cdot\|_\sharp \le \rho^*\|\cdot\|$ implies that there is an optimal control policy

$$\mathbf{u}_t(x(t)) := \arg\min_v \max_{i\in\mathcal{M}} \|A_i x(t) + B_i v\|, \ \forall x(t), \forall\, t, \quad (16)$$

under which we have $\|x(t+1)\| \le \rho^*\|x(t)\|$, hence $\|x(t)\| \le (\rho^*)^t\|x(0)\|$, $\forall\, t \in \mathbb{Z}_+, x(0) \in \mathbb{R}^n, \sigma \in \mathcal{S}$. This implies that the SLCS is nondefective. The following theorem, proved in Appendix C, shows that the converse is also true.

Theorem IV.1. *An extremal norm of the SLCS exists if and only if the SLCS is nondefective.*

We next focus on seminorms that yield tight lower bounds.

Definition IV.2. *A nonzero seminorm* ξ *on* $\mathbb{R}^n$ *is called a lower extremal seminorm if* $\xi_\sharp(\cdot) \ge \rho^* \cdot \xi(\cdot)$.

In Appendix D, we prove the following result.

Theorem IV.2. *If the SLCS is nondefective, then a lower extremal seminorm exists.*

The converse of Theorem IV.2 is not true, at least when $\rho^* = 0$. For example, the (non-switched) LTI system (A, B) with $A = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ and $B = \begin{bmatrix}1\\ 1\end{bmatrix}$ has $\rho^* = 0$; hence any seminorm is a lower extremal seminorm. However, as discussed in Section II, this system is defective.

Some norms can be both upper and lower extremal.

Definition IV.3. *A norm* $\|\cdot\|$ *on* $\mathbb{R}^n$ *is called a* Barabanov norm *if* $\|\cdot\|_\sharp = \rho^*\|\cdot\|$.

The following result is proved in Appendix E.

Theorem IV.3. *If the SLCS is irreducible, then a Barabanov norm exists.*

In Example II.1 with $a_1/b_1 = a_2/b_2$, the 1D SLCS has a Barabanov norm $|\cdot|$. Another example is given below.

Example IV.2. Consider the following SLCS on $\mathbb{R}^2$:

$$A_1 = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \ B_1 = \begin{bmatrix}1\\ 1\end{bmatrix}; \quad A_2 = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \ B_2 = \begin{bmatrix}0\\ 1\end{bmatrix}.$$

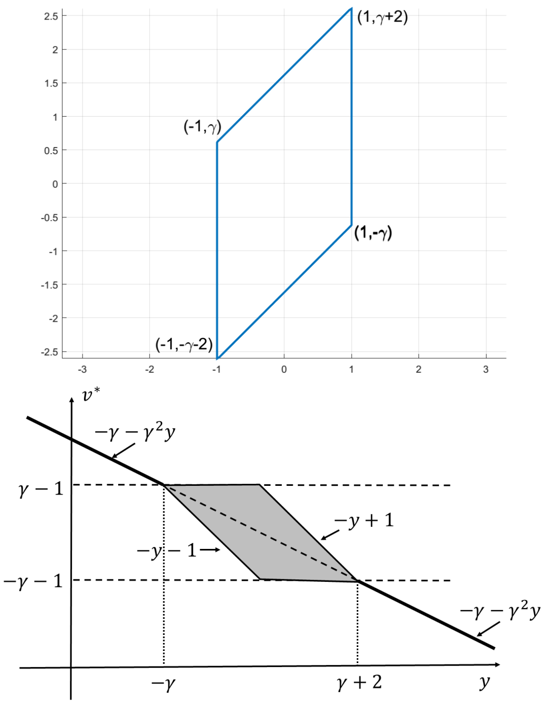

Fig. 1. Top: Unit ball of the norm $\|\cdot\|$ in Example IV.2; Bottom: v^* for $z=(1,y)$, $\forall\, y\in\mathbb{R}$.

Define a norm on $\mathbb{R}^2$ as

$$\|z\| \,:=\, \max\{|z_1|,\, \gamma|z_2-z_1|\}, \quad \forall\, z\in(z_1,z_2)\in\mathbb{R}^2,$$

where $\gamma=\frac{\sqrt{5}-1}{2}$ satisfies $\gamma=1/(\gamma+1)$. We claim that $\|\cdot\|_\sharp=\gamma\|\cdot\|$. By homogeneity, we only need to check this claim for $z=(0,1)$ and for $z=(1,y)$ where $y\in\mathbb{R}$. If $z=(0,1)$, then $\|z\|=\gamma$, and $\|z\|_\sharp=\inf_v\max\{\|(v,v)\|,\|(0,1+v)\|\}=\inf_v\max\{|v|,\gamma|1+v|\}=\gamma^2=\gamma\|z\|$, where the minimum is achieved at $v^*=-\gamma^2$. Suppose $z=(1,y)$. Then $\|z\|=\max\{1,\gamma|y-1|\}$, and $\|z\|_\sharp=\inf_v\max\{|1+v|,\gamma,\gamma|y+v|\}$.

- Case 1: Suppose $y<-\gamma$. Then $\|z\|=\gamma|y-1|$, and $\|z\|_\sharp=\gamma^2(1-y)$ with $v^*=-\gamma-\gamma^2 y$;
- Case 2: Suppose $-\gamma\le y\le\gamma+2$. Then $\|z\|=1$, and $\|z\|_\sharp=\gamma$ where v^* can take any value between $\max\{-y-1,-\gamma-1\}$ and $\min\{-y+1,\gamma-1\}$.
- Case 3: Suppose $y>\gamma+2$. Then $\|z\|=\gamma|y-1|$, and $\|z\|_\sharp=\gamma^2(y-1)$ with $v^*=-\gamma-\gamma^2 y$.

In Fig. 1, we plot the unit ball of $\|\cdot\|$ on the top, and the function $v^*(y)$ at the bottom (the shaded region indicates that the value of v^* is not unique). Note that the optimal control policy can be chosen to be linear: $u^*(t)=[-\gamma \quad -\gamma^2]x(t)$.

Remark IV.1. The notions of extremal and Barabanov norms are originally proposed for the study of the JSR and the stability of autonomous SLSs [23], [27], [32]. We extend them to the resilient stabilization of the SLCS. The proofs of Theorem IV.1 and Theorem IV.3 are inspired by those of [26, Theorem 3] and [23], respectively. See also [25, Theorem 2.1]. Another relevant method is the variational approach [33].

Extremal norms can also be defined in terms of $\mathcal{T}^{(h)}$. Specifically, (i) a nonzero seminorm ξ is lower h-extremal if $\mathcal{T}^{(h)}(\xi)\ge(\rho^*)^h\xi$; (ii) a norm $\|\cdot\|$ is (upper) h-extremal if $\mathcal{T}^{(h)}(\|\cdot\|)\le(\rho^*)^h\|\cdot\|$, and it is an h-Barabanov norm if $\mathcal{T}^{(h)}(\|\cdot\|)=(\rho^*)^h\|\cdot\|$. Although (1-)extremal (semi)norms are also h-extremal, the converse may not be true.

D. Norms under Linear Transformations

The norm bounding techniques introduced in this section are independent of coordinates on $\mathbb{R}^n$. To see this, consider the coordinate change $x=T\tilde{x}$ by a nonsingular matrix $T\in\mathbb{R}^{n\times n}$. A norm (resp. seminorm) ξ in x-coordinates is transformed by T to the norm (resp. seminorm) $\tilde{\xi}:=\xi\circ T$ in the $\tilde{x}$-coordinates. Denote the SLCS $\{(A_i,B_i)\}_{i\in\mathcal{M}}$ in x-coordinates by $\mathfrak{S}$. In $\tilde{x}$-coordinates it has the form $\widetilde{\mathfrak{S}}=\{(\widetilde{A}_i,\widetilde{B}_i)\}_{i\in\mathcal{M}}$ where $\widetilde{A}_i:=T^{-1}A_iT$ and $\widetilde{B}_i:=T^{-1}B_i$. Obviously, $\mathfrak{S}$ and $\widetilde{\mathfrak{S}}$ have the same σ-resilient stabilizing rate, i.e., $\rho^*=\tilde{\rho}^*$. Similar to $\mathcal{T}$ defined in (15) for $\mathfrak{S}$, we define a mapping $\widetilde{\mathcal{T}}$ for $\widetilde{\mathfrak{S}}$ by

$$\widetilde{\mathcal{T}}[\tilde{\xi}](z) \,:=\, \inf_{v\in\mathbb{R}^p}\max_{i\in\mathcal{M}}\tilde{\xi}(\widetilde{A}_iz+\widetilde{B}_iv), \quad \forall\, z\in\mathbb{R}^n,$$

which satisfies $\widetilde{\mathcal{T}}(\xi\circ T)=\mathcal{T}(\xi)\circ T$. Then a seminorm ξ satisfies $\alpha\xi\le\mathcal{T}(\xi)\le\beta\xi$ for some $\alpha,\beta\ge 0$ if and only if $\tilde{\xi}=\xi\circ T$ satisfies $\alpha\tilde{\xi}\le\widetilde{\mathcal{T}}(\tilde{\xi})\le\beta\tilde{\xi}$. In particular, if ξ is an extremal norm (resp. lower extremal seminorm, Barabanov norm) for $\mathfrak{S}$, so is $\tilde{\xi}$ for $\widetilde{\mathfrak{S}}$.

Given two norms ξ and ξ' on $\mathbb{R}^n$, define

$$d(\xi,\xi') \,:=\, \log\left(\frac{\min\{\beta\ge 0\,|\,\xi'\le\beta\xi\}}{\max\{\alpha\ge 0\,|\,\xi'\ge\alpha\xi\}}\right),$$

which measures how similar the unit balls of ξ and ξ' are after proper scalings. Define an equivalence relation for norms on $\mathbb{R}^n$ as $\xi\sim\xi'$ if and only if $\xi'=\gamma\xi$ for some $\gamma>0$, and denote by $[\xi]$ the equivalent class that ξ belongs to. Then $d(\cdot,\cdot)$ specifies a metric on the family of equivalent classes of norms on $\mathbb{R}^n$ (see [34] for a more general metric). The mapping $\mathcal{T}$ (or $\widetilde{\mathcal{T}}$), which preserves this equivalent relation, can be extended to a mapping between equivalent classes of norms.

A norm ξ^* is a Barabanov norm of the SLCS $\mathfrak{S}$ if and only if $d(\xi^*,\mathcal{T}(\xi^*))=0$, or equivalently, if the equivalent class $[\xi^*]$ is a fixed point of $\mathcal{T}$. In the next section, we will search for Barabanov norms in various subsets $\mathcal{K}$ of norms. The distance $d(\mathcal{K},\xi^*)\,:=\,\inf\{d(\xi,\xi^*)\,|\,\xi\in\mathcal{K}\}$ measures quantitatively how well norms in $\mathcal{K}$ approximate the Barabanov norm ξ^* (if exists). In practice, as ξ^* is difficult to find or even nonexistent, one can use $\inf_{\xi\in\mathcal{K}}d(\xi,\mathcal{T}(\xi))$ as an indicator for the proximity of the best norms in $\mathcal{K}$ to being a Barabanov norm.

The following result will be useful in Section V-A.

Lemma IV.4 (Fritz John's Theorem [35]). *Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^n$, and let $\mathcal{K}_e$ be the set of all norms of the form $\|\cdot\|\circ T$ for some nonsingular $T\in\mathbb{R}^{n\times n}$ (such norms are called ellipsoidal norms; see Section V-A). Then for an arbitrary norm ξ on $\mathbb{R}^n$, $d(\mathcal{K}_e,\xi)\le\log(\sqrt{n})$.*

Indeed, one choice of the norm in $\mathcal{K}_e$ with the smallest d-distance to ξ is such that its unit ball is the largest ellipsoid contained in the unit ball of ξ (see [34]).

V. Computing σ-resilient Stabilizing Rate

Using the results in Section IV, we now use certain families of norms to compute bounds on ρ^*. For a given σ-resiliently stabilizable SLCS, the computed norm $\|\cdot\|$ can be used to devise a σ-resiliently stabilizing controller in the form of (16).

A. Ellipsoidal Norms

Denote by $\mathbb{P}_{\succ 0}$ and $\mathbb{P}_{\succeq 0}$ the sets of all $n \times n$ positive definite (P.D.) and positive semidefinite (P.S.D.) matrices, respectively. We write $P \succ 0$ if $P \in \mathbb{P}_{\succ 0}$ and $P \succeq 0$ if $P \in \mathbb{P}_{\succeq 0}$. For each $P \succeq 0$, $\|z\|_P := \sqrt{z^T P z}$ defines a seminorm on $\mathbb{R}^n$. If $P \succ 0$, then $\|\cdot\|_P$ is a norm, called an ellipsoidal norm as its unit ball is an ellipsoid. Note that $\|\cdot\|_P = \|\cdot\|_I \circ T$ where $T = P^{1/2}$.

Applying the results in Section IV to the ellipsoidal norms, we obtain lower and upper bounds on ρ^*. As shown below, the best such bounds are off by at most a factor of $\sqrt{n}$.

Proposition V.1. *Let the SLCS be irreducible. Then there exists an ellipsoidal norm $\|\cdot\|_P$ based on which the lower bound of ρ^* obtained from Proposition IV.1 is at least $\rho^*/\sqrt{n}$ and the upper bound of ρ^* obtained from Proposition IV.2 is at most $\sqrt{n} \cdot \rho^*$.*

Proof. The irreducibility assumption implies that the SLCS has a Barabanov norm ξ^*. By Lemma IV.4, there exists an ellipsoidal norm $\xi = \|\cdot\|_P$ satisfying $\xi \leq \xi^* \leq \sqrt{n}\,\xi$. This implies that $\mathcal{T}(\xi) \leq \mathcal{T}(\xi^*) = \rho^* \xi^* \leq \sqrt{n} \cdot \mathcal{T}(\xi)$. In particular, $\mathcal{T}(\xi) \leq \rho^* \xi^* \leq \rho^* \sqrt{n}\,\xi$ and $\mathcal{T}(\xi) \geq (\rho^*/\sqrt{n})\,\xi^* \geq (\rho^*/\sqrt{n})\,\xi$. This proves the desired results. ☐

In Proposition V.1, both the lower and upper bounds are achieved by the same ellipsoidal norm. Using different ellipsoidal norms, one may obtain tighter bounds. Moreover, if the SLCS is nondefective, then only the second part of the statement regarding the upper bound of ρ^* holds true.

Remark V.1. By using $\mathcal{T}^{(h)}$ with $h > 1$ and Proposition IV.3, the results in Proposition V.1 can be improved: there exists an ellipsoidal norm $\xi = \|\cdot\|_P$ satisfying $[(\rho^*)^h/\sqrt{n}] \cdot \xi \leq \mathcal{T}^{(h)}(\xi) \leq (\rho^*)^h \sqrt{n}\,\xi$, thus providing a lower bound of at least $\rho^*/\sqrt[2h]{n}$ and an upper bound of at most $\sqrt[2h]{n} \cdot \rho^*$ for ρ^*. For a fixed n, as $h \to \infty$, estimate errors can be made arbitrarily small. The drawback of using a large h, however, is the much increased complexity in evaluating $\mathcal{T}^{(h)}(\xi)$.

To find the bounds of ρ^* from the ellipsoidal norms, we introduce the following notation. Let $m = |\mathcal{M}|$, and define

$$\Delta := \left\{\theta \in \mathbb{R}^m \,|\, \theta_i \geq 0, \forall\, i \in \mathcal{M}, \textstyle\sum_{i \in \mathcal{M}} \theta_i = 1\right\}$$

to be the m-simplex. For each $\theta \in \Delta$ and $P \succeq 0$, define

$$\Gamma_\theta(P) := \sum_{i \in \mathcal{M}} \theta_i A_i^T P A_i - \left(\sum_{i \in \mathcal{M}} \theta_i A_i^T P B_i\right) \times \left(\sum_{i \in \mathcal{M}} \theta_i B_i^T P B_i\right)^{\dagger} \left(\sum_{i \in \mathcal{M}} \theta_i B_i^T P A_i\right), \quad (17)$$

where $\dagger$ denotes the matrix pseudo inverse. Note that $\Gamma_\theta(P)$ is the (generalized) Schur complement [36, pp. 28] of the lower right block of the following P.S.D. matrix:

$$\Upsilon_\theta(P) := \sum_{i \in \mathcal{M}} \theta_i \begin{bmatrix} A_i^T P A_i & A_i^T P B_i \\ B_i^T P A_i & B_i^T P B_i \end{bmatrix}. \quad (18)$$

From this we conclude that: (i) $\Gamma_\theta(P) \succeq 0$; and (ii) for a fixed P (resp. θ), $\Gamma_\theta(P)$ is a PSD-concave mapping of θ (resp. P) into $\mathbb{P}_{\succeq 0}$ under the partial order $\succeq$ (cf. [29]). Define the set

$$\Gamma_\Delta(P) := \{\Gamma_\theta(P) \,|\, \theta \in \Delta\} \subset \mathbb{P}_{\succeq 0}.$$

Lemma V.1. *For each $P \succeq 0$, denote $\|\cdot\|_{P\sharp} := \mathcal{T}(\|\cdot\|_P)$ where the operator $\mathcal{T}$ is defined in (15). Then, $\forall\, z \in \mathbb{R}^n$,*

$$\|z\|_{P\sharp} = \sup_{\theta \in \Delta} \|z\|_{\Gamma_\theta(P)} = \sup_{Q \in \Gamma_\Delta(P)} \|z\|_Q. \quad (19)$$

Proof. It follows from (15) that $(\|z\|_{P\sharp})^2$ is the optimal value of the following optimization problem in $r \in \mathbb{R}$ and $v \in \mathbb{R}^p$:

$$\begin{aligned} &\text{minimize} \;\; r \geq 0 && (20)\\ &\text{subject to} \;\; (A_i z + B_i v)^T P (A_i z + B_i v) \leq r, \; \forall\, i \in \mathcal{M}. \end{aligned}$$

By introducing the multipliers (dual variables) $\theta_i \geq 0$ for each $i \in \mathcal{M}$, the dual problem of (20) is easily seen to be

$$\max_{\theta \in \Delta} \; z^T \Gamma_\theta(P) z. \quad (21)$$

Since the optimization problem (20) is both convex (indeed a second order cone programming) and strongly feasible (r can be made arbitrarily large), it has the same optimal value as that of (21). This proves the desired result. ☐

We now apply Proposition IV.1 to the ellipsoidal norm $\|\cdot\|_P$ for $P \succ 0$. By Lemma V.1, the condition $\|\cdot\|_{P\sharp} \geq \alpha \|\cdot\|_P$ is equivalent to $\sup_{\theta \in \Delta} z^T \Gamma_\theta(P) z \geq \alpha^2 z^T P z$, $\forall z$. A sufficient condition for this to hold is $\Gamma_\theta(P) \succeq \alpha^2 P$ for some $\theta \in \Delta$, or by using the Schur complement

$$\Upsilon_\theta(P) - \begin{bmatrix} \alpha^2 P & 0 \\ 0 & 0 \end{bmatrix} \succeq 0 \quad (22)$$

for some $\theta \in \Delta$, where $\Upsilon_\theta(P)$ is defined in (18). Hence, Proposition IV.1 implies the following result.

Proposition V.2. *Suppose the matrix inequality (22) holds for some $\alpha \geq 0$, $P \succ 0$, and $\theta \in \Delta$. Then the σ-resilient stabilizing rate ρ^* satisfies $\rho^* \geq \alpha$.*

If $P \succ 0$ is given, then a lower bound of ρ^* is obtained by finding the largest possible α satisfying (22) for some $\theta \in \Delta$, which is a semidefinite program (SDP) that is easily solvable. To find the best such lower bound, we can solve the bilinear matrix inequality (BMI) problem in (α^2, P, θ):

$$\max_{\alpha^2 \geq 0,\, P \succ 0,\, \theta \in \Delta} \alpha^2, \quad \text{subject to the constraint (22).} \quad (23)$$

We next apply Proposition IV.2 to the ellipsoidal norms. Given $P \succ 0$, the condition $\|\cdot\|_{P\sharp} \leq \beta \|\cdot\|_P$ is equivalent to $\sup_{\theta \in \Delta} z^T \Gamma_\theta(P) z \leq \beta^2 z^T P z$, $\forall z$, or equivalently, $\Gamma_\theta(P) \preceq \beta^2 P$ for all $\theta \in \Delta$. As a result, an upper bound of ρ^* is provided by the solution β^* to the following problem:

$$\min_{\beta \geq 0} \beta, \quad \text{subject to} \;\; \Gamma_\theta(P) \preceq \beta^2 P, \quad \forall\, \theta \in \Delta. \quad (24)$$

The above problem is difficult to solve since it is insufficient to check the constraint at the vertices of the m-simplex Δ only as $\Gamma_\theta(P)$ is concave in θ.

An easily computed upper bound of ρ^* is described as follows. For a given $P \succ 0$, the condition $\|\cdot\|_{P\sharp} \leq \beta\|\cdot\|_P$ for some $\beta \geq 0$ is equivalent to

$$\inf_v \max_i (A_i z + B_i v)^T P(A_i z + B_i v) \leq \beta^2 z^T P z, \ \forall z. \quad (25)$$

Set $v = Kz$ for some $K \in \mathbb{R}^{p\times n}$. Then using Schur complement, a sufficient condition for (25) is given by

$$\begin{bmatrix} \beta Q & (A_i Q + B_i F)^T \\ A_i Q + B_i F & \beta Q \end{bmatrix} \succeq 0, \quad \forall i \in \mathcal{M}, \quad (26)$$

where $Q := P^{-1} \succ 0$ and $F := KP^{-1}$. This leads to the following result previously reported in [20, Remark 7].

Proposition V.3 ([20]). *Suppose $\beta \geq 0$ is such that (26) holds for some $Q \in \mathbb{P}_{\succ 0}$ and $F \in \mathbb{R}^{p\times n}$. Then $\rho^* \leq \beta$.*

For a fixed β, (26) is an LMI feasibility problem that can be solved efficiently. The tightest upper bound β can be obtained by a bisection algorithm. If (26) is satisfied for some $Q \succ 0$, F, and β, then under the linear state feedback controller $\mathbf{u}_t(x(t)) = FQ^{-1}x(t)$, we have $\|x(t+1)\|_{Q^{-1}} \leq \beta\|x(t)\|_{Q^{-1}}$, $\forall t$, for all $x(0)$ and $\sigma \in \mathcal{S}$.

Remark V.2. It is proved in [21] that a constant $\beta \geq 0$ is an upper bound of ρ^* if there exist $Q_i = Q_i^T \in \mathbb{R}^{n\times n}$, $i \in \mathcal{M}$, and $G \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{p\times n}$ such that, for any $i, j \in \mathcal{M}$,

$$\begin{bmatrix} \beta(G + G^T - Q_i) & G^T A_i^T + Y^T B_i^T \\ A_i G + B_i Y & \beta Q_j \end{bmatrix} \succ 0. \quad (27)$$

If (27) holds for some $\beta < 1$, the controller $u(t) = YG^{-1}x(t)$ σ-resiliently stabilizes the SLCS. This test is better than that in Proposition V.3 (see Example V.1), but it remains conservative as it assumes linear controllers (see Example V.2).

B. Polyhedral Norms

For a matrix $C = [c_1, \cdots, c_\ell] \in \mathbb{R}^{n\times \ell}$ with $c_i \in \mathbb{R}^n$, define

$$\xi(z) := \max_{j=1,\ldots,\ell} |c_j^T z|, \quad \forall z \in \mathbb{R}^n.$$

Obviously, ξ is a seminorm on $\mathbb{R}^n$ with the set $\{z \in \mathbb{R}^n \,|\, \xi(z) \leq 1\}$ being a possibly unbounded polyhedron. We call ξ the polyhedral seminorm with parameter C. If the range of C is $\mathbb{R}^n$, then ξ becomes a *polyhedral norm*, denoted by $\|\cdot\|_C$, whose unit ball is a (centrally) symmetric polytope.

Let $\mathcal{K}_p$ be the set of all polyhedral norms on $\mathbb{R}^n$. For any norm ξ on $\mathbb{R}^n$, $d(\mathcal{K}_p, \xi) = 0$, i.e., $\mathcal{K}_p$ is a dense subset of norms [34]. Therefore, bounds on ρ^* obtained from polyhedral norms can be arbitrarily tight. On the other hand, polyhedral norms have high representation complexity. For example, the number of facets of the unit ball of a polyhedral norm ξ on $\mathbb{R}^n$ satisfying $d(\xi, \|\cdot\|) \leq \varepsilon$ for the Euclidean norm $\|\cdot\|$ and a constant $\varepsilon > 0$ increases exponentially in n [34]. As a result, algorithms to be developed in this section based on polyhedral norms are suitable when the state dimension n is small.

The following result is straightforward.

Lemma V.2. *Let ξ and $\tilde{\xi}$ be two polyhedral seminorms on $\mathbb{R}^n$ with the parameters $C = [c_1, \cdots, c_\ell] \in \mathbb{R}^{n\times \ell}$ and $\widetilde{C} = [\tilde{c}_1, \cdots, \tilde{c}_{\tilde{\ell}}] \in \mathbb{R}^{n\times \tilde{\ell}}$, respectively. Then $\xi \leq \tilde{\xi}$ if and only if $\mathrm{co}_{\mathrm{sym}}(C) \subset \mathrm{co}_{\mathrm{sym}}(\widetilde{C})$, where $\mathrm{co}_{\mathrm{sym}}(C)$ denotes the symmetric convex hull generated by $\{c_1, \ldots, c_\ell, -c_1, \ldots, -c_\ell\}$ and similarly for $\mathrm{co}_{\mathrm{sym}}(\widetilde{C})$. As a result, $\xi = \tilde{\xi}$ if and only if $\mathrm{co}_{\mathrm{sym}}(C) = \mathrm{co}_{\mathrm{sym}}(\widetilde{C})$.*

Note that a column c_j of the parameter matrix C of a polyhedral seminorm ξ is redundant if c_j is in the symmetric convex hull generated by all the other columns of C.

Lemma V.3. *Suppose ξ is a polyhedral seminorm on $\mathbb{R}^n$ with the parameter $C = [c_1, \cdots, c_\ell] \in \mathbb{R}^{n\times \ell}$. Then*

$$\xi_\sharp(z) = \max_{\tilde{c}\in\Omega_C} \tilde{c}^T z, \quad \forall z \in \mathbb{R}^n,$$

for some symmetric polytope Ω_C in $\mathbb{R}^n$. In other words, $\xi_\sharp$ is also a polyhedral seminorm on $\mathbb{R}^n$.

Proof. For each $z \in \mathbb{R}^n$, $\xi_\sharp(z)$ defined in (15) is the optimal value of the following linear program:

$$\min_{v\in\mathbb{R}^p,\, r\in\mathbb{R}} r, \quad \text{s.t. } \pm c_j^T(A_i z + B_i v) \leq r, \ \forall i, \ \forall j. \quad (28)$$

Its dual problem, which has the same optimal value, is

$$\max_{\theta_{ij}^+, \theta_{ij}^-} \quad \sum_{i,j} (\theta_{ij}^+ - \theta_{ij}^-) c_j^T A_i z \quad (29)$$

$$\text{subject to} \quad \sum_{ij} (\theta_{ij}^+ - \theta_{ij}^-) c_j^T B_i = 0, \ \forall i, j, \ \text{ and} \quad (30)$$

$$\sum_{ij} (\theta_{ij}^+ + \theta_{ij}^-) = 1, \ \theta_{ij}^+, \theta_{ij}^- \geq 0, \ \forall i, j. \quad (31)$$

The optimal value of problem (29) can be written as $\max\{\tilde{c}^T z \,|\, \tilde{c} \in \Omega_C\}$, where $\Omega_C \subset \mathbb{R}^n$ is given by

$$\Omega_C := \left\{ \sum_{i,j} (\theta_{ij}^+ - \theta_{ij}^-) A_i^T c_j \,\middle|\, (30) \text{ and } (31) \text{ hold} \right\}. \quad (32)$$

Clearly, Ω_C is a bounded convex polytope. It is centrally symmetric because the constraints (30) and (31) are invariant to exchanging θ_{ij}^+ and θ_{ij}^- for each i, j. Let the matrix $C_\sharp$ be such that its columns consist of exactly those vertices of Ω_C in any generic half space. Then $\Omega_C = \mathrm{co}_{\mathrm{sym}}(C_\sharp)$ and $\xi_\sharp$ is exactly the polyhedral seminorm with the parameter $C_\sharp$. ☐

We now apply Proposition IV.1 to the polyhedral seminorms. Let ξ be a polyhedral seminorm on $\mathbb{R}^n$ with the parameter $C = [c_1, \cdots, c_\ell] \in \mathbb{R}^{n\times \ell}$, and let $\xi_\sharp$ be the polyhedral seminorm defined by the set Ω_C in Lemma V.3. By Lemma V.2, $\xi_\sharp \geq \alpha\xi$ for some $\alpha \geq 0$ if and only if $\alpha c_k \in \Omega_C$ for all $k = 1, \ldots, \ell$, or equivalently, if and only if $\alpha \leq \min_{k=1,\ldots,\ell} \alpha_k^*$, where $\alpha_k^* := \sup\{\alpha \geq 0 \,|\, \alpha c_k \in \Omega_C\}$, $k = 1, \ldots, \ell$. By the definition of Ω_C in (32), α_k^* is the optimal value of the following linear program:

$$\max_{\theta_{ij}^+, \theta_{ij}^-, \alpha\geq 0} \quad \alpha \quad (33)$$

$$\text{subject to } (30), (31), \text{ and } \alpha c_k = \sum_{i,j} (\theta_{ij}^+ - \theta_{ij}^-) A_i^T c_j.$$

Consequently, we obtain the following result.

Proposition V.4. *For any $C = [c_1, \cdots, c_\ell] \in \mathbb{R}^{n\times \ell}$, the σ-resilient stabilizing rate ρ^* satisfies $\rho^* \geq \min_{k=1,\ldots,\ell} \alpha_k^*$, where α_k^* is the optimal value of the linear program (33).*

Algorithm 1

Initialize $C \in \mathbb{R}^{n\times\ell}$ with columns c_j, $j = 1, \dots, \ell$
repeat
  for $k = 1, \dots, \ell$ **do**
    Solve the linear program (33) to obtain α_k^*
  end for
  $k_1 \leftarrow \arg\max_k \alpha_k^*$, $k_2 \leftarrow \arg\min_k \alpha_k^*$
  $c_{k_1} \leftarrow \sqrt{\alpha_{k_1}^*/\alpha_{k_2}^*} \cdot c_{k_1}$, $c_{k_2} \leftarrow \sqrt{\alpha_{k_2}^*/\alpha_{k_1}^*} \cdot c_{k_2}$
until $(\max_k \alpha_k^*)/(\min_k \alpha_k^*) \le 1+\varepsilon$ or maximum number of iterations is reached
return $\alpha^* = \min_k \alpha_k^*$

Typically, the closer α_k^*'s are to being identical, the closer ξ is to being a Barabanov norm. Thus those columns c_k of C with larger (resp. smaller) α_k^* should be scaled up (resp. down) for better lower bounds of ρ^*. This leads to Algorithm 1 that updates C iteratively. The algorithm terminates if α_k^*'s are almost identical or a prescribed number of iterations is reached. To find a good initial guess of C, one can first run the algorithms in Section V-A to obtain a good ellipsoidal norm $\|\cdot\|_P$; do a coordinate change $x = P^{-1/2}\tilde{x}$ (see Section IV-D); and in the $\tilde{x}$-coordinates initialize C so that its columns are a uniform quantization of (half of) the unit sphere $\mathbb{S}^{n-1}$.

Proposition IV.2 can also be applied to the polyhedral norms to obtain upper bounds of ρ^*. We first cite a well known fact.

Lemma V.4. *Suppose $C = [c_1, \cdots, c_\ell] \in \mathbb{R}^{n\times\ell}$ has range $\mathbb{R}^n$ so that $\|\cdot\|_C$ is a polyhedral norm whose unit ball is denoted by B. Let $\{z_1, \dots, z_q\}$ be an enumeration of the vertices of B. Then, $\mathrm{co}_{\mathrm{sym}}(C)$ is the polar dual of B, or more precisely,*

$$\mathrm{co}_{\mathrm{sym}}(C) = \left\{c \in \mathbb{R}^n \,\middle|\, |c^T z_k| \le 1,\ k = 1, \dots, q\right\}.$$

For a polyhedral norm $\|\cdot\|_C$, denote $\|\cdot\|_{C\sharp} = \mathcal{T}(\|\cdot\|_C)$. By Lemma V.3, $\|z\|_{C\sharp} = \max_{\tilde{c}\in\Omega_C} \tilde{c}^T z$ with Ω_C defined in (32). For any $\beta \ge 0$, Lemma V.2 implies that $\|\cdot\|_{C\sharp} \le \beta\|\cdot\|_C$ if and only if $\Omega_C \subseteq \mathrm{co}_{\mathrm{sym}}(\beta C)$. By Lemma V.4, the latter is equivalent to $|c^T z_k| \le \beta$ for all $c \in \Omega_C$ and all vertices z_k of the unit ball of $\|\cdot\|_C$. This condition is further equivalent to $\|z_k\|_{C\sharp} \le \beta$ for all z_k. This leads to the following result.

Proposition V.5. *For any $C = [c_1, \cdots, c_\ell] \in \mathbb{R}^{n\times\ell}$ whose range is $\mathbb{R}^n$, the σ-resilient stabilizing rate ρ^* satisfies $\rho^* \le \max_{k=1,\dots,q} \|z_k\|_{C\sharp}$, where z_k, $k = 1, \dots, q$, are the vertices of the closed unit ball of $\|\cdot\|_C$.*

Note that, for each z_k, $\|z_k\|_{C\sharp}$ can be computed by solving the linear program (28) or (29) with z replaced by z_k.

Example V.1. Consider the following SLCS on $\mathbb{R}^2$:

$$A_1 = a_1\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix},\ B_1 = \begin{bmatrix}0\\1\end{bmatrix};\ \ A_2 = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},\ B_2 = \begin{bmatrix}1\\0\end{bmatrix}.$$

In the first case we set $a_1 = 0.5$. By solving the BMI problem (23), the lower bound of ρ^* obtained using ellipsoidal norms is $\alpha^* = 0.8031$. By using Proposition V.3 and a bisection algorithm, the tightest upper bound of ρ^* by using ellipsoidal norms is $\beta^* = 0.8956$. Solving the LMI (27) in Remark V.2 yields a slightly improved upper bound 0.8949. In comparison, by using polyhedral norms, namely, Algorithm 1 and Proposition V.5, with $C \in \mathbb{R}^{2\times 36}$ initialized to have columns that are uniform samplings of the unit circle, we find that ρ^* has the lower bound 0.8660 and the upper bound 0.8732, both better than the results from ellipsoidal norms.

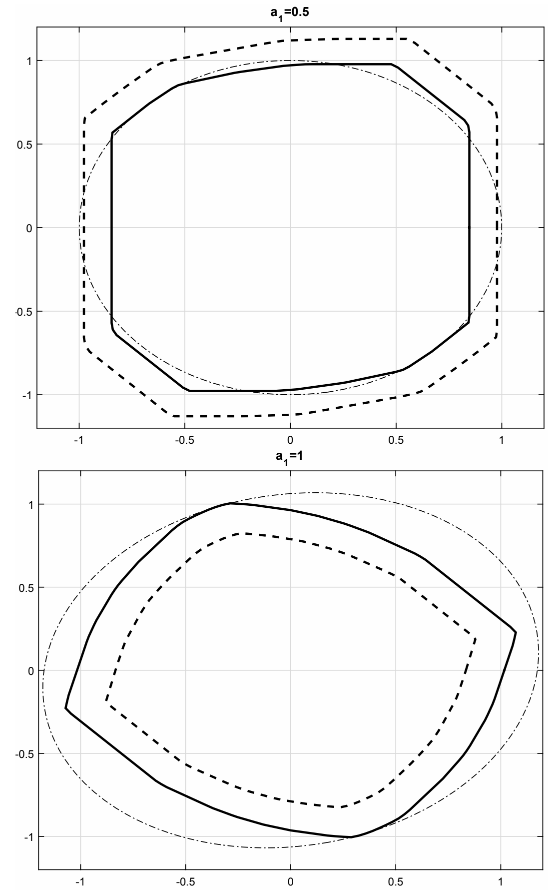

Fig. 2. Unit spheres of the polyhedral norms $\|\cdot\|_C$ (bold lines) and $\|\cdot\|_\sharp$ (dashed lines) obtained by Algorithm 1 for the SLCS in Example V.1 when $a_1 = 0.5$ (top) and $a_1 = 1$ (bottom). Unit spheres of the ellipsoidal norms computed by Proposition V.3 are also plotted (dash-dotted lines).

In the second case we set $a_1 = 1$. The best lower and upper bounds obtained by solving problem (23) and by using Proposition V.3 are 1.1305 and 1.2927, respectively. Solving the problem (27) yields the upper bound 1.2910. Using Algorithm 1 and Proposition V.5 with the same initial C as in the case of $a_1 = 0.5$, the lower and upper bounds of ρ^* obtained by polyhedral norms are 1.2183 and 1.2239, respectively.

The unit spheres of the computed polyhedral and ellipsoidal norms are plotted in Fig. 2. The former is close to being a Barabanov norm, while the latter has some general semblance.

Example V.2. This example shows that the optimal user control policy is in general nonlinear. Consider the SLCS:

$$A_1 = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\ B_1 = \begin{bmatrix}1\\1\end{bmatrix};\ \ A_2 = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix},\ B_2 = \begin{bmatrix}-1\\1\end{bmatrix}.$$

The unit spheres of the computed polyhedral norms $\|\cdot\|_C$ by Algorithm 1 with $\ell = 144$ and the corresponding $\|\cdot\|_{C\sharp}$ are displayed on the top of Fig. 3. Using $\|\cdot\|_C$ in Propositions V.4 and V.5 yield $0.6302 \le \rho^* \le 0.6309$. The corresponding

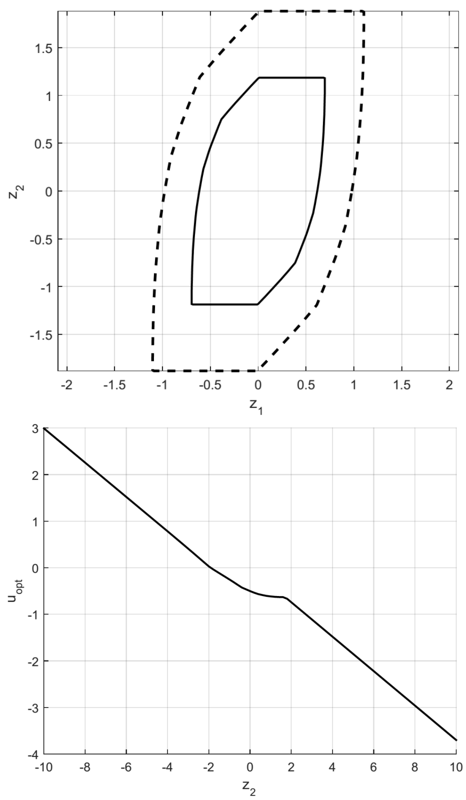

Fig. 3. Top: Unit spheres of the polyhedral norms $\|\cdot\|_C$ (in bold lines) and $\|\cdot\|_{C^\sharp}$ (in dashed lines) obtained by Algorithm 1 for the SLCS in Example V.2. Bottom: optimal u^* for $z=(1,z_2)$ where $z_2\in[-10,10]$.

optimal user control $u^*(z)$ with $z=(1,z_2)$ for $z_2\in[-10,10]$ is shown at the bottom of Fig. 3, which is clearly nonlinear.

We now show formally that a linear control policy $u(t)=Kx(t)=\begin{bmatrix}k_1 & k_2\end{bmatrix}x(t)$ is not optimal. Under this policy, the SLCS becomes the SLS $\{\bar{A}_i\}_{i=1,2}$, where $\bar{A}_1=A_1+B_1K=\begin{bmatrix}1+k_1 & k_2\\ k_1 & k_2\end{bmatrix}$ and $\bar{A}_2=A_2+B_2K=\begin{bmatrix}-k_1 & -k_2\\ k_1 & 1+k_2\end{bmatrix}$. Let ρ_i^* be the spectral radius of $\bar{A}_i$ for $i=1,2$. We show next that $\max\{\rho_1^*,\rho_2^*\}>\rho_0=\sqrt{0.4}\approx 0.6325$ for any choice of K.

Define $\Delta_1:=(1+k_1+k_2)^2-4k_2$ and $\Delta_2:=(1-k_1+k_2)^2+4k_1$. The pairs of (possibly complex) eigenvalues of $\bar{A}_1$ and $\bar{A}_2$ are given by $\mu_{1,2}=(1+k_1+k_2\pm\sqrt{\Delta_1})/2$ and $\lambda_{1,2}=(1-k_1+k_2\pm\sqrt{\Delta_2})/2$, respectively. Suppose there exist k_1,k_2 such that $\max\{|\mu_1|,|\mu_2|,|\lambda_1|,|\lambda_2|\}\le\rho_0$. This implies that $|k_2|=|\det(\bar{A}_1)|\le(\rho_0)^2=0.4$ and $|k_1|=|\det(\bar{A}_2)|\le(\rho_0)^2=0.4$, i.e., $k_1,k_2\in[-0.4,0.4]$.

Consider the following two cases:

Case 1: $k_2\ge 0$. Since Δ_2 as a function of k_1 has the minimum $\Delta_{2,\min}=4k_2\ge 0$, $\lambda_1=(1-k_1+k_2+\sqrt{\Delta_2})/2$ is real. For each fixed $k_1\in[-0.4,0.4]$, in view of $1-k_1+k_2\ge 0$, Δ_2 (and hence λ_1) is nondecreasing in k_2. Therefore, $\lambda_1\ge\lim_{k_2\downarrow 0}\lambda_1=1$ for any $|k_1|\le 0.4$, contradicting $|\lambda_1|\le\rho_0$.

Case 2: $k_2<0$. Then, $\Delta_1\ge 0$ and $\mu_1=(1+k_1+k_2+\sqrt{\Delta_1})/2\in\mathbb{R}$. Since $1+k_1+k_2\ge 0$, Δ_1 and μ_1 are nondecreasing in k_1. Thus, $\mu_1\ge\lim_{k_1\to -0.4}\mu_1=f(k_2):=\frac{1}{2}\big(0.6+k_2+\sqrt{(0.6+k_2)^2-4k_2}\big)\ge 0$. As $f(k_2)$ is strictly decreasing in $k_2\in[-0.4,0]$ with $f(-0.0558)=\rho_0$, we need $k_2\in[-0.0558,0)$ for $\mu_1\le\rho_0$. For such k_2, $\Delta_2>0$, $\forall\, k_1\in[-0.4,0.4]$. Thus $\lambda_1\in\mathbb{R}$ and $\lambda_1\ge\lim_{k_2\to -0.0558}\lambda_1=\frac{1}{2}\big(0.9442-k_1+\sqrt{(0.9442-k_1)^2+4k_1}\big)\ge 0.8995$ for any $|k_1|\le 0.4$, a contradiction to the assumption $|\lambda_1|\le\rho_0$.

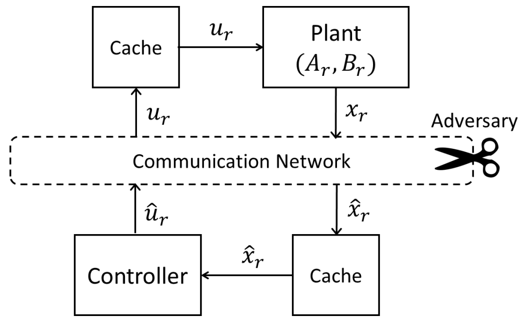

Fig. 4. A networked control system with data package drops [16].

To sum up, the stabilizing rate achieved by any linear control policy, i.e., the JSR of $\{\bar{A}_1,\bar{A}_2\}$, is at least 0.6325 and outside the interval $[0.6302, 0.6309]$ containing ρ^*. In fact, the gap is even bigger than it appears, as $\max\{\rho_1^*,\rho_2^*\}$ is in general strictly less than the JSR of $\{\bar{A}_1,\bar{A}_2\}$. For example, it is found numerically that $\max\{\rho_1^*,\rho_2^*\}$ attains its minimum value 0.6489 at $k_1=-0.4211$ and $k_2=-0.1294$. The JSR of the resulting $\{\bar{A}_1,\bar{A}_2\}$, on the other hand, is at least 0.7156.

VI. Applications in Networked Control Systems

Consider the networked control system with data package drops studied in [16]. Suppose a plant with the state $x_r\in\mathbb{R}^n$ and the input $u_r\in\mathbb{R}^p$ follows the dynamics $x_r(t+1)=A_rx_r(t)+B_ru_r(t)$, $t\in\mathbb{Z}_+$, for some given constant matrices A_r and B_r. At time $t_0=0$, the state $x(0)$ of the plant is transmitted successfully via a communication network to a remote control site. The state received by the control site, $\hat{x}_r(0):=x_r(0)$, is stored in a cache and used by a controller to produce the control command $\hat{u}_r(0)$, which is then transmitted successfully back to the plant. Upon receiving the control command $u_r(0):=\hat{u}_r(0)$, the plant stores it in cache and used it as the control input so that $x_r(1)=A_rx_r(0)+B_ru_r(0)$. However, starting from time $t=1$ on, an adversary blocks the communications between the plant and the controller by a duration of at most $m-1$ time steps, where $m\in\mathbb{N}$ is given. Then the next successful communication reassumes at a time $t_1\in\{1,2,\ldots,m\}$. Between t_0 and t_1, the plant keeps using the last received control command $u_r(0)$ stored in its cache as its control input, resulting in $x_r(t_1)=(A_r)^{t_1}x_r(0)+\sum_{t=0}^{t_1-1}(A_r)^tB_ru_r(0)$. This process is then repeated.

Denote by $0=t_0<t_1<t_2<\cdots$ the sequence of times at which the communications between the plant and the controller are successful, and define $x(k):=x_r(t_k)$, $u(k):=u_r(t_k)$, $k\in\mathbb{Z}_+$. Then the dynamics of $x(k)$ are given by the following SLCS:

$$x(k+1)=A_{\sigma(k)}x(k)+B_{\sigma(k)}u(k),\quad k\in\mathbb{Z}_+, \tag{34}$$

where the mode $\sigma(k):=t_{k+1}-t_k\in\mathcal{M}=\{1,\ldots,m\}$ is determined by the adversary, and for each $i\in\mathcal{M}$,

$$A_i=(A_r)^i,\quad B_i:=\big[(A_r)^{i-1}+(A_r)^{i-2}+\cdots+I\big]B_r.$$

The problem of stabilizing the plant regardless of how the adversary blocks the communication network becomes the resilient stabilization problem of the SLCS (34).

Example VI.1. Suppose the system matrices of the plant are

$$A_r = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & 0 \end{bmatrix}, \quad B_r = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and the adversary blocks up to two rounds of communication consecutively, i.e., $m = 3$. For the resulting SLCS (34), the bisection algorithm derived from Proposition V.3 returns an upper bound of ρ^* as 1.0911 and an associated solution

$$P = Q^{-1} = \begin{bmatrix} 162.5237 & 328.9297 & 166.4328 \\ -328.9297 & 666.4824 & -337.5884 \\ 166.4328 & 337.5884 & 171.1676 \end{bmatrix}.$$

Using this P in the LMI problem (22) results in a lower bound 0.9170. Furthermore, using P as the initial guess and solving the BMI problem (23) yields a lower bound 0.9505. Hence, using ellipsoidal norms, $\rho^* \in [0.9505, 1.0911]$.

Algorithm 1 is then applied to the SLCS after the coordinate change $x = P^{-1/2}\tilde{x}$. By initializing $C \in \mathbb{R}^{3\times 85}$ so that its columns are a roughly uniform sampling of a half of the unit sphere $\mathbb{S}^2$, Algorithm 1 returns a lower bound 0.9881 of ρ^*. See Fig. 5 for the plots (in $\tilde{x}$-coordinates) of the unit ball of the returned polyhedral norm. This same norm yields via Proposition V.5 an upper bound 1.0590 of ρ^*. By perturbing C locally, a better upper bound 1.0510 is obtained. Thus, using polyhedral norms, we conclude $\rho^* \in [0.9881, 1.0510]$.

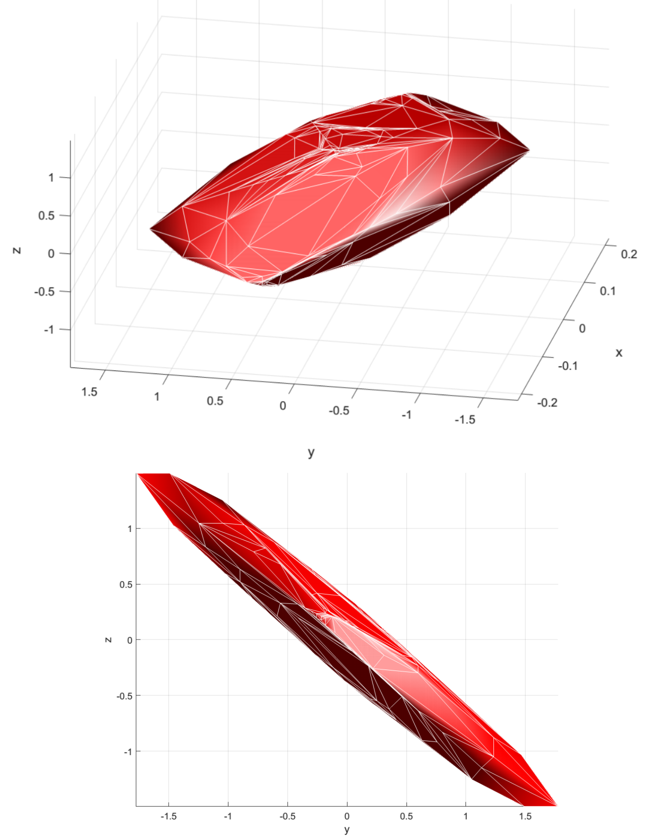

Fig. 5. Unit ball of the polyhedral norm returned by Algorithm 1 for Example VI.1.

VII. Conclusions and Future Directions

The switching-resilient stabilization problem of discrete-time switched linear control systems is formulated. Both theoretical results and practical bounding techniques are derived for characterizing a stabilizability metric. Examples are presented to demonstrate the obtained results. Future research includes extensions to the case with bounded continuous control inputs and different information structures.

Appendix A
Proof of Theorem II.1

Proof. We only prove one direction as the other is trivial. Let the SLCS be σ-resiliently asymptotically stabilized by a control policy $\mathbf{u} \in \mathcal{U}$. Let the z be any nonzero initial state. Without loss of generality, assume that z is in the unit sphere $\mathbb{S}^{n-1}$. Then $x(t;\sigma,\mathbf{u},z) \to 0$ as $t \to \infty$ for any $\sigma \in \mathcal{S}$.

Claim: there exists $N_z \in \mathbb{Z}_+$ such that for any $\sigma \in \mathcal{S}$,

$$\|x(t_\sigma;\sigma,\mathbf{u},z)\| < \frac{1}{2} \text{ for some } t_\sigma \le N_z. \tag{35}$$

Suppose otherwise. Then there exist an increasing sequence of times $N_1 < N_2 < \cdots$ and a sequence of switching sequences $\sigma^{(1)}, \sigma^{(2)}, \ldots \in \mathcal{S}$ such that $\|x(t;\sigma^{(k)},\mathbf{u},z)\| \ge \frac{1}{2}$, $\forall\, t = 0, \ldots, N_k$ for each $k \in \mathbb{N}$. At each fixed time t, since $\sigma^{(k)}(t)$, $k = 1, 2, \ldots$, take values in the finite set $\mathcal{M}$, at least one value, denoted by $\sigma^{(\infty)}(t)$, must appear infinitely often. Assemble $\sigma^{(\infty)}(t)$ for all t into a switching sequence and denote it by $\sigma^{(\infty)} \in \mathcal{S}$. By taking progressively finer subsequences of $\sigma^{(k)}$, $k \in \mathbb{N}$, and using induction on the time t, we obtain $\|x(t;\sigma^{(\infty)},\mathbf{u},z)\| \ge \frac{1}{2}$ at all $t \in \mathbb{Z}_+$. This contradicts the assumption that $\mathbf{u}$ is stabilizing, thus proving the claim in (35).

We next show that N_z in Claim (35) can be chosen to be uniformly bounded for all $z \in \mathbb{S}^{n-1}$. To this end, fix an arbitrary $z \in \mathbb{S}^{n-1}$. For any $\sigma \in \mathcal{S}$, denote by $u_z(t;\sigma) \in \mathbb{R}^p$ the actual control input implemented at time $t \in \mathbb{Z}_+$ when generating $x(t;\sigma,\mathbf{u},z)$. Since $\mathbf{u}$ is admissible, so is the control policy $\mathbf{u}_z := (u_z(0;\sigma), u_z(1;\sigma), \ldots)$. In fact, $\mathbf{u}_z$ is obtained by running a simulator of the SLCS with identical dynamics but the fixed initial state z and mimicking the controls produced by the simulator under the control policy $\mathbf{u}$ in response to any $\sigma \in \mathcal{S}$. By Claim (35), under $\mathbf{u}_z$ and for any $\sigma \in \mathcal{S}$, $\|x(t;\sigma,\mathbf{u}_z,z)\| < \frac{1}{2}$ for some $t \le N_z$. Since the solution $x(t;\sigma,\mathbf{u}_z,z')$ is continuous in its initial state z' for any fixed $t \in \mathbb{Z}_+$ and $\sigma \in \mathcal{S}$, there exists an open neighborhood U_z of z such that for any $z' \in U_z \cap \mathbb{S}^{n-1}$ and any $\sigma \in \mathcal{S}$, $\|x(t;\sigma,\mathbf{u}_z,z')\| < \frac{1}{2}$ for some $t \le N_z$. The family of all such open sets U_z for $z \in \mathbb{S}^{n-1}$ is an open cover of the compact set $\mathbb{S}^{n-1}$; thus there exist finitely many $z^{(1)}, \ldots, z^{(p)}$ in $\mathbb{S}^{n-1}$ such that the union of the corresponding neighborhoods $U_{z^{(i)}}$ covers $\mathbb{S}^{n-1}$. A feasible control policy $\mathbf{u}^*$ can then be constructed by patching together the control policies $\mathbf{u}_{z^{(i)}}$: if the initial state $x(0) \in \mathbb{S}^{n-1}$ belongs to some $U_{z^{(i)}}$, then $\mathbf{u}_{z^{(i)}}$ is invoked.

Let $N_{\max} := \max_{i=1,\ldots,p} N_{z^{(i)}} < \infty$. Under this $\mathbf{u}^*$, for any $z \in \mathbb{S}^{n-1}$ and any $\sigma \in \mathcal{S}$, $\|x(t;\sigma,\mathbf{u}^*,z)\| < \frac{1}{2}$ for some $t \le N_{\max}$. By a homogeneous extension of $\mathbf{u}^*$ from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n \setminus \{0\}$, we conclude that $\|x(t;\sigma,\mathbf{u}^*,z)\| < \frac{1}{2}\|z\|$ for some $t \le N_{\max}$, $\forall\, z \neq 0$, $\forall\, \sigma \in \mathcal{S}$. By restarting $\mathbf{u}^*$ whenever this occurs and using a standard argument (e.g., [37, Proposition 2.1]), we obtain an admissible control policy that σ-resiliently exponentially stabilizes the SLCS. ☐

APPENDIX B
PROOF OF THEOREM III.1

Proof. By suitably scaling the matrices A_i's, we can assume without loss of generality that $\rho^* = 1$.

We first show that $\mathcal{W}$ defined in (8) is control σ-invariant. For any $z \in \mathcal{W}$, $\zeta(z) < \infty$ implies that there exist a policy $\mathbf{u} = (\mathbf{u}_0, \mathbf{u}_1, \ldots) \in \mathcal{U}$ and $\kappa_z \in [0,\infty)$ such that $\|x(t;\sigma,\mathbf{u},z)\| \le \kappa_z$, $\forall\, t$, $\forall\, \sigma = (\sigma_0,\sigma_1,\ldots) \in \mathcal{S}$. Let $v = \mathbf{u}_0(z)$ and let $\sigma_0 = i$ be arbitrary. Then the solution starting from $x(1) = A_i z + B_i v$ under the control policy $\mathbf{u}_+ := (\mathbf{u}_1,\mathbf{u}_2,\ldots)$ satisfies $\|x(t;\sigma_+,\mathbf{u}_+,x(1))\| = \|x(t+1;\sigma,\mathbf{u},z)\| \le \kappa_z$ for all t and all $\sigma_+ := (\sigma_1,\sigma_2,\ldots) \in \mathcal{S}$. As a result, $\zeta(x(1)) \le \kappa_z < \infty$ and thus $x(1) \in \mathcal{W}$. This proves that $\mathcal{W}$ is control σ-invariant. As the SLCS is irreducible, $\mathcal{W}$ is either $\{0\}$ or $\mathbb{R}^n$. We show by contradiction that the former is impossible. Suppose $\mathcal{W} = \{0\}$. Then for any $z \in \mathbb{S}^{n-1}$ and any $\mathbf{u} \in \mathcal{U}$, there exist some $\sigma \in \mathcal{S}$ and $s_{z,\mathbf{u},\sigma} \in \mathbb{Z}_+$ such that $\|x(s_{z,\mathbf{u},\sigma};\sigma,\mathbf{u},z)\| > 2$. We claim that the times $s_{z,\mathbf{u},\sigma}$ are uniformly bounded in z and $\mathbf{u}$:

$$\textbf{Claim: } \exists\, N \in \mathbb{Z}_+ \text{ such that } \forall\, \mathbf{u} \in \mathcal{U},\ \forall\, z \in \mathbb{S}^{n-1}, \quad \|x(t;\sigma,\mathbf{u},z)\| > 2 \text{ for some } \sigma \in \mathcal{S} \text{ and } t \le N. \tag{36}$$

Suppose Claim (36) fails. Then there exist a sequence (z_k) in $\mathbb{S}^{n-1}$, a sequence of control policies $(\mathbf{u}^k)$ in $\mathcal{U}$, and a strictly increasing sequence of times (s_k) such that for any $\sigma \in \mathcal{S}$, $\|x(t;\sigma,\mathbf{u}^k,z_k)\| \le 2$, $\forall\, t = 0,\ldots,s_k$ for each $k \in \mathbb{N}$. By passing to a subsequence if necessary, we assume that (z_k) converges to some $z_* \in \mathbb{S}^{n-1}$. We next construct a control policy $\mathbf{u}^*$ under which $\|x(t;\sigma,\mathbf{u}^*,z_*)\| \le 2$ for all $t \in \mathbb{Z}_+$ and all $\sigma \in \mathcal{S}$. To this purpose, for each k, we denote by $u^k_{z_k,\sigma}(t)$ the actual control at time t produced by the control policy $\mathbf{u}^k$ for the initial state z_k in response to an arbitrary switching sequence $\sigma \in \mathcal{S}$. We assume without loss of generality that $u^k_{z_k,\sigma}(t)$ lies in the orthogonal complement of $\cap_{i\in\mathcal{M}}\mathcal{N}(B_i)$ since the component of $u^k_{z_k,\sigma}(t)$ in $\cap_{i\in\mathcal{M}}\mathcal{N}(B_i)$ will not affect the state dynamics, where $\mathcal{N}(\cdot)$ denotes the null space of a matrix. For any k and each $t = 0,\ldots,s_k - 1$, it follows from $\|x(t;\sigma,\mathbf{u}^k,z_k)\| \le 2$ and $\|x(t+1;\sigma,\mathbf{u}^k,z_k)\| = \|A_{\sigma(t)}x(t;\sigma,\mathbf{u}^k,z_k) + B_{\sigma(t)}u^k_{z_k,\sigma}(t)\| \le 2$, $\forall \sigma(t) \in \mathcal{M}$ that $\max_{i\in\mathcal{M}} \|B_i u^k_{z_k,\sigma}(t)\| \le 2\big(\max_{i\in\mathcal{M}}\|A_i\| + 1\big)$, which in turn implies that $u^k_{z_k,\sigma}(t)$ is uniformly bounded in $(\cap_{i\in\mathcal{M}}\mathcal{N}(B_i))^\perp$. Thus, for each fixed $\sigma \in \mathcal{S}$ and $t \in \mathbb{Z}_+$, $\big(u^k_{z_k,\sigma}(t)\big)$ has a convergent subsequence whose limit is denoted by $u^*_{z_*,\sigma}(t)$. Let $\mathbf{u}^*$ be the control policy that produces the actual control $u^*_{z_*,\sigma}(t)$ at time t for the initial state z_* in response to any $\sigma \in \mathcal{S}$, which is feasible since it is the limit of a sequence of feasible control policies $(\mathbf{u}^k)$. By the continuity of the state solution in initial state and control input for each fixed σ, we deduce that $\|x(t;\sigma,\mathbf{u}^*,z_*)\| \le 2$ for all $t \in \mathbb{Z}_+$ and all $\sigma \in \mathcal{S}$. It follows from $\rho^* = 1$ that $\zeta(z_*) \le 2$ and hence $z_* \in \mathcal{W}$, a contradiction to $\mathcal{W} = \{0\}$. This proves Claim (36).

Claim (36) then implies that, regardless of $x(0)$ and $\mathbf{u}$, there always exists a switching sequence of length no more than N under which the state norm is at least doubled. Repeating this switching strategy by the adversary, we conclude that $\rho^* > 1$, which contradicts the assumption $\rho^* = 1$. Therefore, $\mathcal{W} \neq \{0\}$. This implies $\mathcal{W} = \mathbb{R}^n$, i.e., $\zeta(\cdot)$ is finite everywhere on $\mathbb{R}^n$. Thus $\zeta(\cdot)$ is a norm on $\mathbb{R}^n$, and $\zeta(\cdot) \le \kappa\|\cdot\|$ for some constant $\kappa > 0$. This shows that the SLCS is nondefective. ☐

APPENDIX C
PROOF OF THEOREM IV.1

Proof. The "only if" part has been proved; we prove the "if" part as follows. Suppose the SLCS is nondefective with $\rho^* > 0$. Then $\zeta(\cdot)$ defined in (7) is finite everywhere and is thus a norm on $\mathbb{R}^n$. Given $\mathbf{u} \in \mathcal{U}$ and $\sigma \in \mathcal{S}$, let $(\mathbf{u}_0,\mathbf{u}_+)$ be a decomposition of $\mathbf{u}$ and $(\sigma(0),\sigma_+)$ a decomposition of σ. For each $z \in \mathbb{R}^n$, we can rewrite

$$\begin{aligned}\zeta(z) &= \inf_{\mathbf{u}_0}\inf_{\mathbf{u}_+}\sup_{\sigma(0)}\sup_{\sigma_+}\sup_{t\in\mathbb{Z}_+} \frac{\|x(t;\sigma,\mathbf{u},z)\|}{(\rho^*)^t} \\ &= \inf_{\mathbf{u}_0}\sup_{\sigma(0)}\inf_{\mathbf{u}_+}\sup_{\sigma_+}\sup_{t\in\mathbb{Z}_+} \frac{\|x(t;\sigma,\mathbf{u},z)\|}{(\rho^*)^t}.\end{aligned}$$

The reason that $\sup_{\sigma(0)}$ and $\inf_{\mathbf{u}_+}$ can switch order is due to the observation in Remark II.1: for the objective of maximizing $\sup_{t\in\mathbb{Z}_+}\|x(t)\|/(\rho^*)^t$, knowing the optimal state feedback control policy $\mathbf{u}_+$ after time 0 gives no extra advantage to the adversary for its decision on $\sigma(0)$. By denoting $\mathbf{u}_0(z) = v$ and $\sigma(0) = i$, and using $x(t+1;\sigma,\mathbf{u},z) = x(t;\sigma_+,\mathbf{u}_+,x_{i,v}(1))$ where $x_{i,v}(1) := A_i z + B_i v$, we have, for any $z \in \mathbb{R}^n$,

$$\begin{aligned}&\zeta(z) \\ &= \inf_v \sup_i \inf_{\mathbf{u}_+}\sup_{\sigma_+}\max\left(\|z\|,\ \sup_{t\in\mathbb{Z}_+}\frac{\|x(t;\sigma_+,\mathbf{u}_+,x_{i,v}(1))\|}{(\rho^*)^{t+1}}\right) \\ &= \inf_v\sup_i \max\Big(\|z\|, \zeta(A_i z + B_i v)/\rho^*\Big) \\ &= \max\Big(\|z\|, \zeta_\sharp(z)/\rho^*\Big).\end{aligned}$$

It follows then that $\zeta(\cdot) \ge \zeta_\sharp(\cdot)/\rho^*$, i.e., $\zeta_\sharp(\cdot) \le \rho^*\cdot\zeta(\cdot)$, making $\zeta(\cdot)$ an extremal norm of the SLCS.

Finally, if the SLCS is nondefective with $\rho^* = 0$, then for any $z \in \mathbb{R}^n$, there exists $v \in \mathbb{R}^p$ such that $A_i z + B_i v = 0, \forall\, i \in \mathcal{M}$. This means that any norm $\|\cdot\|$ on $\mathbb{R}^n$ is an extremal norm since $\|\cdot\|_\sharp \equiv 0$. ☐

APPENDIX D
PROOF OF THEOREM IV.2

Proof. Suppose the SLCS is nondefective and $\rho^* > 0$. Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$, and use $\mathcal{T}$ in (15) to define a sequence of seminorms on $\mathbb{R}^n$ as $\xi^{(0)}(\cdot) := \|\cdot\|$, and $\xi^{(t)}(\cdot) := \underbrace{\mathcal{T}\circ\cdots\circ\mathcal{T}}_{t \text{ times}}(\|\cdot\|)$ for each $t \in \mathbb{N}$. By induction,

$$\xi^{(t)}(z) = \inf_{\mathbf{u}\in\mathcal{U}}\sup_{\sigma\in\mathcal{S}}\|x(t;\sigma,\mathbf{u},z)\|, \quad \forall\, z \in \mathbb{R}^n,\ t \in \mathbb{N}. \tag{37}$$

Therefore, $\xi^{(t)}(z)/(\rho^*)^t \leq \zeta(z)$, $\forall\, t \in \mathbb{N}$, $\forall\, z \in \mathbb{R}^n$, where ζ is defined in (7). Since the SLCS is nondefective, ζ is pointwise finite on $\mathbb{R}^n$; thus for each $s \in \mathbb{N}$, $\sup_{t\geq s} \xi^{(t)}/(\rho^*)^t$ is pointwise finite (and easily seen to be a seminorm) on $\mathbb{R}^n$. Consider the following function defined for $z \in \mathbb{R}^n$:

$$\eta(z) := \limsup_{t\to\infty} \frac{\xi^{(t)}(z)}{(\rho^*)^t} = \inf_{s\in\mathbb{N}} \left(\sup_{t\geq s} \frac{\xi^{(t)}(z)}{(\rho^*)^t} \right). \tag{38}$$

Clearly, $\eta \leq \zeta$. Being the limit of a nonincreasing sequence of seminorms $\sup_{t\geq s} \xi^{(t)}/(\rho^*)^t$ as $s \to \infty$, η is a seminorm on $\mathbb{R}^n$. We next show that $\eta \not\equiv 0$. Suppose otherwise. Then for any $z \in \mathbb{S}^{n-1}$, we have $\lim_{t\to\infty} \xi^{(t)}(z)/(\rho^*)^t \to 0$. By (37), there exist $N_z \in \mathbb{N}$ and $\mathbf{u}_z \in \mathcal{U}$ such that for any $\sigma \in \mathcal{S}$, $\|x(t;\sigma,\mathbf{u}_z,z)\| < \frac{1}{2}(\rho^*)^t$ for some $t \leq N_z$. Similar to the proof of Theorem II.1, we can first modify $\mathbf{u}_z$ in an open neighborhood U_z of z in $\mathbb{S}^{n-1}$ to obtain a policy $\widetilde{\mathbf{u}}_z$ so that for any $z' \in U_z$ and any $\sigma \in \mathcal{S}$, $\|x(t;\sigma,\widetilde{\mathbf{u}}_z,z')\| < \frac{1}{2}(\rho^*)^t$ for some $t \leq N_z$; obtain finitely many such U_z's to cover $\mathbb{S}^{n-1}$; patch their corresponding $\widetilde{\mathbf{u}}_z$ together to form an overall control policy $\widetilde{\mathbf{u}} \in \mathcal{U}$ and a finite uniform time bound $N_{\max}$ such that for any $z \in \mathbb{S}^{n-1}$ and any $\sigma \in \mathcal{S}$, $\|x(t;\sigma,\widetilde{\mathbf{u}},z)\| < \frac{1}{2}(\rho^*)^t$ for some $t \leq N_{\max}$. Repeating this argument via induction, we deduce that the σ-resilient stabilizing rate is strictly less than ρ^*, a contradiction. Hence η is a nonzero seminorm on $\mathbb{R}^n$. By applying $\mathcal{T}$ to both sides of (38) and using the monotone continuity property established in Lemma IV.3, we have

$$\begin{aligned}\eta_\sharp &= \inf_{s\in\mathbb{N}} \mathcal{T}\left[\sup_{t\geq s} \frac{\xi^{(t)}}{(\rho^*)^t}\right] \geq \inf_{s\in\mathbb{N}} \left(\sup_{t\geq s} \frac{\xi^{(t)}_\sharp}{(\rho^*)^t}\right) \\ &= \inf_{s\in\mathbb{N}} \left(\sup_{t\geq s} \frac{\xi^{(t+1)}}{(\rho^*)^t}\right) = \rho^* \cdot \inf_{s\in\mathbb{N}} \left(\sup_{t\geq s+1} \frac{\xi^{(t)}}{(\rho^*)^t}\right) = \rho^* \cdot \eta,\end{aligned}$$

where the second step follows from Lemma IV.3. Therefore, η is a lower extremal seminorm of the SLCS.

When the SLCS is nondefective with $\rho^* = 0$, it is easily verified that any seminorm ξ on $\mathbb{R}^n$ satisfies $\xi_\sharp \equiv 0$ and thus is a lower extremal seminorm. □

Appendix E
Proof of Theorem IV.3

Proof. Suppose that the SLCS is irreducible with $\rho^* > 0$. By Theorem III.1, the SLCS is also nondefective; thus the function ζ defined in (7) is pointwise finite on $\mathbb{R}^n$. Define

$$\chi(z) := \inf_{\mathbf{u}\in\mathcal{U}} \sup_{\sigma\in\mathcal{S}} \limsup_{t\to\infty} \frac{\|x(t;\sigma,\mathbf{u},z)\|}{(\rho^*)^t}, \quad \forall\, z \in \mathbb{R}^n.$$

Obviously, $\chi \leq \zeta$. Hence χ is pointwise finite on $\mathbb{R}^n$. Further, it is easy to see that χ is convex and positively homogeneous on $\mathbb{R}^n$. Therefore, χ is a seminorm on $\mathbb{R}^n$, whose kernel $\mathcal{N}_\chi := \{z \,|\, \chi(z) = 0\}$ is a subspace of $\mathbb{R}^n$. Note that $\chi(z+z') = \chi(z)$ for all $z \in \mathbb{R}^n$ and all $z' \in \mathcal{N}_\chi$.

We claim that $\mathcal{N}_\chi$ is control σ-invariant. Fix an arbitrary $z \in \mathcal{N}_\chi$. For any $\varepsilon > 0$, there exists a control policy $\mathbf{u}^\varepsilon = (\mathbf{u}^\varepsilon_0, \mathbf{u}^\varepsilon_1, \ldots) \in \mathcal{U}$ such that $\limsup_{t\to\infty} \|x(t;\sigma,\mathbf{u}^\varepsilon,z)\|/(\rho^*)^t < \varepsilon$, $\forall\, \sigma \in \mathcal{S}$. Let $v^\varepsilon := \mathbf{u}^\varepsilon_0(z)$ be the control input at $t = 0$ specified by $\mathbf{u}^\varepsilon$ and let $\sigma(0) = i \in \mathcal{M}$ be arbitrary. Then the solution starting from $x(1) := A_i z + B_i v^\varepsilon$ under $\mathbf{u}^\varepsilon_+ := (\mathbf{u}^\varepsilon_1, \mathbf{u}^\varepsilon_2, \ldots)$ satisfies $x(t;\sigma_+,\mathbf{u}^\varepsilon_+,x(1)) = x(t+1;\sigma,\mathbf{u}^\varepsilon,z)$ and hence $\limsup_{t\to\infty} \|x(t;\sigma_+,\mathbf{u}^\varepsilon_+,x(1))\|/(\rho^*)^t \leq \rho^* \cdot \varepsilon$, for all $\sigma_+ := (\sigma(1),\sigma(2),\ldots) \in \mathcal{S}$. This shows that $\chi(A_i z + B_i v^\varepsilon) \leq \rho^* \cdot \varepsilon$ for all $i \in \mathcal{M}$. Let $\varepsilon_k > 0$, $k \in \mathbb{N}$, be such that $\varepsilon_k \downarrow 0$, and let $v_k := v^{\varepsilon_k}$ for each k, which satisfies

$$\chi(A_i z + B_i v_k) \leq \rho^* \cdot \varepsilon_k, \quad \forall\, i \in \mathcal{M}. \tag{39}$$

Define the subspace $\mathcal{V} := \{v \,|\, \chi(B_i v) = 0, \forall\, i \in \mathcal{M}\} \subset \mathbb{R}^p$, and denote by $\mathcal{V}^\perp$ its orthogonal complement. Let v'_k be the projection of v_k onto $\mathcal{V}^\perp$ for each k. Using (39) and the argument in the proof of Lemma IV.2, we conclude that the sequence (v'_k) is bounded. Hence a subsequence of (v'_k) converges to some $v_* \in \mathcal{V}^\perp$. Taking the limit of (39) for this subsequence with v_k replaced by v'_k which does not change the value of $\chi(\cdot)$, we obtain via the continuity of the seminorm $\chi(\cdot)$ that $\chi(A_i z + B_i v_*) = 0$ for all $i \in \mathcal{M}$. This shows that $\mathcal{N}_\chi$ is a control σ-invariant subspace.

As the SLCS is irreducible, $\mathcal{N}_\chi$ is either $\{0\}$ or $\mathbb{R}^n$. We rule out the latter via contradiction. Suppose $\chi \equiv 0$ on $\mathbb{R}^n$. Consider the scaled SLCS $\{(A_i/\rho^*, B_i/\rho^*)\}_{i\in\mathcal{M}}$ whose solutions $\tilde{x}(t;\sigma,\mathbf{u},z) = x(t;\sigma,\mathbf{u},z)/(\rho^*)^t$. Fix an arbitrary $z \in \mathbb{S}^{n-1}$. For any $\varepsilon > 0$, there exists $\mathbf{u}_z \in \mathcal{U}$ such that for any $\sigma \in \mathcal{S}$, there exists $t_{z,\sigma} \in \mathbb{Z}_+$ such that $\|\tilde{x}(t;\sigma,\mathbf{u}_z,z)\| < \varepsilon$ for all $t \geq t_{z,\sigma}$. Following the argument in the proof of Claim (35) in Theorem II.1, we conclude that there exists a uniform $T_z \in \mathbb{Z}_+$ such that for any $\sigma \in \mathcal{S}$, $\|\tilde{x}(t;\sigma,\mathbf{u}_z,z)\| < \varepsilon$ for some $t \leq T_z$. By setting $\varepsilon = 1/2$ and using a similar argument as in Theorem II.1, we deduce that the scaled SLCS is σ-resiliently exponentially stabilizable. Hence its σ-resilient stabilizing rate $\tilde{\rho}^* < 1$. By Lemma II.1, the σ-resilient stabilizing rate of the original SLCS is $\rho^* \cdot \tilde{\rho}^*$ which is strictly less than ρ^*, a contradiction. This shows that $\mathcal{N}_\chi = \{0\}$, i.e., χ is a norm on $\mathbb{R}^n$.

To show $\chi_\sharp = \rho^* \cdot \chi$, let $z \in \mathbb{R}^n$ be arbitrary. Decompose $\mathbf{u} \in \mathcal{U}$ and $\sigma \in \mathcal{S}$ as $\mathbf{u} = (\mathbf{u}_0, \mathbf{u}_+)$ and $\sigma = (\sigma(0), \sigma_+)$ as before. Denote $v := \mathbf{u}_0(z)$ and $i := \sigma(0)$. Then

$$\begin{aligned}\chi(z) &= \inf_{\mathbf{u}_0} \sup_{\sigma(0)} \inf_{\mathbf{u}_+} \sup_{\sigma_+} \limsup_{t\to\infty} \frac{\|x(t;\sigma,\mathbf{u},z)\|}{(\rho^*)^t} \\ &= \inf_{v\in\mathbb{R}^p} \sup_{i\in\mathcal{M}} \inf_{\mathbf{u}_+} \sup_{\sigma_+} \limsup_{t\to\infty} \frac{\|x(t;\sigma_+,\mathbf{u}_+,A_i z + B_i v)\|}{(\rho^*)^{t+1}} \\ &= \inf_{v\in\mathbb{R}^p} \sup_{i\in\mathcal{M}} \frac{\chi(A_i z + B_i v)}{\rho^*} = \frac{\chi_\sharp(z)}{\rho^*}.\end{aligned}$$

Again, in deriving the first equality we use the observation in Remark II.1 to exchange the order of $\inf_{\mathbf{u}_+}$ and $\sup_{\sigma(0)}$. As a result, $\chi_\sharp = \rho^* \cdot \chi$, proving that χ is a Barabanov norm.

If an irreducible SLCS has $\rho^* = 0$, then any norm $\|\cdot\|$ satisfies $\|\cdot\|_\sharp = 0$ and is a Barabanov norm. □

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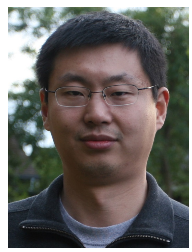

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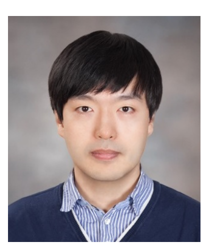

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