Periodic Stabilization of Discrete-Time Switched Linear Systems

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***Abstract*—The goal of this paper is to study the exponential stabilization problem for autonomous discrete-time switched linear systems (SLSs), where only the discrete mode can be controlled. Our approach is based on periodic control Lyapunov functions whose value decreases periodically instead of at each time step as in the classical control Lyapunov functions. Using periodic control Lyapunov functions, we develop the stabilizability analysis and controller synthesis conditions that are less conservative than existing results in that they apply to a larger class of SLSs. Utilizing recent results on the switched optimal control problems, a constructive way to find periodic control Lyapunov functions is presented.**

I. Introduction

Switched linear systems (SLSs) are an important class of hybrid systems where the system dynamics matrix varies within a finite set of subsystem matrices (or modes) according to a switching signal. The study of SLSs has attracted much attention in the past decades [3]. For SLSs (and general hybrid systems), a fundamental problem is to analyze their stability/stabilizability and design the stabilizing controls [4]. The most commonly used approach for this problem is the Lyapunov methods. In the simplest case, common quadratic Lyapunov functions (CQLF) can be used, which generally lead to conservative conditions as a large class of stable or stabilizable systems exist that do not admit a CQLF.

The development of more general Lyapunov functions has been an active research topic [5]. Some important classes of Lyapunov functions include multiple Lyapunov functions [6], piecewise quadratic Lyapunov functions (PWQLF) [7]–[13], polyhedral or polytopic Lyapunov functions [14], sum-of-squares polynomial Lyapunov functions [15], convex hull Lyapunov functions [16], [17], and switched Lyapunov functions [18], [19]. Besides the Lyapunov methods, other approaches include the joint spectral radius (JSP) [20], the generating functions approach [21], to name a few.

Another progress of the Lyapunov method is the so-called non-monotonic Lyapunov functions, whose value may not decrease at each time step along the state trajectories as in the case of classical Lyapunov functions. For this reason, they generally yield less conservative stability certificates for a broader class of the SLSs. To the authors' knowledge, the non-monotonic Lyapunov functions were first proposed in [22], [23] for nonlinear and switching systems. They were recently generalized in [24] to graph Lyapunov functions consisting of a finite set of non-monotonic Lyapunov functions. A special class of the non-monotonic Lyapunov functions is the periodic Lyapunov functions (PLF) whose value decreases periodically in time. Originally proposed to study periodic systems [25]–[27], PLFs were recently found useful in the study of non-periodic systems. For example, PLFs along with periodically time-varying state-feedback controllers were used in the stability/stabilization study of discrete-time uncertain linear systems [28], [29] and nonlinear systems [30]. It was found that, in general, PLFs provide less conservative stability analysis and control synthesis conditions, and can improve the performances of the control systems, such as their robustness and H_∞ performance. The continuous-time counterparts were also investigated in [31], [32]. Other recent applications of PLFs can be found in [33] for systems with network induced input delays, [34] for sampled-data systems under asynchronous samplings, and [35] for impulsive systems. The concept of PLFs was also used in [36] for the stability analysis of continuous-time SLSs with minimum dwell-time constraint.

Inspired by the aforementioned researches, the goal of this paper is to investigate the periodic control Lyapunov functions (PCLF) in the stabilization problem of the discrete-time autonomous SLSs. In the associated state-feedback switching policy, at every h time steps, a switching sequence of length h is generated and applied to minimize the PCLF's value after h steps; and this process is repeated every h steps. If a PCLF exists whose value decreases every h (but not necessarily each) time steps, then the SLS is switching stabilizable. It is worth mentioning that work along the same direction has been reported recently in [37]–[39]. In [37], the PCLFs were used to develop a set-theoretic/geometric necessary and sufficient condition for the switching stabilizability of the autonomous SLSs. Furthermore, the PCLFs were generalized to aperiodic control Lyapunov functions in [38], [39], where sufficient linear matrix inequality (LMI) and bilinear matrix inequality (BMI) conditions were developed to check the stabilizability and reduce the inherent complexity of the geometric approach in [37]. Connections between the developed LMI/BMI conditions and the geometric conditions in [37] were also established.

The main contributions of this paper consist of the following: 1) A periodic control Lyapunov theorem and its converse are derived, which show that a PCLF with a sufficiently large period h exists if and only if the SLS is switching stabilizable; 2) Inspired by the dynamic programming and relaxation schemes in [11], [21], efficient computation methods based on LMI/BMI conditions are developed for computing

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the Lyapunov functions and the stabilizing controllers; 3) The path-following algorithm [40] is applied for solving the BMI conditions, whose efficacy is demonstrated by examples.

Compared to the recent advances in [38], [39] which studied more general stabilizability certificates, this paper focuses on the particular class of quadratic PCLFs and specifically their efficient computation and performance analysis. For example, we derive more explicit bounds in the converse periodic control Lyapunov theorem; provide further refined conservativeness analysis of the LMI/BMI-based sufficient conditions; propose computational algorithms with significantly reduced complexity; and establish the superiority of the proposed algorithms against the existing piecewise quadratic control Lyapunov functions (PWQLF) method [11], [21].

This paper is organized as follows. Some preliminary definitions and the problem formulation are given in Section II. In Section III, the notion of the PCLFs and the associated (converse) Lyapunov theorems are introduced. In Section IV, we study the methods and algorithms based on convex optimizations to compute the PCLFs. In Section V, a comparative analysis of the proposed PCLF and the PWQLFs is provided. A numerical relaxation method to reduce the computational cost is presented in Section VI and some numerical examples are given in Section VII.

II. Preliminaries and Problem Formulation

A. Notation

The adopted notations are as follows. $\mathbb{N}$ and $\mathbb{N}_+$: sets of nonnegative and positive integers, respectively; $\mathbb{R}$: set of real numbers; $\mathbb{R}_+$ and $\mathbb{R}_{++}$: sets of nonnegative and positive real numbers, respectively; $\mathbb{R}^n$: n-dimensional Euclidean space; $\mathbb{R}^{n\times m}$: set of all $n\times m$ real matrices; A^T: transpose of matrix A; $A \succ 0$ (resp. $A \prec 0$, $A \succeq 0$, and $A \preceq 0$): symmetric positive definite (resp. negative definite, positive semi-definite, and negative semi-definite) matrix A; I_n: $n\times n$ identity matrix; $||\cdot||$: Euclidean norm of a vector or spectral norm of a matrix; $\mathbb{S}^n$ (resp. $\mathbb{S}^n_+$, $\mathbb{S}^n_{++}$): set of symmetric (resp. positive semi-definite, positive definite) $n\times n$ matrices; $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$: minimum and maximum eigenvalues of symmetric matrix A, respectively; $\lceil x \rceil$: the minimum integer greater than $x\in\mathbb{R}$; $\lfloor x \rfloor$: the maximum integer less than $x\in\mathbb{R}$; $\mathrm{cond}(P)$: condition number $\lambda_{\max}(P)/\lambda_{\min}(P)$ of $P\in\mathbb{S}^n_{++}$; $\rho(A)$: spectral radius of a square matrix A; $\mathrm{conv}(\cdot)$: convex hull; $t \mod h$: remainder of t divided by h for $t, h\in\mathbb{N}_+$.

B. Problem formulation

Consider the discrete-time autonomous SLS (1)

$$x(k+1) = A_{\sigma_k} x(k), \quad x(0) = z \in \mathbb{R}^n, \tag{1}$$

where for $k\in\mathbb{N}$, $x(k)\in\mathbb{R}^n$ is the state, $\sigma_k \in \mathcal{M} := \{1, 2, \ldots, N\}$ is called the mode, and $A_i\in\mathbb{R}^{n\times n}$, $i\in\mathcal{M}$, are the subsystem matrices. Starting from $x(0) = z\in\mathbb{R}^n$ and under the infinite-horizon switching sequence $\sigma^\infty := (\sigma_0, \sigma_1, \ldots)\in\mathcal{M}^\infty$ or the finite-horizon switching sequence $\sigma^h := (\sigma_0, \sigma_1, \ldots, \sigma_{h-1}) \in \mathcal{M}^h$ for some $h\in\mathbb{N}_+$, the solution of the SLS (1) is denoted by $x(k;\, z,\, \sigma^\infty)$ or $x(k;\, z,\, \sigma^h)$, respectively. For convenience, we denote $A_{\sigma^h} := A_{\sigma_{h-1}}\cdots A_{\sigma_1} A_{\sigma_0}$ for $\sigma^h = (\sigma_0, \sigma_1, \ldots, \sigma_{h-1})\in\mathcal{M}^h$.

Definition 1. *([21, Definition 1]) The SLS (1) is called*

1) *asymptotically switching stabilizable if for any $z\in\mathbb{R}^n$, there exists a switching sequence $\sigma^\infty(z)$ under which $\lim_{k\to\infty} ||x(k;\, z,\, \sigma^\infty(z))|| = 0$.*
2) *exponentially switching stabilizable (with the parameters a and c) if there exist $a\geq 1$ and $c\in[0, 1)$ such that for any $z\in\mathbb{R}^n$, there exists a switching sequence $\sigma^\infty(z)$ satisfying*

$$||x(k;\, z,\, \sigma^\infty(z))|| \leq ac^k ||z||, \quad \forall k\in\mathbb{N}. \tag{2}$$

It is proved in [21, Theorem 1] that the above two notions of switching stabilizability are equivalent. Therefore, we will refer to either of them as switching stabilizability or simply stabilizability throughout the paper. A trivial result is that, if one of the subsystem matrices is Schur stable, the SLS (1) is stabilizable. Thus, to avoid triviality, the following assumption is made in this paper.

Assumption 1. *Each of the subsystem matrix A_i, $i\in\mathcal{M}$, is not Schur stable.*

As a result, we have

$$\phi := \max_{i\in\mathcal{M}} \|A_i\| \geq 1. \tag{3}$$

The goal of this paper is to solve the following problem.

Problem 1 (Switching stabilization)**.** *Determine if the SLS (1) is stabilizable and, if yes, find a stabilizing switching policy.*

More generally, for (not necessarily stabilizable) SLSs, any $c\in\mathbb{R}_+$ (possibly $c > 1$) satisfying (2) for some $a\geq 1$ will be called an *exponential convergence rate*. The *exponential stabilizing rate*, denoted by $c^*\in\mathbb{R}_+$, is the infimum of all such exponential convergence rates. Note that c^* provides a quantitative metric of the SLS's stabilizability. A secondary goal of this paper is to give some characterizations of the rate c^*. Further details on the stabilizing rate can be found for autonomous SLSs in [41] and non-autonomous SLSs in [42].

III. Periodic Stabilization

A. Periodic Control Lyapunov Functions

We now introduce the notion of periodic control Lyapunov functions and the associated Lyapunov theorem for the stabilzability of the SLS (1).

Definition 2 (h-PCLF)**.** *Let $h\in\mathbb{N}_+$. A continuous function $V:\mathbb{R}^n\to\mathbb{R}_+$ is called an h-*periodic control Lyapunov function *(h-PCLF) of the SLS (1) if there exist an h-horizon state-feedback switching policy $\sigma^h:\mathbb{R}^n\to\mathcal{M}^h$, positive constants κ_1, $\kappa_2\in\mathbb{R}_{++}$, and $\kappa_3 < 0$ such that the following conditions hold:*

$$\kappa_1 \|z\|^2 \leq V(z) \leq \kappa_2 \|z\|^2, \quad \forall z \tag{4}$$

$$V(x(h;\, z,\, \sigma^h(z))) - V(z) \leq \kappa_3 \|z\|^2, \quad \forall z. \tag{5}$$

Conditions (4) and (5) hint that the state trajectories of the SLS can be sliced into pieces of length h, each controlled by an h-horizon switching policy defined as follows.

Definition 3 (h-SP)**.** *For a function $V : \mathbb{R}^n \to \mathbb{R}_+$, define*

$$V_h(z) := \min_{\sigma^h \in \mathcal{M}^h} V(A_{\sigma^h} z), \quad \forall z \in \mathbb{R}^n. \tag{6}$$

The (state-feedback) h-horizon switching policy (h-SP) is defined as the mapping $\tilde{\sigma}^h : \mathbb{R}^n \to \mathcal{M}^h$ such that

$$\tilde{\sigma}^h(z) := \arg\min_{\sigma^h \in \mathcal{M}^h} V(A_{\sigma^h} z), \quad \forall z \in \mathbb{R}^n. \tag{7}$$

The h-SP generates the current and future switching signals of length h to minimize the value of $V(\cdot)$ after h steps. When it is applied to the system repeatedly every h steps, we obtain the following infinite-horizon switching policy.

Definition 4 (h-PSP)**.** *For a function $V : \mathbb{R}^n \to \mathbb{R}_+$, the h-periodic infinite-horizon switching policy (h-PSP) $\tilde{\sigma}^h_\infty$ is obtained by repeating the h-SP $\tilde{\sigma}^h$ every h steps:*

$$\tilde{\sigma}^h_\infty(z) = (\tilde{\sigma}^h(z),\, \tilde{\sigma}^h(x(h)),\, \tilde{\sigma}^h(x(2h)), \ldots),\ \forall z \in \mathbb{R}^n, \tag{8}$$

where $x((\ell+1)h) = x(h;\, x(\ell h),\, \tilde{\sigma}^h(x(\ell h)))$, $\ell \in \mathbb{N}_+$.

In other words, the first h switching signals are generated as $\tilde{\sigma}^h(z)$, under which the system state evolves to $x(h)$. Then the next h switching signals are generated as $\tilde{\sigma}^h(x(h))$, driving the state to $x(2h)$. This process is then repeated indefinitely.

The next theorem shows that the existence of an h-PCLF is a sufficient condition for the stabilizability of the SLS.

Theorem 1 (PCLF Theorem)**.** *If there exists an h-PCLF $V(\cdot)$, then the SLS (1) under the corresponding h-PSP $\tilde{\sigma}^h_\infty$ is stabilized with the parameters $a = \phi^h \left(\frac{\kappa_2}{\kappa_1}\right)^{\frac{1}{2}} \left(\frac{\kappa_2}{\kappa_2+\kappa_3}\right) \in [1, \infty)$ and $c = \left(1 + \frac{\kappa_3}{\kappa_2}\right)^{\frac{1}{2h}} \in [0, 1)$, where ϕ is defined in (3).*

Proof. Consider the sequence $\xi(t) := x(ht;\, z,\, \tilde{\sigma}^h_\infty(z))$, $t \in \mathbb{N}$, where $\xi(0) = x(0) = z$ is arbitrary. The conditions (4) and (5) and the definition of the h-PSP $\tilde{\sigma}^\infty_h$ imply that, for $t \in \mathbb{N}$,

$$V(\xi(t)) - V(\xi(t+1)) \geq -\kappa_3 \|\xi(t)\|^2 \geq -\frac{\kappa_3}{\kappa_2} V(\xi(t))$$
$$\Rightarrow \quad V(\xi(t+1)) \leq \left(1 + \frac{\kappa_3}{\kappa_2}\right) V(\xi(t)).$$

Note that $1 + \kappa_3/\kappa_2 \in [0,\, 1)$. By induction, this leads to

$$V(\xi(t)) \leq \left(1 + \frac{\kappa_3}{\kappa_2}\right)^t V(z), \quad \forall t \in \mathbb{N}.$$

Using (4) and the fact that $\xi(t) = x(ht;\, z,\, \tilde{\sigma}^h_\infty)$, we obtain

$$\left\|x(ht;\, z,\, \tilde{\sigma}^h_\infty)\right\| \leq \left(\frac{\kappa_2}{\kappa_1}\right)^{\frac{1}{2}} \left(1 + \frac{\kappa_3}{\kappa_2}\right)^{\frac{t}{2}} \|z\|, \quad \forall t.$$

Noting that $k = h\lfloor k/h \rfloor + (k \mod h)$ where $\lfloor k/h \rfloor \geq k/h - 1$ and $(k \mod h) \leq h$ for $k \in \mathbb{N}$, we have

$$\begin{aligned} &\left\|x(k;\, z,\, \tilde{\sigma}^h_\infty)\right\| = \left\|x(h\lfloor k/h \rfloor + (k \mod h);\, z,\, \tilde{\sigma}^h_\infty)\right\| \\ &\leq \phi^{(k \mod h)} \left\|x(h\lfloor k/h \rfloor;\, z,\, \tilde{\sigma}^h_\infty)\right\| \\ &\leq \phi^h \left(\frac{\kappa_2}{\kappa_1}\right)^{\frac{1}{2}} \left(1 + \frac{\kappa_3}{\kappa_2}\right)^{\frac{1}{2}\lfloor k/h \rfloor} \|z\| \leq ac^k \|z\| \end{aligned}$$

for all $z \in \mathbb{R}^n$, which is the desired conclusion. $\square$

For a (not necessarily stabilizable) SLS and given $h \in \mathbb{N}_+$, its exponential stabilizing rate by h-periodic switching policies $\tilde{\sigma}^h_\infty$ of the form (8) is defined as

$$\begin{aligned} c^*_h := \inf_{\tilde{\sigma}^h_\infty(\cdot)} \{c \geq 0 :\ &\text{there exists } a < \infty \text{ such that} \\ &\left\|x(k;\, z,\, \tilde{\sigma}^h_\infty(z))\right\| \leq ac^k \|z\|,\ \forall z \in \mathbb{R}^n, \forall k \in \mathbb{N}\}. \end{aligned}$$

Obviously, $c^* \leq c^*_h$ holds. Theorem 1 can also be modified to estimate c^*_h as follows. Let $V : \mathbb{R}^n \to \mathbb{R}_+$ be a function satisfying all the conditions of h-PCLF with the only exception being that the constant κ_3 in (5) may be positive. Then the proof of Theorem 1 implies that $c^*_h \leq \left(1 + \frac{\kappa_3}{\kappa_1}\right)^{\frac{1}{2h}}$.

B. Periodic Quadratic Control Lyapunov Functions

In Theorem 1, the h-PCLF $V(\cdot)$ is a generic positive definite function. In the rest of the paper, we will consider only quadratic $V(z) = z^T P z$ given by $P \in \mathbb{S}^n_{++}$. As will be shown later on, there is no loss of generality in doing so.

Definition 5 (h-PQCLF)**.** *An h-PCLF $V(\cdot)$ of the form $V(z) = z^T P z$ for some $P \in \mathbb{S}^n_{++}$ is called an h-periodic quadratic control Lyapunov function (h-PQCLF).*

For any quadratic function $V(z) = z^T P z$ with $P \in \mathbb{S}^n_{++}$, define the quantity $w_h \in \mathbb{R}$ as

$$w_h := \sup_{z \in \mathbb{R}^n,\, \|z\|=1} [V_h(z) - V(z)], \tag{9}$$

where $V_h(\cdot)$ is defined in (6). By the compactness of the set $\{z \in \mathbb{R}^n : \|z\| = 1\}$ and the continuity of the function inside the bracket, the supremum in (9) can be replaced with maximum. In the sequel, the notation $w_h(P)$ is occasionally used to emphasize the dependence of w_h on P.

Note that w_h is the smallest value of $w \in \mathbb{R}$ such that $V(A_{\sigma^h} z) - V(z) \leq w\|z\|^2$ for all $\sigma^h \in \mathcal{M}^h$ and all $z \in \mathbb{R}^n$. In a sense, w_h measures quantitatively the degree of satisfaction of the condition (5) by the quadratic function $V(\cdot)$. When $w_h < 0$, $V(\cdot)$ becomes an h-PQCLF.

Theorem 2 (PQCLF Theorem I)**.** *Suppose for a quadratic function $V(z) = z^T P z$ with $P \in \mathbb{S}^n_{++}$, the quantity w_h defined in (9) satisfies $w_h < 0$. Then $V(\cdot)$ is an h-PQCLF of the SLS (1), and the SLS is stabilizable with the parameters $a = \phi^h \mathrm{cond}(P)^{1/2} \left(\frac{\lambda_{\max}(P)}{\lambda_{\max}(P)+w_h}\right)$ and $c = \left(1 + \frac{w_h}{\lambda_{\max}(P)}\right)^{\frac{1}{2h}}$.*

Proof. By (9), $\min_{\sigma^h \in \mathcal{M}^h} V(A_{\sigma^h} z) - V(z) \leq w_h \cdot \|z\|^2 < 0$ for all $z \in \mathbb{R}^n$. Therefore, with $\tilde{\sigma}^h(\cdot)$ defined in (7), the conditions (4) and (5) of Definition 2 are satisfied with $\kappa_1 = \lambda_{\min}(P)$, $\kappa_2 = \lambda_{\max}(P)$, and $\kappa_3 = w_h$. The desired conclusion then follows immediately from Theorem 1. $\square$

Remark 1. *Even if $w_h \geq 0$, we can still obtain an estimate of the exponential stabilizing rate c^*_h as $c^*_h \leq \left(1 + \frac{w_h}{\lambda_{\min}(P)}\right)^{\frac{1}{2h}}$.*

Due to the importance of w_h in determining the stabilizability of the SLS, several of its properties are listed below.

Proposition 1. *The following statements hold:*

a) (Monotonicity): If $w_h \leq 0$, then $w_h \geq w_{2h} \geq \cdots$;

b) *If $c \in \mathbb{R}_+$ is an exponential convergence rate of the SLS, i.e., if (2) holds with the given c and some $a \in [0, \infty)$, then $-\lambda_{\min}(P) \le w_h \le \lambda_{\max}(P) a^2 c^{2h} - \lambda_{\min}(P)$;*
c) *If the SLS (1) is stabilizable with the parameters $a \in [1, \infty)$ and $c \in [0, 1)$, then $\lim_{h\to\infty} w_h = -\lambda_{\min}(P)$;*
d) *With $P = I_n$, $w_h \le \min_{\sigma^h \in \mathcal{M}^h} \|A_{\sigma^h}\|^2 - 1$;*
e) *With $P = I_n$, $c_h^* \le \min_{\sigma^h \in \mathcal{M}^h} \|A_{\sigma^h}\|^{\frac{1}{h}}$. In particular, $\lim_{h\to\infty} c_h^* \le \hat{\rho}$, where $\hat{\rho} := \lim_{h\to\infty} \min_{\sigma^h \in \mathcal{M}^h} \|A_{\sigma^h}\|^{\frac{1}{h}}$ is the joint spectral subradius of the matrix set $\{A_i\}_{i\in\mathcal{M}}$ [20, pp. 7].*

Proof. Let the h-SP $\tilde{\sigma}^h$ be defined as in (7).
a): Suppose $w_h \le 0$. Denote $x(h) := x(h; z, \tilde{\sigma}^h(z))$ and $x(2h) := x(h; x(h), \tilde{\sigma}^h(x(h)))$. Then $V(x(2h)) - V(z) \le [V(x(2h)) - V(x(h))] + [V(x(h)) - V(z)] \le w_h\|x(h)\|^2 + w_h\|z\|^2 \le w_h\|z\|^2$, which implies $w_{2h} \le w_h$. The rest of the inequalities can be obtained by induction.
b): The lower bound is obtained from $w_h \ge \sup_{z\in\mathbb{R}^n, \|z\|=1}[-V(z)] = -\lambda_{\min}(P)$, and the upper bound follows from $V_h(z) - V(z) = V(x(h; z, \tilde{\sigma}^h)) - V(z) \le \lambda_{\max}(P) a^2 c^{2h} \|z\|^2 - \lambda_{\min}(P) \|z\|^2$.
c): This follows from b) by letting $h \to \infty$.
d): Exchanging the order of sup and min, we have $\sup_{z\in\mathbb{R}^n, ||z||=1} V_h(z) \le \min_{\sigma^h\in\mathcal{M}^h} \|A_{\sigma^h}\|^2$. The result immediately follows from the definition (9) of w_h.
e): The first inequality is obtained from the upper bound on w_h in part d) with the upper bound on c_h^* in Remark 1. The second inequality follows by taking the limit $h \to \infty$. □

Part a) of Proposition 1 leads to the following result.

Corollary 1 (Periodic monotonicity I)**.** *If $V(\cdot)$ is an h-PQCLF of the SLS (1), then it is a kh-PQCLF for all $k \in \mathbb{N}_+$.*

The switching strategy $\tilde{\sigma}^h$ given in (7) can be viewed as the solution of a finite-horizon switched LQR problem [12] with the terminal cost function $V(\cdot)$ and zero running cost. In this context, $V_h(\cdot)$ is the h-horizon value function. By the Bellman equation [43], $V_h(\cdot)$ can be obtained recursively from the one-stage value iteration $V_{k+1}(z) = \min_{i\in\mathcal{M}} V_k(A_i z)$ with $V_0(z) = V(z)$. Denote by $V_\infty(\cdot)$ the pointwise limit (whenever it exists) of the sequence of the functions $\{V_k(\cdot)\}_{k=0}^\infty$. Some properties of the functions $V_h(\cdot)$ and $V_\infty(\cdot)$ are given below.

Proposition 2. *$V_h(\cdot)$ and $V_\infty(\cdot)$ have the following properties.*
1) *$V_{h+s}(z) = \min\limits_{\sigma^s\in\mathcal{M}^s} V_h(A_{\sigma^s} z)$, $\forall z \in \mathbb{R}^n$, $s \in \mathbb{N}_+$.*
2) *If the SLS (1) is stabilizable, then $V_\infty(\cdot) \equiv 0$ on $\mathbb{R}^n$.*

Proof. The statement 1) can be proved by repeating the one-stage value iterations. To prove 2), we note that the assumption implies that there exist constants $a \in [1, \infty)$, $c \in [0, 1)$, and a switching sequence $\tilde{\sigma}^\infty(z)$ for any $z \in \mathbb{R}^n$ such that $||x(k; z, \tilde{\sigma}^\infty(z))|| \le ac^k||z||$, $\forall k \in \mathbb{N}$. Then, it follows that $0 \le V_h(z) \le \lambda_{\max}(P) \|x(h; z, \tilde{\sigma}^\infty)\|^2 \le \lambda_{\max}(P) a^2 c^{2h} \|z\|^2$, which implies $\lim_{h\to\infty} V_h(z) = 0$. □

C. Converse PQCLF Theorem

By Theorem 2, if a PQCLF exists, then the SLS (1) is stabilized by the h-PSP (8). We next study the converse problem, namely, if the SLS is stabilizable, does there always exist a PQCLF? The following theorem says that this is indeed the case; in fact, $V(\cdot)$ with any given $P \in \mathbb{S}_{++}^n$ can become a PQCLF for a sufficiently large $h \in \mathbb{N}_+$.

Theorem 3 (Converse PQCLF Theorem I)**.** *Suppose that the SLS (1) is stabilizable with the parameters $a \in [1, \infty)$ and $c \in [0, 1)$. Then, for any $P \in \mathbb{S}_{++}^n$, $V(z) = z^T P z$ is a PQCLF of the SLS for all $h > \bar{h}(P, a, c)$, where $\bar{h}(P, a, c)$ is defined by $\bar{h}(P, a, c) = \left\lceil \frac{\ln(\mathrm{cond}(P)) + \ln(a^2)}{\ln(1/c^2)} \right\rceil$.*

Proof. By part b) of Proposition 1, $w_h \le \lambda_{\max}(P) a^2 c^{2h} - \lambda_{\min}(P)$. Thus $V(\cdot)$ is a PQCLF if $\lambda_{\max}(P) a^2 c^{2h} < \lambda_{\min}(P)$, or equivalently, if $h > \ln(\mathrm{cond}(P) a^2)/\ln(1/c^2)$. □

We have now established that the SLS is stabilizable if and only if it can be stabilized by a periodic switching policy σ_∞^h of the form (8), namely, an h-PSP. A natural question is whether the exponential stabilizing rate c^* defined in Section II can be achieved by using an h-PSP, i.e., whether $c^* = c_h^*$. The next result shows that asymptotically this is indeed the case.

Proposition 3. *It holds that $\lim_{h\to\infty} c_h^* = c^*$.*

Proof. Without loss of generality, let $P = I_n$. By Remark 1, $c_h^* \le (1 + w_h)^{\frac{1}{2h}}$. From the definition of c^*, for any $\varepsilon > 0$, there exists a constant $a \in [0, \infty)$ such that (2) holds. Thus by part b) of Proposition 1, we have $w_h \le a^2(c^* + \varepsilon)^{2h} - 1$. Combining the two inequalities, we have $c_h^* \le a^{\frac{1}{h}}(c^* + \varepsilon)$. Taking the limit $h \to \infty$ and noting that $c_h^* \ge c^*$ and that $\varepsilon > 0$ is arbitrary, we obtain the desired conclusion. □

Despite the above result, it is possible that $c_h^* > c^*$ for any finite $h \in \mathbb{N}_+$, i.e., the exponential stabilizing rate c^* cannot be exactly achieved by a finite horizon h-PSP. This is illustrated by the following example of SLS (1) taken from [41]:

$$A_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad x(0) = e := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

As observed in [41], $x_2(k)$ is non-decreasing; hence the SLS is not stabilizable, i.e., $c^* \ge 1$. Now set $P = I_n$. For any $h \in \mathbb{N}_+$, it is easy to see that, starting from $x(0) = e$, the h-SP $\tilde{\sigma}^h(\cdot)$ produces the switching sequence $\tilde{\sigma}^h(e) = (1, 1, \ldots, 1, 2)$, under which we have $x(h) = A_2 A_1^{h-1} e = A_2 e = 2e$. By induction and the homogeneity of $\tilde{\sigma}^h(\cdot)$, we have $x(\ell h) = 2^\ell e$, $\forall \ell \in \mathbb{N}$; hence $c_h^* \ge 2^{\frac{1}{h}} > 1$. On the other hand, part e) of Proposition 1 implies that $c_h^* \le \|A_2 A_1^{h-1}\|^{\frac{1}{h}} = \|A_2\|^{\frac{1}{h}} = 6^{\frac{1}{2h}}$; hence $c^* = \lim_{h\to\infty} c_h^* \le 1$. Combining, we have shown that $c^* = 1$ and that $c_h^* > c^*$ for all h. Indeed, [41, Prop. 2] showed that the same conclusion holds for any 0-homogeneous state-feedback switching policy.

IV. Computation of PQCLFs

In this section, inspired by [11], [21], an efficient numerical method is proposed to compute the PQCLFs.

A. Computing Overestimates of w_h

For given $P \in \mathbb{S}^n_{++}$ and $h \in \mathbb{N}_+$, a sufficient condition for $V(z) = z^T P z$ to be an h-PQCLF is that (an overestimate of) the quantity w_h defined in (9) is negative. We will now compute overestimates of w_h. First note that the function $V_h(\cdot)$ defined in (6) can be represented as

$$V_h(z) = \min_{H \in \mathcal{P}_h(P)} z^T H z,$$

where $\mathcal{P}_h(P)$ is the matrix set $\{A^T_{\sigma^h} P A_{\sigma^h}\}_{\sigma^h \in \mathcal{M}^h}$. As an example, for $N = 2$ and $h = 2$, $\mathcal{P}_h(P)$ consists of the matrices $(A_1A_1)^T P(A_1A_1)$, $(A_1A_2)^T P(A_1A_2)$, $(A_2A_1)^T P(A_2A_1)$, and $(A_2A_2)^T P(A_2A_2)$. Alternatively, $\mathcal{P}_h(P)$ is obtained from the iteration [11]: $\mathcal{P}_0(P) = \{P\}$, and for $k \in \{1, 2, \dots, h\}$,

$$\mathcal{P}_k(P) = \{A^T_i H A_i : H \in \mathcal{P}_{k-1}(P)\}. \tag{10}$$

In the sequel, $\mathcal{P}_k(P)$ will simply be denoted by $\mathcal{P}_k$ if there is no confusion. We now introduce a semidefinite programming (SDP) problem whose solutions provide overestimates of w_h.

Problem 2. *Let* $\mathrm{conv}(\mathcal{P}_h)$ *be the convex hull of* $\mathcal{P}_h$*. Find*

$$\tilde{w}_h := \min\{w \in \mathbb{R} : \exists\, P' \in \mathrm{conv}(\mathcal{P}_h) \text{ s.t. } P' - P \preceq w I_n\}.$$

Proposition 4. $w_h \le \tilde{w}_h$ *holds.*

Proof. Fix any $w \in \mathbb{R}$ such that $P' - P \preceq w I_n$ for some $P' \in \mathrm{conv}(\mathcal{P}_h)$, i.e., $P' = \sum_{i=1}^k \alpha_i F^{(i)}$ where $\{F^{(i)}\}_{i=1}^k$ is an enumeration of $\mathcal{P}_h$ and $(\alpha_1, \dots, \alpha_k)$ is in the k-simplex $\Delta_k := \left\{(\alpha_1, \dots, \alpha_k) \mid \sum_{i=1}^k \alpha_i = 1,\ \alpha_i \ge 0,\ i = 1, \dots, k\right\}$. For any $z \in \mathbb{R}^n$, since $V_h(z) = \min_{i=1,\dots,k} z^T F^{(i)} z \le z^T P' z$, we have $V_h(z) - z^T P z \le z^T(P' - P)z \le w\|z\|^2$. By the definition of w_h, this implies $w_h \le w$; hence $w_h \le \tilde{w}_h$. $\square$

Thus, a sufficient stabilizability condition can be obtained.

Corollary 2. *If* $\tilde{w}_h < 0$ *for given* $P \in \mathbb{S}^n_{++}$ *and* $h \in \mathbb{N}_+$*, then* $V(z) = z^T P z$ *is an* h*-PQCLF and the SLS (1) is stabilizable.*

Remark 2. *The condition in Problem 2 is similar to the Lyapunov-Metzler inequalities in [10] (see [39] for more general conditions). For example, [10] also employed Lyapunov functions that are the minimum of a finite number of functions and over-approximated their minimum by their weighted averages. This technique is frequently used in the study of switched systems, e.g., [9], [11], [12].*

The following result shows that if $\tilde{w}_h$ can be made negative, then the choice of P can be arbitrary with a large enough h.

Proposition 5. *Suppose there exist* $P_0 \in \mathbb{S}^n_{++}$ *and* $h \in \mathbb{N}_+$ *such that* $\tilde{w}_h(P_0) < 0$*. Then for any* $P \in \mathbb{S}^n_{++}$*,* $\tilde{w}_{ht}(P) < 0$ *for some large enough* $t \in \mathbb{N}_+$*.*

Proof. Let $h \in \mathbb{N}_+$ be as given. For any $P \in \mathbb{S}^n_{++}$, an enumeration of $\mathcal{P}_h$ is given by $\{F^{(i)}(P)\}_{i=1}^k$ where $k = |\mathcal{M}|^h$ and each $F^{(i)}(P)$ is of the form $A^T_{\sigma^h} P A_{\sigma^h}$ for some $\sigma^h \in \mathcal{M}^h$. Note that $F^{(i)}(P)$ is $\mathbb{S}^n_+$-monotone in P: $P \preceq P'$ implies $F^{(i)}(P) \preceq F^{(i)}(P')$.

Suppose there exists some $P = P_0 \in \mathbb{S}^n_{++}$ such that $\tilde{w}_h = c_0 < 0$. Then $\sum_{i=1}^k \alpha_i F^{(i)}(P_0) - P_0 \preceq c_0 I_n$ for some $\alpha \in \Delta_k$, or equivalently, $L(P_0) := \sum_{i=1}^k \alpha_i F^{(i)}(P_0) \preceq \beta P_0$ where $\beta := (1 + c_0/\lambda_{\min}(P_0)) \in [0, 1)$. Note that $L(\cdot)$ thus defined is an $\mathbb{S}^n_+$-monotonic map from $\mathbb{S}^n_+$ to $\mathbb{S}^n_+$. Its t-time composition, $L^{(t)} := L \circ \cdots \circ L$ where $t \in \mathbb{N}_+$, is also $\mathbb{S}^n_+$-monotone.

For any $P \in \mathbb{S}^n_{++}$, we have $\alpha_1 P_0 \preceq P \preceq \alpha_2 P_0$ where $\alpha_1 := \lambda_{\min}(P)/\lambda_{\max}(P_0)$ and $\alpha_2 := \lambda_{\max}(P)/\lambda_{\min}(P_0)$. Then $L^{(t)}(P) \preceq L^{(t)}(\alpha_2 P_0) \preceq \alpha_2 \beta^t P_0 \preceq (\alpha_2/\alpha_1)\beta^t P$. By choosing $t \ge \log(\alpha_2/\alpha_1)/\log(1/\beta)$, we have $(\alpha_2/\alpha_1)\beta^t < 1$ and hence $L^{(t)}(P) - P \prec 0$. Since $L^{(t)}(P) \in \mathrm{conv}(\mathcal{P}_{ht}(P))$, we conclude that $\tilde{w}_{ht}(P) < 0$. $\square$

B. Computing Periodic Switching Policies

We now describe the computation of the switching policies. Suppose an h-PQCLF $V(z) = z^T P z$ has been found. To compute the stabilizing h-SP $\tilde{\sigma}^h(\cdot)$ defined in (7), a naive approach is to enumeate $V(A_{\sigma^h} z)$ for all $\sigma^h \in \mathcal{M}^h$ and z. Alternatively, the dynamic programming approach in [12, Theorem 1] can be used. Specifically, the statement 1) of Proposition 2 yields the following whose proof is straightforward.

Proposition 6. *For* $z \in \mathbb{R}^n$ *and* $k \in \{1, 2, \dots, h\}$*, define*

$$i^*_k(z) := \arg\min_{i \in \mathcal{M}} \min_{H \in \mathcal{P}_{k-1}} z^T A^T_i H A_i z, \tag{11}$$

where $\mathcal{P}_k$ *is given in (10). Then the* h*-SP at state* $z \in \mathbb{R}^n$ *is*

$$\tilde{\sigma}^h(z) = (i^*_h(z),\ i^*_{h-1}(x(1)), \dots,\ i^*_1(x(h-1))), \tag{12}$$

where $(x(0), x(1), \dots, x(h-1))$ *is the state trajectory driven by the switching policy (11).*

C. Conservativeness of Overestimates $\tilde{w}_h$

We have shown in Proposition 1 that if the SLS is stabilizable, then $\lim_{h\to\infty} w_h < 0$ for any choice of $P \in \mathbb{S}_{++}$. With $\tilde{w}_h$ being an easier-to-compute overestimate of w_h, a natural question is whether $\lim_{h\to\infty} \tilde{w}_h < 0$ also holds. In the following, we will show that the answer to this question is in general negative via a counterexample, namely, a stabilizable SLS for which $\tilde{w}_h \ge 0$ for all $h \in \mathbb{N}_+$. This will imply that, compared to Theorem 2, Corollary 2 provides only a sufficient stabilizability test; and its conservativeness may not be completely eliminated by increasing h.

Consider the following SLS taken from [2]:

$$A_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} \cos(\pi/20) & -\sin(\pi/20) \\ \sin(\pi/20) & \cos(\pi/20) \end{bmatrix},$$

which satisfies $\det(A_1) = \det(A_2) = 1$. Define the subset $U := \{x \in \mathbb{R}^2 : 2\pi - \pi/10 \le \angle x \le 2\pi\}$ where $\angle x$ denotes the phase angle of $x \in \mathbb{R}^2 \simeq \mathbb{C}$. It is proved in [2] that the SLS is stabilized by the switching policy $\sigma(x) = 1$ if $x \in U$ and $\sigma(x) = 2$ if otherwise. (see [21, Sec. IV] for another similar example). Hence, by Proposition 1, $w_h < 0$ for sufficiently large h. On the other hand, the next (more general) result shows that $\tilde{w}_h$ cannot be made negative.

Proposition 7. *Given a SLS for which* $|\det(A_i)| \ge 1$ *for all* $i \in \mathcal{M}$*. Then* $\tilde{w}_h \ge 0$ *for all* $P \in \mathbb{S}^n_{++}$ *and all* $h \in \mathbb{N}_+$*.*

Proof. Using the inequality of arithmetic and geometric means, for any $H \in \mathbb{S}^n_+$, we have $\frac{1}{n}\mathrm{trace}(H) \ge \sqrt[n]{\det(H)}$. Therefore, for any $P \in \mathbb{S}^n_{++}$, $h \in \mathbb{N}_+$, and $\sigma^h \in \mathcal{M}^h$,

$$\mathrm{trace}\left(I_n - P^{-1/2} A^T_{\sigma^h} P A_{\sigma^h} P^{-1/2}\right)$$

$$\leq n - n\left[\det(P^{-1/2}A_{\sigma^h}^T P A_{\sigma^h} P^{-1/2})\right]^{\frac{1}{n}}$$
$$= n - n\left[\det(A_{\sigma_0})^2 \cdots \det(A_{\sigma_{h-1}})^2\right]^{\frac{1}{n}} \leq 0,$$

where the last inequality follows from the assumption that $|\det(A_i)| \geq 1$ for all $i \in \mathcal{M}$. This implies that the convex hull of the set $\{I_n - P^{-1/2}A_{\sigma^h}^T P A_{\sigma^h} P^{-1/2}\}_{\sigma^h \in \mathcal{M}^h}$ does not intersect $\mathbb{S}_{++}^n$ as matrices in the latter set have positive trace. This in turn implies that the convex hull of the set $\{P - A_{\sigma^h}^T P A_{\sigma^h}\}_{\sigma^h \in \mathcal{M}^h}$ does not intersect $\mathbb{S}_{++}^n$, either. By the definition of $\tilde{w}_h$, we have $\tilde{w}_h \geq 0$. □

Remark 3. *Proposition 7 generalizes Example 17 in [39] to a class of stabilizable SLSs that fail the SDP stabilizability test in Problem 2. We also note that Proposition 5 is similar to [39, Theorem 23], while in its proof gives an explicit lower bound on the period h for satisfying the test.*

D. Mean-Square Stabilizability

The next result shows that Corollary 2 also ensures the open-loop mean square stabilizability [21, Definition 2] of the SLS under a suitably chosen random switching policy. Recall that the SLS (1) under a random switching policy is called mean-square (MS) exponentially stable with the parameters $a \geq 0$ and $c \in [0, 1)$ if for any $x(0) = z \in \mathbb{R}^n$, the expectation $\mathbf{E}[\|x(k)\|^2] \leq ac^k \|x(0)\|^2$ for all $k \in \mathbb{N}$.

Theorem 4. *Given $P \in \mathbb{S}_{++}^n$ and $h \in \mathbb{N}_+$, suppose that*

1) *$\tilde{w}_h < 0$ with the minimum of Problem 2 achieved as $\sum_{j=1}^k \alpha_j A_{\sigma^{h,j}}^T P A_{\sigma^{h,j}} - P \preceq \tilde{w}_h I_n$, where $k = |\mathcal{M}^h|$, $\alpha \in \Delta_k$, and $\{\sigma^{h,j}\}_{j=1}^k$ is an enumeration of $\mathcal{M}^h$;*
2) *At each time $k = ht$, $t \in \mathbb{N}$, the h-horizon stochastic switching sequence $\sigma^h(k) \in \mathcal{M}^h$ is drawn independently randomly from $\mathcal{M}^h$ with the probabilities $\mathbf{P}\{\sigma^h(k) = \sigma^{h,j}\} = \alpha_j$. The resulting infinite horizon stochastic switching policy is denoted by σ_α^∞.*

Then, under σ_α^∞, the SLS is MS exponentially stable.

Proof. Let $V(z) = z^T P z$. By assumptions, with $x(0) = z$, $\mathbf{E}[V(x(h))] = \sum_{j=1}^k \alpha_j V(A_{\sigma^{h,j}} z) \leq V(z) + \tilde{w}_h \|z\|^2 \leq \beta V(z)$, where $\beta := 1 + \tilde{w}_h/\lambda_{\min}(P) \in [0, 1)$. Repeating the same argument, we have $\mathbf{E}[V(x(th))] \leq \beta^t V(z) \leq \beta^t \lambda_{\max}(P) \|z\|^2$. To bound the state expectation at the times inside each period, we can use the same argument as in the proof of Theorem 1. This concludes the proof. □

Remark 4. *Connections between the Lyapunov-Metzler inequalities (see Remark 2) and the mean-square stability of Markov jump linear systems (MJLSs) were discussed in [10]. The SLS under the stochastic switching policy σ_α^∞ can be viewed as a special case of the MJLSs, and Theorem 4 establishes the condition $\tilde{w}_h < 0$ as a (conservative) sufficient condition for its open-loop MS stabilizability. Indeed, [39, Theorem 22] showed further that the SLS is open-loop stabilizable by deterministic periodic switching policies if $\tilde{w}_h < 0$.*

E. BMI Problem Formulation

In solving Problem 2 to test the stabilizability of the SLS, one can choose any fixed $P \in \mathbb{S}_{++}$, which makes the problem a convex one. A convenient choice is $P = I_n$ as it has the smallest condition number, hence the smallest complexity bound $\bar{h}(P, a, c)$ in Theorem 3. In practice, better P can be found by simultaneously solving for $P \in \mathbb{S}_{++}^n$ and $\alpha \in \mathbb{R}^k$ in Problem 2, resulting in the following (nonconvex) bilinear matrix inequality (BMI) problem.

Problem 3. *Solve the BMI optimization*

$$\begin{aligned} \widehat{w}_h := \min \;\; & w \in \mathbb{R} \\ \text{s.t.} \;\; & P' - P \preceq wI_n \;\text{ for some } P' \in \operatorname{conv}(\mathcal{P}_h(P)) \\ & P \in \mathbb{S}_{++}^n. \end{aligned}$$

The above BMI problem can be solved by, e.g., the path-following method [40], projection algorithm [1], or techniques such as BMI optimization [44] and polynomial optimization [45], [46]. In the following, we describe the key step for applying the path-following method for its solution.

Problem 4. *For given $h \in \mathbb{N}_+$, $\varepsilon \in \mathbb{R}_{++}$, $\alpha \in \mathbb{R}^k$, and $P \in \mathbb{S}_{++}^n$, solve the following optimization for $\Delta P \in \mathbb{S}^n$, $\Delta\alpha \in \mathbb{R}^k$, and $w \in \mathbb{R}$:*

$$\begin{aligned} (\Delta\alpha^*, \Delta P^*, w^*) := & \operatorname*{arg\,min}_{\Delta\alpha \in \mathbb{R}^k,\, \Delta P \in \mathbb{S}^n,\, w \in \mathbb{R}} \; w \\ \text{subject to} \;\; & \alpha + \Delta\alpha \in \Delta_k, \quad P + \Delta P \succ 0, \\ & ||\Delta P|| \leq \varepsilon, \quad ||\Delta\alpha|| \leq \varepsilon, \\ & \sum\nolimits_{j=1}^k \alpha_j F^{(j)} + \sum\nolimits_{j=1}^k \Delta\alpha_j F^{(j)} \\ & + \sum\nolimits_{j=1}^k \alpha_j \Delta F^{(j)} - P - \Delta P \preceq wI_n. \end{aligned}$$

Here, $\{F^{(j)}\}_{j=1}^k$ is an enumeration of $\mathcal{P}_h(P)$, and $\{\Delta F^{(j)}\}_{j=1}^k$ is an enumeration of $\mathcal{P}_h(\Delta P)$.

Note that Problem 4 is an LMI problem and can be solved using convex optimizations [47]. The overall path-following algorithm is briefly summarized in Algorithm 1.

Algorithm 1 Path-following algorithm for solving Problem 3.

```
1: Set h_max ∈ N_+.
2: for h = 1 to h_max do
3:     P ← I_n; set sufficiently small ε; initialize α ∈ Δ_k.
4:     repeat
5:         Solve Problem 4 for the given P and α.
6:         P ← P + ΔP*,   α ← α + Δα*
7:     until ||ΔP*|| and ||Δa*|| are sufficiently small
8:     if w* < 0 then
9:         Stop and return P, h and P_h.
10:    end if
11: end for
```

V. Comparison with Piecewise Quadratic Control Lyapunov Functions

In this section, the proposed PQCLF method is compared against the piecewise quadratic control Lyapunov function (PWQCLF) approach in [11], [21]. We will show that the PQCLF approach performs better (i.e., is less conservative) and establish the connections between the two approaches.

A. Piecewise Quadratic Control Lyapunov Functions

We review some notions in [21]. The (non-discounted) weak generating function $H : \mathbb{R}^n \to \mathbb{R}_+ \cup \{\infty\}$ of the SLS (1) is

$$H(z) := \inf_{\sigma^\infty} \sum_{k=0}^{\infty} \|x(k;\, z,\, \sigma^\infty)\|^2, \quad \forall z \in \mathbb{R}^n,$$

where the infimum is over all infinite-horizon switching sequences. It is proved in [21, Prop. 8] that the SLS (1) is stabilizable if and only if $H(\cdot) < \infty$, i.e., if and only if

$$\eta := \sup_{\|z\|=1} H(z) \in [1, \infty). \tag{13}$$

Assume (13) holds. Then, $H(\cdot)$ is a control Lyapunov function:

$$\|z\|^2 \le H(z) \le \eta \|z\|^2, \tag{14}$$
$$\min_{i\in\mathcal{M}} H(A_i z) - H(z) = -\|z\|^2, \quad \forall z \in \mathbb{R}^n. \tag{15}$$

For $h \in \mathbb{N}$, the h-truncated version of $H(\cdot)$ is defined as

$$H^h(z) := \min_{\sigma^h} \sum_{k=0}^{h} \left\|x(k;\, z,\, \sigma^h)\right\|^2, \quad \forall z \in \mathbb{R}^n.$$

$H^h(\cdot)$ can be obtained iteratively from the Bellman equation

$$H^h(z) = \|z\|^2 + \min_{i\in\mathcal{M}} H^{h-1}(A_i z), \quad \forall z \in \mathbb{R}^n, \tag{16}$$

with $H^0(z) = \|z\|^2$. Equivalently, $H^h(z) = \min_{H\in\mathcal{H}_h} z^T H z$ is a piecewise quadratic function, where $\mathcal{H}_h$ is a sequence of sets of positive definite matrices obtained from $\mathcal{H}_0 = \{I\}$ and

$$\mathcal{H}_h = \{I_n + A_i^T H A_i :\, H \in \mathcal{H}_{h-1},\, i \in \mathcal{M}\},\ h \in \mathbb{N}_+. \tag{17}$$

For $h \in \mathbb{N}_+$, define the quantity

$$\tau_h := \sup_{\|z\|=1} \left[\min_{i\in\mathcal{M}} H^{h-1}(A_i z) - H^{h-1}(z)\right]. \tag{18}$$

Then, $H^{h-1}(\cdot)$ is a control Lyapunov function (called a *piecewise quadratic control Lyapunov function*, or *PWQCLF)* of the SLS if and only if $\tau_h < 0$. As τ_h is difficult to compute, we introduce an overestimate of it. Define the set

$$\mathcal{G}_h := \{A_i^T H A_i \,:\, H \in \mathcal{H}_{h-1},\, i \in \mathcal{M}\}. \tag{19}$$

Let $\tilde{\tau}_h$ be the infimum of all $w \in \mathbb{R}$ satisfying

$$\forall H \in \mathcal{H}_{h-1},\ \exists \text{ some } S \in \text{conv}(\mathcal{G}_h) \text{ s.t. } S - H \preceq w I_n. \tag{20}$$

It is easy to see that that $\tilde{\tau}_h \ge \tau_h$ and that $\tilde{\tau}_h$ can be computed by solving a number of SDP problems, one for each $H \in \mathcal{H}_{h-1}$. This yields the following sufficient stabilizability test.

Lemma 1 ([11, Corollary 1])**.** *If $\tilde{\tau}_h < 0$, then $H^{h-1}(\cdot)$ is a PWQCLF and the SLS is stabilizable.*

Conversely, if the SLS is stabilizable, the following result implies that $H^{h-1}(\cdot)$ for large enough h is a PWQCLF.

Proposition 8. *Suppose the SLS is stabilizable. Then,*

$$\tau_h \le \mu_h := \eta^2(1-\eta^{-1})^h - 1, \quad \forall h \in \mathbb{N}_+, \tag{21}$$

where η is defined in (13).

Proof. By [21, Prop. 11], as $h \to \infty$, $H^{h-1}(\cdot)$ converges to $H(\cdot)$ non-decreasingly and uniformly exponentially fast on the unit sphere: $0 \le H(z) - H^{h-1}(z) \le \eta^2(1-\eta^{-1})^h\|z\|^2$, $\forall z$. For $h \in \mathbb{N}_+$, we then have

$$\begin{aligned}
&\min_{i\in\mathcal{M}} H^{h-1}(A_i z) - H^{h-1}(z) \le \min_{i\in\mathcal{M}} H(A_i z) - H^{h-1}(z)\\
&\quad = \min_{i\in\mathcal{M}} H(A_i z) - H(z) + \left[H(z) - H^{h-1}(z)\right]\\
&\quad \le [\eta^2(1-\eta^{-1})^h - 1]\,\|z\|^2 .
\end{aligned}$$

Here, (15) is used in the last step. □

Note that for h large enough, $\mu_h < 0$; hence $\tau_h < 0$.

B. Comparison of PQCLF and PWQCLF Methods

The following result states that the use of the PQCLF is no more conservative than that of the PWQCLF.

Proposition 9. *Let $h \in \mathbb{N}_+$ and consider $P = I_n$ and $V(z) = z^T P z = \|z\|^2$. Then, the function $V_h(\cdot)$ defined in (6) satisfies*

$$V_h(z) - V(z) \le \min_{i\in\mathcal{M}} H^{h-1}(A_i z) - H^{h-1}(z). \tag{22}$$

That is, $w_h \le \tau_h$. As a consequence, if $H^{h-1}(z)$ is a PWQCLF, then $V(z)$ is an h-PQCLF.

Proof. Using the Bellman equation (16), we have

$$\begin{aligned}
&V_h(z) - V(z) = V_h(z) + \min_{i\in\mathcal{M}} H^{h-1}(A_i z) - H^h(z)\\
&= \min_{i\in\mathcal{M}} H^{h-1}(A_i z) - H^{h-1}(z) + V_h(z) + H^{h-1}(z) - H^h(z).
\end{aligned}$$

The desired conclusion follows since $V_h(z) + H^{h-1}(z) = \inf_{\sigma^h} \|x(h; z, \sigma^h)\|^2 + \inf_{\sigma^h} \sum_{k=0}^{h-1} \|x(k; z, \sigma^h)\|^2$ is no larger than $H^h(z) = \inf_{\sigma^h} \sum_{k=0}^{h} \|x(k; z, \sigma^h)\|^2$. □

For stabilizable SLS, $\lim_{h\to\infty} V_h(\cdot) = 0$ by Proposition 2. The convergence rate can be estimated as follows.

Lemma 2. *Suppose the SLS is stabilizable and $V(z) = z^T P z$ for some $P \in \mathbb{S}^n_{++}$. Then,*

$$V_h(z) \le \lambda_{\max}(P)\eta(1-\eta^{-1})^h \|z\|^2, \quad \forall z \in \mathbb{R}^n,\ h \in \mathbb{N}_+.$$

Proof. The proof follows the same line as the proof of [21, Prop. 11]. The Bellman equation (16) together with $H^h(z)/\eta \le \|z\|^2$ yields $\min_{i\in\mathcal{M}} H^{h-1}(A_i z) \le (1-\eta^{-1})H^h(z)$. Applying this step repeatedly, we have

$$\begin{aligned}
V_h(z) &= \min_{\sigma^h\in\mathcal{M}^h} V(A_{\sigma^h} z)\\
&\le \min_{\sigma^h\in\mathcal{M}^h} H^0(A_{\sigma^h} z) \cdot \lambda_{\max}(P)\\
&\le \left(1-\eta^{-1}\right) \min_{\sigma^{h-1}\in\mathcal{M}^{h-1}} H^1(A_{\sigma^{h-1}} z) \cdot \lambda_{\max}(P)\\
&\le \cdots \le \left(1-\eta^{-1}\right)^h H^h(z) \cdot \lambda_{\max}(P)\\
&\le \eta \left(1-\eta^{-1}\right)^h \|z\|^2 \lambda_{\max}(P).
\end{aligned}$$

This concludes the proof. □

Using Lemma 2, we can prove a version of the converse PQCLF theorem different from Theorem 3.

Theorem 5 (Converse PQCLF Theorem II)**.** *Suppose the SLS is stabilizable and $P = I_n$, i.e., $V(z) = \|z\|^2$. Then, $V_h(z) - V(z) \leq \nu_h \|z\|^2, \forall z \in \mathbb{R}^n$, where*

$$\nu_h := \eta(1-\eta^{-1})^h - 1, \quad \forall h \in \mathbb{N}_+. \tag{23}$$

Thus, $V(\cdot)$ is an h-PQCLF if $h \geq \bar{h}(\eta) := \left\lceil \frac{\ln \eta}{\ln \eta - \ln(\eta-1)} \right\rceil$.

Proof. The conclusions readily follow from Lemma 2. □

Both μ_h defined in (21) and ν_n defined in (23) are upper bounds of w_h defined in (9). For stabilizable SLS, since $\eta \geq 1$, we have $\mu_h \geq \nu_h$; hence ν_h is a tighter upper bound of w_h. The next result shows that the condition $\tilde{w}_h < 0$ in Corollary 2 is no more conservative than the condition $\tilde{\tau}_h < 0$ in Lemma 1 under a certain condition.

Proposition 10. *For each $H^{(i)} \in \mathcal{H}_{h-1}$, $i = 1, \ldots, N^{h-1}$, let $\alpha^{(i)} \in \Delta_{N^h}$ be such that it achieves the minimum in (20), and define the column stochastic matrix (nonnegative matrix with each column adding up to one) $B := \begin{bmatrix}\alpha^{(1)}\mathbf{1}_N^T & \cdots & \alpha^{(k)}\mathbf{1}_N^T\end{bmatrix} \in \mathbb{R}^{N^h \times N^h}$, where $\mathbf{1}_N \in \mathbb{R}^N$ is the vector whose entries are ones. Suppose that there exists a stationary distribution $v \in \Delta_{N^{h-1}}$, the N^{h-1}-simplex, such that $Bv = v$. With $P = I_n$, $\tilde{w}_h \leq \tilde{\tau}_h$ for all $h \in \mathbb{N}_+$.*

Proof. See Appendix A. □

Remark 5. *In practice, by perturbing the optimal solution w to (20) slightly, α hence B can be assumed to be positive (all entries being positive). By a standard result of Markov chain, the stationary distribution v in Proposition 10 is guaranteed to exist.*

VI. Complexity Reduction via Relaxation

Although Theorem 3 shows that a PQCLF can always be found when the SLS is stabilizable, the required period h could be large. As h increases, the size of the set $\mathcal{P}_h$ computed via the iteration (10) grows exponentially fast. To reduce computational complexity, the relaxation method suggested in [11]–[13] can be adopted. For any set $\mathcal{F} \subset \mathbb{S}_+^n$, a subset $\mathcal{F}^\varepsilon \subseteq \mathcal{F}$ is called ε-equivalent to $\mathcal{F}$ for some $\varepsilon > 0$ if

$$\min_{H \in \mathcal{F}} z^T H z \leq \min_{H \in \mathcal{F}^\varepsilon} z^T H z \leq \min_{H \in \mathcal{F}} z^T H z + \varepsilon \|z\|^2, \forall z \in \mathbb{R}^2.$$

Each $H \in \mathcal{F}$ is called ε-redundant if $\mathcal{F} \setminus \{H\}$ is ε-equivalent to $\mathcal{F}$. A sufficient condition for H to be ε-redundant is given by the convex condition $H \succeq P - \varepsilon I_n$ where P is some convex combination of the matrices in $\mathcal{F} \setminus \{H\}$. By repeatedly removing ε-redundant matrices from $\mathcal{F}$, one can obtain ε-equivalent subsets of $\mathcal{F}$.

Now modify the iteration (10) as follows: $\mathcal{P}_0^\varepsilon := \{P\}$, and for $k = 1, 2, \ldots$, let $\mathcal{P}_k^\varepsilon$ be an ε-equivalent subset of $\{A_i^T S A_i : S \in \mathcal{P}_{k-1}^\varepsilon, i \in \mathcal{M}\}$. Define the ε-*relaxed value function*

$$V_k^\varepsilon(z) := \min_{F \in \mathcal{P}_k^\varepsilon} z^T F z.$$

Clearly, $V_k(z) \leq V_k^\varepsilon(z)$. The next result provides an upper bound of $V_k^\varepsilon(z)$ using the weak generating function $H(z)$.

Proposition 11. *For $k \in \{1, 2, \ldots\}$ and $z \in \mathbb{R}^n$,*

$$V_k(z) \leq V_k^\varepsilon(z) \leq (1+\varepsilon)H^k(z) - H^{k-1}(z) - \varepsilon V_k(z). \tag{24}$$

Proof. The proof follows a similar line to that of [21, Prop. 12]. Obviously, (24) holds for $k = 1$ as $V_1^\varepsilon(z) \leq V_1(z) + \varepsilon||z||^2 = (1+\varepsilon)[V_1(z) + ||z||^2] - ||z||^2 - \varepsilon V_1(z) = (1+\varepsilon)H^1(z) - H^0(z) - \varepsilon V_1(z)$, $\forall z$. Suppose it holds for the $k-1$ case, i.e., $V_{k-1}^\varepsilon(z) \leq (1+\varepsilon)H^{k-1}(z) - H^{k-2}(z) - \varepsilon V_{k-1}(z)$, $\forall z$. Then, $\forall z \in \mathbb{R}^n$,

$$\begin{aligned}
&\tilde{V}_k^\varepsilon(z) := \min_{i \in \mathcal{M}} V_{k-1}^\varepsilon(A_i z) \\
&\leq \min_{i \in \mathcal{M}} [(1+\varepsilon)H^{k-1}(A_i z) - H^{k-2}(A_i z) - \varepsilon V_{k-1}(A_i z)] \\
&\leq (1+\varepsilon)\min_{i \in \mathcal{M}} H^{k-1}(A_i z) - \min_{i \in \mathcal{M}} H^{k-2}(A_i z) - \varepsilon V_k(z).
\end{aligned}$$

By the construction of $\mathcal{P}_k^\varepsilon$, we have

$$\begin{aligned}
V_k^\varepsilon(z) &\leq \tilde{V}_k^\varepsilon(z) + \varepsilon||z||^2 \\
&\leq (1+\varepsilon)\min_{i \in \mathcal{M}} \left[H^{k-1}(A_i z) + \|z\|^2\right] \\
&\quad - \min_{i \in \mathcal{M}} \left[H^{k-2}(A_i z) + \|z\|^2\right] - \varepsilon V_k(z) \\
&= (1+\varepsilon)H^k(z) - H^{k-1}(z) - \varepsilon V_k(z)
\end{aligned}$$

for all $z \in \mathbb{R}^n$. This completes the proof. □

To proceed, define

$$w_{h,\varepsilon} := \sup_{\|z\|=1} [V_h^\varepsilon(z) - V(z)]. \tag{25}$$

Then, relaxed versions of the Lyapunov theorem and its converse theorem can be established as follows.

Theorem 6 (ε-Relaxed (Converse) PQCLF theorems)**.** *Let $P \in \mathbb{S}_{++}^n$ and $h \in \mathbb{N}_+$ be given. Define $V(z) = z^T P z, \forall z$.*

1) *If $w_{h,\varepsilon} < 0$, then $V(\cdot)$ is an h-PQCLF and the SLS (1) is stabilizable.*
2) *Suppose that the SLS (1) is stabilizable and $\varepsilon > 0$ is small enough so that $\varepsilon\eta - \lambda_{\min}(P) < 0$, where η is defined in (13). Then, $V(\cdot)$ is an h-PQCLF satisfying $w_{h,\varepsilon} < 0$ for all $h \geq \bar{h}(\varepsilon, \eta, P)$, where*

$$\bar{h}(\varepsilon, \eta, P) := \left\lceil \frac{\ln(\eta^2) - \ln(\lambda_{\min}(P) - \varepsilon\eta)}{\ln(1/(1-\eta^{-1}))} \right\rceil.$$

Proof. 1) If $w_{h,\varepsilon} < 0$, then $V_h^\varepsilon(z) < V(z)$, hence $V_h(z) < V(z)$, for all $z \neq 0$. By Theorem 2, the SLS is stabilizable.

2) By Proposition 11, we have

$$\begin{aligned}
&V_h^\varepsilon(z) - V(z) \\
&\leq (1+\varepsilon)H^h(z) - H^{h-1}(z) - \varepsilon V_h(z) - V(z) \\
&\leq H^h(z) + \varepsilon H^h(z) - H^{h-1}(z) - \lambda_{\min}(P)||z||^2 \\
&\leq \min_{i \in \mathcal{M}} H^{h-1}(A_i z) - H^{h-1}(z) \\
&\qquad + \varepsilon H^h(z) + (1 - \lambda_{\min}(P))||z||^2 \\
&\leq \mu_h||z||^2 + \varepsilon H^h(z) + (1 - \lambda_{\min}(P))||z||^2 \\
&\leq [\mu_h + \varepsilon\eta + 1 - \lambda_{\min}(P)]\,||z||^2 \\
&= \left[\eta^2(1-\eta^{-1})^h + \varepsilon\eta - \lambda_{\min}(P)\right]||z||^2,
\end{aligned}$$

where the Bellman equation (16) is used in the third inequality and Proposition 8 is used in the forth inequality. It is easily verified that the right-hand side of the above inequality is negative for $z \neq 0$ if $h > \bar{h}(\varepsilon, \eta, P)$. □

Similar to $\tilde{w}_h$ in Problem 2 being an overestimate of w_h, an overestimate of $w_{h,\varepsilon}$ can be obtained by solving the SDP

$$\tilde{w}_{h,\varepsilon} := \min\{w \in \mathbb{R} : \exists P' \in \operatorname{conv}(\mathcal{P}_h^\varepsilon) \text{ s.t. } P' - P \preceq wI_n\}.$$

Since $\tilde{w}_{h,\varepsilon} \geq w_{h,\varepsilon} \geq w_h$, if $\tilde{w}_{h,\varepsilon} < 0$, then $V(z) = z^T P z$ is an h-PQCLF of the SLS.

To generate a stabilizing switching policy based on $V_k^\varepsilon(z)$, one can simply replace $\mathcal{P}_h$ in Proposition 6 with $\mathcal{P}_h^\varepsilon$.

Definition 6 (ε-Relaxed h-SP)**.** *For $z \in \mathbb{R}^n$ and $t \in \{1, 2, \ldots, h\}$, define*

$$i_t^*(z) := \arg\min_{i\in\mathcal{M}} \min_{S\in\mathcal{P}_{t-1}^\varepsilon} z^T A_i^T S A_i z. \tag{26}$$

Then the ε-relaxed h-SP is defined by

$$\sigma_h(z) = (i_h^*(z),\, i_{h-1}^*(x(1)), \ldots, i_1^*(x(h-1))), \tag{27}$$

where $(x(0), x(1), \ldots, x(h-1))$ is the state trajectory driven by the switching policy (26).

Theorem 7. *Let $P \in \mathbb{S}_{++}^n$ and $h \in \mathbb{N}_+$ be given. If $w_{h,\varepsilon} < 0$, then the SLS (1) under the h-PSP $\hat{\sigma}_\infty^h(\cdot)$ obtained by concatenating the ε-relaxed h-SP (27) is exponentially stable.*

Proof. Let $\hat{\sigma}_h(z)$ be the ε-relaxed h-SP (27). By the definition of $w_{h,\varepsilon}$, we have, for any $z \in \mathbb{R}^n$,

$$\begin{aligned}
&w_{h,\varepsilon}\|z\|^2 \geq \min_{H\in\mathcal{P}_h^\varepsilon} z^T H z - V(z)\\
&\geq \min_{H\in\mathcal{P}_{h-1}^\varepsilon,\, i\in\mathcal{M}} z^T A_i^T H A_i z - V(z)\\
&= \min_{H\in\mathcal{P}_{h-1}^\varepsilon} x(1;\, z,\, \hat{\sigma}_h(z))^T H x(1;\, z,\, \hat{\sigma}_h(z)) - V(z)\\
&\geq \min_{H\in\mathcal{P}_{h-2}^\varepsilon,\, i\in\mathcal{M}} x(1;\, z,\, \hat{\sigma}_h(z))^T A_i^T H A_i x(1;\, z,\, \hat{\sigma}_h(z))\\
&- V(z) \geq \cdots \geq V(x(h;\, z,\, \hat{\sigma}_h(z))) - V(z).
\end{aligned}$$

Thus, $w_{h,\varepsilon} < 0$ implies $V(x(h;\, z,\, \hat{\sigma}_h(z))) - V(z) \leq -|\omega_{h,\varepsilon}| \cdot \|z\|^2$. The conclusion then follows from Theorem 1. ☐

Remark 6. *Results in the previous section can be extended to the stabilization of the SLSs $x(k+1) = A_{\sigma_k} x(k) + B_{\sigma_k} u_k$ controlled by the switching sequence σ_k and continuous control input u_k. By using a mode-dependent linear state-feedback policy $u_k = F_{\sigma_k} x(k)$ with the gain matrices $\{F_i\}_{i\in\mathcal{M}}$, the closed-loop system becomes the autonomous SLS with subsystems matrices $\{A_i + B_i F_i\}_{i\in\mathcal{M}}$. In this case, the following nonlinear matrix inequality feasibility problem similar to Problem 3 can be formulated to find a stabilizing controller: find $\{F_i \in \mathbb{R}^{m\times n}\}_{i=1}^N$, $P \in \mathbb{S}^n$, and $\{\alpha_{\sigma_h}\}_{\sigma_h\in\mathcal{M}^h}$ such that*

$$\begin{aligned}
&\sum_{\sigma_h\in\mathcal{M}^h} \alpha_{\sigma_h} \tilde{A}_{\sigma_h}^T P \tilde{A}_{\sigma_h} - P \prec 0,\\
&P \succ 0, \quad \sum_{\sigma_h\in\mathcal{M}^h} \alpha_{\sigma_h} = 1, \quad \alpha_{\sigma_h} > 0, \quad \forall \sigma_h \in \mathcal{M}^h,
\end{aligned}$$

where $\tilde{A}_i := A_i + B_i F_i$. By repeatedly applying the extended Schur complement [48, Theorem 1] to the above inequality, a BMI feasibility problem can be obtained. The condition may be generally conservative, and the size of the BMI grows exponentially fast as h increases. Another approach [49] is to use the dynamic programming strategy as in [11]–[13].

VII. Examples

We now present some numerical examples. All examples were solved by MATLAB R2008a running on a Windows 7 PC with Intel Core i5-4210M 2.6GHz CPU, 4 GB RAM.

Example 1. *Consider the SLS (1) with*

$$A_1 = \begin{bmatrix} 0.9995 & 0.0656 \\ 0.1312 & 0.4089 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.9788 & 0.1514 \\ -0.3030 & 2.1905 \end{bmatrix}. \tag{28}$$

The eigenvalues are $\lambda = 1.0137, 0.3947$ for A_1 and $\lambda = 1.0179, 2.1514$ for A_2. By solving Problem 2 with $P = I_n$, the solution $\tilde{w}_h \geq 0$ for $h \in \{1, \ldots, 7\}$, while $\tilde{w}_h < 0$ for $h = 8$. Thus, an 8-PQCLF exists. Using this PQCLF and the h-PSP in Proposition 6, the state trajectory of the SLS starting from $x(0) = [0.7271,\, 0.3093]^T$ and the time history of the PQCLF along this state trajectory are plotted in Fig. 1 (a). Moreover, with $\varepsilon = 0.1$ and $P = I_n$, $\tilde{w}_{h,\varepsilon}$ defined in (25) is computed. It is found that $\tilde{w}_{h,\varepsilon} < 0$ is infeasible for $h \in \{1, \ldots, 7\}$, while feasible for $h = 8$. The comparison of $|\mathcal{P}_h(I_n)|$ and $|\mathcal{P}_h^\varepsilon(I_n)|$ for $h \in \{1, \ldots, 8\}$ is given in Table I below. As can be seen, a modest relaxation ($\varepsilon = 0.1$) can significantly reduce the complexity growth. The simulation result of the SLS under the ε-relaxed h-SP in Definition 6 is depicted in Fig. 1 (b). For this example, a comparison of $\tilde{w}_h$ in Problem 2 and $\tilde{\tau}_h$ defined right after (19) is shown in Fig. 2 for $h \in \{2, 3, \ldots, 8\}$. The results verify the conclusion of Proposition 10.

TABLE I
Example 1. Comparison of $|\mathcal{P}_h(I_n)|$ and $|\mathcal{P}_h^\varepsilon(I_n)|$.

h	$\lvert\mathcal{P}_h(I_n)\rvert$	$\lvert\mathcal{P}_h^\varepsilon(I_n)\rvert$
1	2	1
2	4	2
3	8	3
4	16	4
5	32	6
6	64	8
7	128	11
8	256	15

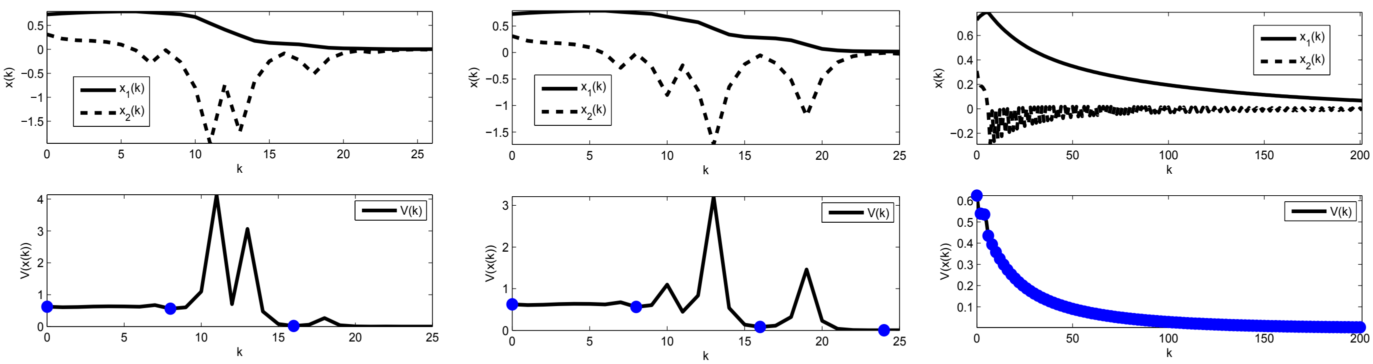
On the other hand, if Algorithm 1 (BMI approach) is applied with $h = 2$, then after two iterations and a computation time of 10.3 seconds, a feasible solution is found as $P = \begin{bmatrix} 0.7140 & 0.2756 \\ 0.2756 & 1.2841 \end{bmatrix}$, whose corresponding $\widehat{w}_h < 0$ in Problem 3. Using this PQCLF and the 2-PSP in Proposition 6, the state trajectory of the SLS with and the corresponding time history of the PQCLF are plotted in Fig. 1 (c).

Lastly, starting from the same initial state as before, the state trajectory of the SLS under the random switching policy in Theorem 4 and the corresponding time history of the PQCLF along the state trajectory are illustrated in Fig. 3.

Remark 7. *A drawback of the PQCLF approach is that the stabilized state trajectory may still exhibit large fluctuations inside each period, as can be observed in Figs. 1. This is also the case for the minimum dwell-time control problem in [36].*

Example 2. *Let us consider another SLS with*

$$A_1 = \begin{bmatrix} 1.5506 & 0.0566 & -1.0985 & -0.1757 \\ -0.3396 & 0.4109 & -0.2967 & 0.1151 \\ -0.3447 & 0.4109 & 1.1601 & -0.0102 \\ 0.6737 & 0.2137 & -0.1765 & 0.3472 \end{bmatrix},$$



(a) Using the h-PSP (b) Using the ε-relaxed h-PSP (c) Using the 2-PSP from Algorithm 1

Fig. 1. Example 1 using three different stabilizing switching policies. In each case, the upper subplot is the trajectory of the state variable $x(k)$; the lower subplot is the time history of the PQCLF, with the solid dots representing the values of the PQCLF at time instants $0,\ h,\ 2h, \ldots$.

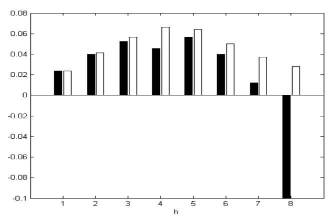


Fig. 2. Example 1. Comparison of $\tilde{w}_h$ (black bar) and $\tilde{\tau}_h$ (white bar).

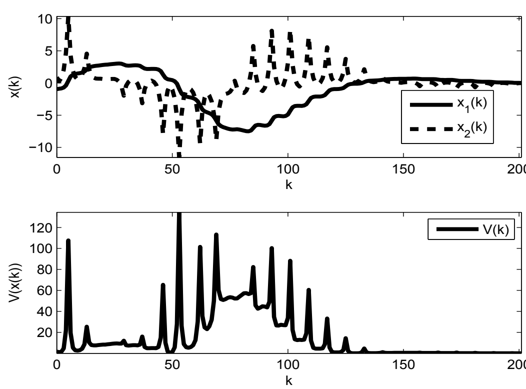


Fig. 3. Example 1 under the random switching policy in Theorem 4. Upper subplot: State trajectory. Lower subplot: Time histories of the PQCLF.

$$A_2 = \begin{bmatrix} 0.3445 & 0.1366 & 0.0016 & 0.0261 \\ 0.0468 & 0.5284 & -0.2634 & -0.6819 \\ -0.3462 & -0.5099 & 0.5676 & 0.0361 \\ -0.0052 & 0.3774 & -0.5784 & 1.2199 \end{bmatrix},$$

both of which are unstable. In this case, with $P = I_n$, $\tilde{w}_h < 0$ is infeasible for $h \in \{1, \ldots, 7\}$ and feasible for $h = 8$. Thus, the SLS is stabilizable using the 8-PQCLF $V(z) = \|z\|^2$. If Algorithm 1 is applied, after 7 iterations and a computation of 32.3270 seconds, a feasible solution is found as $h = 3$ and

$$P = \begin{bmatrix} 0.6671 & 0.0845 & -0.0466 & -0.2303 \\ 0.0845 & 1.3138 & -0.1240 & -0.3137 \\ -0.0466 & -0.1240 & 0.6747 & 0.4218 \\ -0.2303 & -0.3137 & 0.4218 & 0.9720 \end{bmatrix},$$

which satisfies $\widehat{w}_h < 0$ in Problem 3.

Example 3 (Inverted Pendulum). *Consider the linearized inverted pendulum system*

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{(M+m)g}{Ml} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-1}{Ml} \end{bmatrix} u(t),$$

where $x_1(t)$ and $x_2(t)$ denote the angle of the pendulum from the vertical and its angular velocity, $g = 9.8m/s^2$ is the gravity constant, m (resp. M) is the mass of the pendulum (resp. the cart), l is the length of the pendulum, and $u(t)$ is the force applied to the cart. Assume that $x_1(t)$ can be measurable at all times t while $x_2(t)$ is measured at the time instants $t = kT$, $k \in \mathbb{N}$, for some $T > 0$.

A feedback controller $u(t) = Fx_1(t)$ with $F \in \mathbb{R}$ cannot stabilize the system. Consider now the switching control policy

$$u(t) = \begin{cases} Fx_1(t), & \text{if } \sigma(x(kT)) = 1 \\ -Fx_1(t), & \text{if } \sigma(x(kT)) = 2 \end{cases}, \ \forall t \in [kT, (k+1)T),$$

for $k \in \mathbb{N}$. Here, $\sigma(x(t))$ is a state-feedback switching policy that only switches at the time instants $t = kT$. Define $G_1 = \begin{bmatrix} 0 & 1 \\ \frac{(M+m)g}{Ml} - \frac{F}{Ml} & 0 \end{bmatrix}$, $G_2 = \begin{bmatrix} 0 & 1 \\ \frac{(M+m)g}{Ml} + \frac{F}{Ml} & 0 \end{bmatrix}$. Then the sampled state $x(k) = x(kT)$ follows a discrete-time SLS with the subsystem dynamics $A_1 := \exp(G_1 T)$, $A_2 := \exp(G_2 T)$. Suppose the parameters are given by $M = 1\,\mathrm{kg}$, $m = 10\,\mathrm{kg}$, $l = 10\,\mathrm{m}$, $T = 0.1\,\mathrm{s}$, and $F = (M+m)g + 10$. It is found that, with $P = I_n$, $\tilde{w}_h < 0$ is infeasible for $h \in \{1, 2\}$, while feasible for $h = 3$. With the initial state $z = [0, -0.5]^T$, the state trajectory of the SLS under the h-PSP (8) with $P = I_n$ and $h = 3$ is plotted in Fig. 4.

Conclusion

The PQCLF framework is studied for the exponential stabilization of SLSs. Algorithms to search for the PQCLFs have been developed. Compared to some competing methods, the proposed algorithms produce no less conservative results in theory and favorable results in numerical experiments.

Appendix A
Proof of Proposition 10

The set $\mathcal{H}_{h-1}$ defined iteratively by (17) has the cardinality $k = N^{h-1}$ and an enumration $\{H^{(i)}\}_{i=1}^{k}$. Let

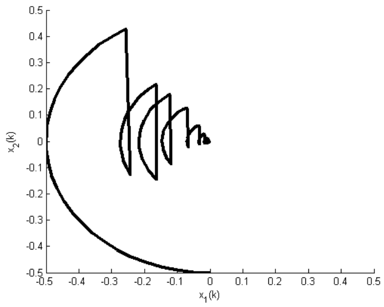


Fig. 4. State trajectory of Example 3 under the h-PSP ($h = 3$).

$\{G^{(i)}\}_{i=1}^{N^h}$ be an enumeration of $\mathcal{G}_h$ defined in (19). Then $\tilde{\tau}_h = \max_{H\in\mathcal{H}_{h-1}} w(H)$, where

$$w(H) = \min\left\{ w \ : \exists \alpha \in \Delta_{N^h} \text{ s.t. } \sum_{j=1}^{N^h} \alpha_j G^{(j)} - H \preceq w I_n \right\}.$$

For each $H^{(i)} \in \mathcal{H}_{h-1}$, $i = 1, \dots, k$, let $\alpha^{(i)} \in \Delta_{N^h}$ be such that it achieves the minimum in $w(H^{(i)})$:

$$\sum_{j=1}^{N^h} \alpha_j^{(i)} G^{(j)} - H^{(i)} \preceq w(H^{(i)}) I_n. \tag{29}$$

Next, note that each $G^{(j)} \in \mathcal{G}_h$ can be decomposed into $G^{(j)} = F^{(j)} + R^{(j)} - I_n$, where $F^{(j)} \in \mathcal{P}_h$ and $R^{(j)} \in \mathcal{H}_{h-1}$. Therefore, $\sum_{j=1}^{N^h} \alpha_j^{(i)} G^{(j)} = \sum_{j=1}^{N^h} \alpha_j^{(i)} F^{(j)} + \sum_{j=1}^{N^h} \alpha_j^{(i)} R^{(j)} - I_n$, or equivalently,

$$(\alpha^{(i)} \otimes I_n)^T \bar{G} = (\alpha^{(i)} \otimes I_n)^T \bar{F} + (\alpha^{(i)} \otimes I_n)^T \bar{R} - I_n, \tag{30}$$

where $\otimes$ stands for the Kronecker's product and

$$\bar{G} := \begin{bmatrix} G^{(1)} \\ \vdots \\ G^{(N^h)} \end{bmatrix}, \bar{F} := \begin{bmatrix} F^{(1)} \\ \vdots \\ F^{(N^h)} \end{bmatrix}, \bar{R} := \begin{bmatrix} R^{(1)} \\ \vdots \\ R^{(N^h)} \end{bmatrix}.$$

Without loss of generality, we can rearrange the order of the matrices $G^{(j)}$, $F^{(j)}$, $R^{(j)}$ in $\bar{G}$, $\bar{F}$, $\bar{R}$, respectively, so that

$$\bar{R} = \begin{bmatrix} R^{(1)} \\ \vdots \\ R^{(N^h)} \end{bmatrix} = \begin{bmatrix} \mathbf{1}_N \otimes H^{(1)} \\ \vdots \\ \mathbf{1}_N \otimes H^{(k)} \end{bmatrix}, \tag{31}$$

where $\mathbf{1}_N \in \mathbb{R}^N$ is the vector whose entries are ones. Similarly, one can represent (29) as $(\alpha^{(i)} \otimes I_n)^T \bar{G} - H^{(i)} \preceq w(H^{(i)}) I_n$, $\forall i \in \{1, \dots, k\}$, or equivalently,

$$\begin{bmatrix} \mathbf{1}_N \otimes [(\alpha^{(1)} \otimes I_n)^T \bar{G} - H^{(1)}] \\ \vdots \\ \mathbf{1}_N \otimes [(\alpha^{(k)} \otimes I_n)^T \bar{G} - H^{(k)}] \end{bmatrix} \preceq \begin{bmatrix} \mathbf{1}_N \otimes w(H^{(1)}) I_n \\ \vdots \\ \mathbf{1}_N \otimes w(H^{(k)}) I_n \end{bmatrix}. \tag{32}$$

Note that in the above inequality, "$\preceq$" should be interpreted as n-by-n block-wise comparison. Next, define the nonnegative column stochastic matrix $B := \begin{bmatrix} \alpha^{(1)} \mathbf{1}_N^T & \cdots & \alpha^{(k)} \mathbf{1}_N^T \end{bmatrix} \in \mathbb{R}^{N^h \times N^h}$. By assumption, there is a vector $\begin{bmatrix} v_1 & \cdots & v_{N^h} \end{bmatrix} \in \Delta_{N^h}$ such that $Bv = v$. Since $v \in \Delta_{N^h}$, (32) implies that

$$\begin{aligned} &(v \otimes I_n)^T \begin{bmatrix} \mathbf{1}_N \otimes [(\alpha^{(1)} \otimes I_n)^T \bar{G} - H^{(1)}] \\ \vdots \\ \mathbf{1}_N \otimes [(\alpha^{(k)} \otimes I_n)^T \bar{G} - H^{(k)}] \end{bmatrix} \\ &\preceq (v \otimes I_n)^T \begin{bmatrix} \mathbf{1}_N \otimes w(H^{(1)}) I_n \\ \vdots \\ \mathbf{1}_N \otimes w(H^{(k)}) I_n \end{bmatrix}. \end{aligned} \tag{33}$$

By plugging (30) into the above inequality, the left-hand side can be simplified to

$$\begin{aligned} &\bar{F}^T (Bv \otimes I_n) + \bar{R}^T (Bv \otimes I_n) - I_n - \bar{R}^T (v \otimes I_n) \\ &= \bar{F}^T (v \otimes I_n) + \bar{R}^T (v \otimes I_n) - I_n - \bar{R}^T (v \otimes I_n) \\ &= \bar{F}^T (v \otimes I_n) - I_n = \sum_{j=1}^{N^h} v_j F^{(j)} - I_n, \end{aligned}$$

where $Bv = v$ is used in the first equality. Moreover, noting the definition of $\bar{R}$ in (31), the right-hand side of (33) is $\sum_{j=1}^{N^h} v_j w(R^{(j)}) I_n$. Thus, it follows from (33) that $\sum_{j=1}^{N^h} v_j F^{(j)} - I_n \preceq \sum_{j=1}^{N^h} v_j w(R^{(j)}) I_n \preceq \tilde{\tau}_h I_n$. This in turn implies the desired conclusion $\tilde{w}_h \le \tilde{\tau}_h$.

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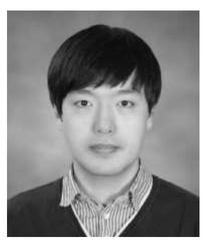

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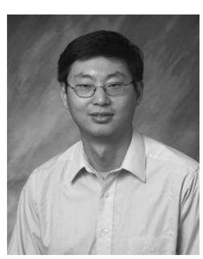

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