On the Optimal Solutions of the Infinite-Horizon Linear Sensor Scheduling Problem

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***Abstract*—This paper studies the infinite-horizon sensor scheduling problem for linear Gaussian processes with linear measurement functions. Several important properties of the optimal infinite-horizon schedules are derived. In particular, it is proved that under some mild conditions, both the optimal infinite-horizon average-per-stage cost and the corresponding optimal sensor schedules are independent of the covariance matrix of the initial state. It is also proved that the optimal estimation cost can be approximated arbitrarily closely by a periodic schedule with a finite period. Moreover, it is shown that the sequence of the average-per-stage costs of the optimal schedule must converge. These theoretical results provide valuable insights into the design and analysis of various infinite-horizon sensor scheduling algorithms.**

I. Introduction

The sensor scheduling problem seeks an optimal schedule over a certain time horizon to activate/deactivate a subset of available sensors to improve the estimation performance and reduce the estimation cost (e.g. energy consumption and communication overheads). It has numerous applications in various engineering fields [8], [10], [11].

Previous research has mainly focused on the finite-horizon sensor scheduling problem for linear Gaussian processes. In this case, a straightforward solution is to enumerate all the possible finite-horizon schedules [10]. The complexity of such an approach grows exponentially fast as the horizon size increases. Various methods have been proposed in the literature to tackle this challenge. These methods can be roughly divided into the following three categories: (i) methods that focus on certain simple special classes of schedules, such as myopic schedules that only consider immediate performance at each time step instead of the overall performance over the whole horizon [9], [13]; (ii) methods that "embed" discrete schedules into a larger class of schedules with continuously-variable sensor indices [4], [12]; (iii) and methods that prune the search tree based on certain properties of the Riccati recursions [3], [15].

The methods in the first category are often easy to implement, but provide no guarantees on the overall estimation performance. The "embedding" approach in the second category is a common trick to tackle complex discrete optimization or optimal control problems [1], [6]. The resulting relaxed schedule can often be interpreted as the time-average "frequencies" or "probabilities" for using different sensors. It has been recently proved that, in continuous time, the performance of the optimal relaxed schedule can be approximated with arbitrary accuracy by a discrete schedule through fast switchings [12]. This is analogous to the result derived in [1] for solving the optimal control problem of switched systems using embedding. However, in discrete time, the result no longer holds as the switching rate is fixed; in this case, the relaxed schedule can only be implemented probabilistically [4], resulting in a random scheduling of the sensors. The pruning methods in the third category make essential use of the monotonicity and concavity properties of the Riccati mapping to obtain conditions under which the exploration of certain branches can be avoided without losing the optimal schedule. In our earlier paper [15], an efficient sub-optimal algorithm was proposed to prune out not only the non-optimal branches but also less important ones to further reduce the complexity. Further error bounds associated with this pruning algorithm have also been derived in [16].

Different from most previous research, this paper studies the infinite-horizon sensor scheduling problem for discrete-time linear Gaussian processes observed by linear sensors. The problem is much more challenging than its finite-horizon counterpart and has not been adequately investigated in the literature. Instead of proposing a specific scheduling algorithm, we focus on deriving several properties of the problem which are of fundamental importance for the design and analysis of various infinite-horizon sensor scheduling algorithms. In particular, it is proved that under some mild conditions, both the optimal infinite-horizon average-per-stage cost and the corresponding optimal sensor schedule are independent of the covariance matrix of the initial state. It is also proved that the optimal estimation cost can be approximated arbitrarily closely by a periodic schedule with a finite period. Furthermore, it is concluded that the sequence of the average-per-stage costs of the optimal schedule must converge. These theoretical properties provide us valuable insight into the infinite-horizon sensor scheduling problem and will be useful for developing algorithms. In addition, the existence of a periodic suboptimal schedule justifies the experimental results of many finite-horizon scheduling algorithms [5], [15] that yield periodic schedules for relatively large horizons.

The rest of the paper is organized as follows. The infinite-horizon sensor scheduling problem is formulated in Section II. Some important properties of the difference Riccati recursion are reviewed in Section III. These properties are then used in Section IV to prove the universal approximation property of the periodic schedule. Finally, some concluding remarks are given in Section V.

Notation: Let $\mathcal{A}$ be the semi-definite cone, namely, the set of all the positive semidefinite matrices. Denote by $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ the smallest and the largest eigenvalues, respectively, of a given matrix in $\mathcal{A}$. Let $\mathbb{R}_+$ and $\mathbb{Z}_+$ be the set of nonnegative real numbers and integers, respectively. Denote by $|\cdot|$ the standard Euclidean norm of vectors or absolute value of numbers, and $\|\cdot\|$ the vector-induced matrix norm. For any $\phi_c \in \mathcal{A}$ and $r > 0$, define $\mathcal{B}(\phi_c; r) := \{\phi \in \mathcal{A} : \|\phi - \phi_c\| \leq r\}$. Denote by I_n the identity matrix of dimension n, and diag$\{.,.\}$ the diagonal matrix composed of the input arguments.

II. Problem Formulation

Consider the following linear time-invariant stochastic system:

$$x(t+1) = Ax(t) + w(t),\ t \in \mathbb{Z}_+, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system and $w(t)$ is the process noise. The initial state, $x(0)$, is assumed to be Gaussian with zero mean and covariance matrix ϕ_0, i.e., $x(0) \sim \mathcal{N}(0, \phi_0)$. There are M different sensors attached to the process. At each time step, we assume that only one sensor is available to take measurements. The measurement of the ith sensor is given by:

$$y_i(t) = C_i x(t) + v_i(t),\ t \in \mathbb{Z}_+, \tag{2}$$

where $y_i(t) \in \mathbb{R}^p$ and $v_i(t) \in \mathbb{R}^p$ are the measurement output and measurement noise of the ith sensor at time t, respectively. We assume that the process noise and all the measurement noises are mutually independent Gaussian white noises given by:

$$w(t) \sim \mathcal{N}(0, \Phi^w), \quad v_i(t) \sim \mathcal{N}(0, \Phi_i^v),$$

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all of which are also independent of the initial state $x(0)$.

Define $\lambda_w^- = \lambda_{\min}(\Phi^w)$ and $\lambda_v^- = \min_{i \in \mathbb{M}}\{\lambda_{\min}(\Phi_i^v)\}$. Assume that $\lambda_w^- > 0$ and $\lambda_v^- > 0$. Let $\mathbb{M} := \{1, \ldots, M\}$ be the set of sensor indices. For each $N \in \mathbb{Z}_+$, denote by $\mathbb{M}^N$ the set of all the sequences of sensor indices of length N. An element $\sigma \in \mathbb{M}^N$ is called an N-horizon *sensor schedule*. The set of all infinite-horizon sensor schedules is denoted by $\mathbb{M}^\infty$. An infinite-horizon schedule $\sigma \in \mathbb{M}^\infty$ is called periodic with a period $l \in \mathbb{Z}_+$ if $\sigma(t) = \sigma(t+l)$ for all $t \in \mathbb{Z}_+$. Under a given sensor schedule $\sigma \in \mathbb{M}^\infty$, the measurement sequence is determined by:

$$y(t) = y_{\sigma(t)}(t) = C_{\sigma(t)}x(t) + v_{\sigma(t)}(t), \quad t \in \mathbb{Z}_+.$$

For each $t_1 \le t_2 < \infty$, denote by $\hat{x}^\sigma(t_2|t_1)$ the minimum mean-square error (MMSE) estimate of $x(t_2)$ given the measurements $\{y(0), \ldots, y(t_1)\}$, the initial covariance ϕ_0 and the sensor schedule $\sigma \in \mathbb{M}^\infty$. Define the prediction error $e^\sigma(t|t-1)$ by

$$e^\sigma(t|t-1) = x(t) - A\hat{x}^\sigma(t-1|t-1),$$

and let $\Sigma_t^\sigma(\phi_0)$ be its covariance matrix. When no ambiguity arises, we may drop the dependence on the initial covariance matrix and simply write Σ_t^σ. By a standard result of linear estimation theory, the prediction error covariance can be updated recursively using the Riccati map:

$$\begin{aligned} \Sigma_{t+1}^\sigma &= \Phi^w + A\Sigma_t^\sigma A^T - A\Sigma_t^\sigma C_{\sigma(t)}^T \\ &\quad \times \left(C_{\sigma(t)}\Sigma_t^\sigma C_{\sigma(t)}^T + \Phi_{\sigma(t)}^v\right)^{-1} C_{\sigma(t)}\Sigma_t^\sigma A^T. \end{aligned} \tag{3}$$

For any finite integer N, the performance of an N-horizon sensor schedule $\sigma \in \mathbb{M}^N$ can be evaluated according to the total estimation error defined by:

$$J_N(\sigma; \phi_0) \triangleq \sum_{t=1}^{N} \mathrm{tr}(\Sigma_t^\sigma(\phi_0)), \tag{4}$$

or according to the average-per-stage estimation error defined by:

$$\bar{J}_N(\sigma; \phi_0) \triangleq \frac{1}{N} J_N(\sigma; \phi_0). \tag{5}$$

When N is finite, the two cost functions J_N and $\bar{J}_N$ are equivalent in the sense that they produce the same set of optimal solutions. However, the total cost $J_N(\sigma; \phi_0) \to \infty$ as $N \to \infty$ for all $\sigma \in \mathbb{M}^\infty$ and $\phi_0 \in \mathcal{A}$, because the system is constantly perturbed by a nontrivial Gaussian noise $w(t)$. Thus, the performance of an infinite-horizon sensor schedule is usually measured by the limsup of the N-horizon average-per-stage cost:

$$\bar{J}_\infty(\sigma; \phi_0) \triangleq \limsup_{N \to \infty} \bar{J}_N(\sigma; \phi_0).$$

This cost function has been extensively used for studying various infinite-horizon optimal control and estimation problems [12], [2]. However, this cost function depends only on the limiting behavior of the schedule, which may lead to rather unexpected optimal solutions. For example, one can manipulate a finite portion of an optimal schedule to create an arbitrary transient behavior for the error trajectory without affecting the optimality of the schedule. In some extreme cases, the trajectory of the error covariance under an optimal schedule may even grow unbounded. The following example illustrates such a situation.

Example 1. (Unbounded Optimal Schedule) Consider a simple 2-dimensional system with $A =$diag $\{\lambda, 0\}$, $\lambda > 1$, $C_1 = [1,\ 0]$, $C_2 = [0,\ 1]$, and $\Phi^w =$ diag $\{c, c\} \succ 0$. For simplicity, let $\Phi_1^v = \Phi_2^v = 0$. Note that the system is detectable under sensor 1, but undetectable if using only sensor 2. One optimal schedule can be easily identified as using exclusively sensor 1, which leads to a minimum cost of $\mathrm{tr}(\Phi^w) = 2c$. Now consider an infinite-horizon sensor schedule $\hat{\sigma}$ that alternates between sensor 1 and sensor 2 on time intervals $\mathcal{I}_k^1$ and $\mathcal{I}_k^2$, respectively. Let the length of $\mathcal{I}_k^1$ be $k\lambda^{2k}$, and the length of $\mathcal{I}_k^2$ be k. Define t_{12}^k as the kth switching instants from sensor 2 to sensor 1 and t_{21}^k the other way around. The switching times can be determined recursively as: $t_{21}^k = t_{12}^k + k$ and $t_{12}^{k+1} = t_{21}^k + k\lambda^{2k}$, where $k = 1, 2, \ldots$ and $t_{12}^0 = 0$. Therefore, $\mathcal{I}_k^2 = [t_{12}^k, t_{21}^k)$, $\mathcal{I}_k^1 = [t_{21}^k, t_{12}^{k+1})$, and $\hat{\sigma}$ can be represented as

$$\hat{\sigma}(t) = \begin{cases} 2, & t \in [t_{12}^k, t_{21}^k) \\ 1, & t \in [t_{21}^k, t_{12}^{k+1}), \end{cases}$$

where $t \in \mathbb{Z}_+$, $t_{12}^0 = 0$ and $k = 1, 2, \ldots$ It can be calculated that the average-per-stage cost on $\cup_{i=1}^k \mathcal{I}_k$ with $\mathcal{I}_k = \mathcal{I}_k^2 \cup \mathcal{I}_k^1$ goes to $2c$ as $k \to \infty$, which is the same as the optimal cost. However, the subsequence consisting of error covariances $\Sigma_{t_{21}^k - 1}^\sigma$ diverges as $k \to \infty$.

To exclude such abnormalities, we introduce the following feasible set of sensor schedules which yield a bounded trajectory under a given initial condition $\phi \in \mathcal{A}$:

$$\mathbb{M}_\phi^\infty = \{\sigma \in \mathbb{M}^\infty \colon \exists \beta < \infty,\ \text{s.t. } \Sigma_t^\sigma(\phi) \preceq \beta I_n, \forall t \in \mathbb{Z}_+\}.$$

An infinite-horizon sensor schedule σ is called *feasible* for $\phi \in \mathcal{A}$ if $\sigma \in \mathbb{M}_\phi^\infty$. The following assumption is adopted throughout this paper.

Assumption 1. *$\mathbb{M}_\phi^\infty \neq \emptyset$, $\forall \phi \in \mathcal{A}$.*

Remark 1. The assumption requires that for any initial covariance, there always exists an infinite-horizon schedule that can keep the estimation error covariance bounded for all time. This is a reasonable assumption for typical estimation applications. It can be guaranteed if, for example, one of the subsystems is detectable.

Problem 1. For a given $\phi_0 \in \mathcal{A}$, solve the following problem

$$\bar{V}^*(\phi_0) \triangleq \inf_{\sigma \in \mathbb{M}_{\phi_0}^\infty} \limsup_{N \to \infty} \bar{J}_N(\sigma; \phi_0) \tag{6}$$

Assumption 1 implies that $\bar{V}^*(\phi_0)$ is finite for all $\phi_0 \in \mathcal{A}$. The function $\bar{V}^* : \mathcal{A} \to \mathbb{R}_+$ defined implicitly by equation (6) is called the optimal infinite-horizon cost function. For a general $\phi \in \mathcal{A}$, a schedule that achieves the cost $V^*(\phi)$ will be referred to as an *optimal schedule* for ϕ.

III. The Sequential Riccati Mapping and Its Stability

The Riccati recursion in (3) can be viewed as a mapping from $\Sigma_t^\sigma \in \mathcal{A}$ to $\Sigma_{t+1}^\sigma \in \mathcal{A}$ depending on the sensor index chosen at time t. In general, for each sensor $i \in \mathbb{M}$, we can define the *Riccati mapping* as

$$\begin{aligned} \rho_i(Q) &= \Phi^w + AQA^T \\ &\quad - AQC_i^T \left(C_iQC_i^T + \Phi_i^v\right)^{-1} C_iQA^T,\ Q \in \mathcal{A}. \end{aligned}$$

With this notation, for a generic initial covariance matrix $\phi \in \mathcal{A}$, the covariance matrix $\Sigma_t^\sigma(\phi)$, defined in (3), is the trajectory of the following matrix-valued time-varying nonlinear system:

$$\Sigma_{t+1}^\sigma = \rho_{\sigma(t)}\left(\Sigma_t^\sigma\right),\ \text{ for } t \in \mathbb{Z}_+,\ \text{ with } \Sigma_0^\sigma = \phi. \tag{7}$$

One can also view $\Sigma_t^\sigma(\cdot)$ as the composition of a sequence of Riccati mappings, i.e.

$$\Sigma_t^\sigma = \rho_{\sigma(t-1)} \circ \rho_{\sigma(t-2)} \cdots \circ \rho_{\sigma(0)}, \quad t \in \mathbb{Z}_+. \tag{8}$$

We will also refer to Σ_t^σ as the composite Riccati map associated with σ.

To solve Problem 1, it is critical to understand the dynamic behavior of the matrix-valued nonlinear system (7) under different infinite-horizon schedules. Two well-known properties of the Riccati mapping are useful for this purpose.

Lemma 1. *For any $i \in \mathbb{M}$, $Q_1, Q_2 \in \mathcal{A}$ and $c \in [0, 1]$, we have*

(i) $Q_1 \preceq Q_2 \Rightarrow \rho_i(Q_1) \preceq \rho_i(Q_2)$;
(ii) $\rho_i(cQ_1 + (1-c)Q_2) \succeq c\rho_i(Q_1) + (1-c)\rho_i(Q_2)$.

Remark 2. The lemma indicates that the Riccati mapping is *monotone* and *concave*. The monotonicity property is a well-known result and its proof can be found in [7]. The concavity property is an immediate consequence of Lemma 1-(e) in [14].

Based on these two properties, one can prove the following results.

Proposition 1. *(Theorem 5 of [17]) For any $\phi \in \mathcal{A}$, $\epsilon \in \mathbb{R}^+$, $\sigma \in \mathbb{M}^\infty$, and $t \in \mathbb{Z}_+$, we have $\Sigma_t^\sigma(\phi + \epsilon I_n) \preceq \Sigma_t^\sigma(\phi) + g_t^\sigma(\phi) \cdot \epsilon$, where $g_t^\sigma(\phi)$ is the directional derivative of the t-step Riccati mapping Σ_t^σ at ϕ along direction I_n. Furthermore, if $\Sigma_t^\sigma(\phi) \preceq \beta I_n$ for all $t \in \mathbb{Z}_+$ and for some $\beta < \infty$, then $tr(g_t^\sigma(\phi)) \leq n\beta\eta^t/\lambda_w^-$, $\forall t \in \mathbb{Z}_+$, where*

$$\eta = \frac{1}{1 + \alpha\lambda_w^-} < 1 \quad \text{and} \quad \alpha = \frac{\lambda_w^-}{\|A\|^2\beta^2 + \lambda_w^-\beta}. \tag{9}$$

The above theorem reveals an important property of system (7), namely, that boundedness of its trajectory implies an exponential disturbance attenuation. This property plays a crucial role in the derivation of the various properties of Problem 1 in Section IV.

IV. Main Results

In this section, we will use the properties of the sequential Riccati mapping derived in the last section to gain some insights into the solution of Problem 1.

A. Independence of the Initial Covariance

We first show that the feasible set is independent of the initial covariance.

Lemma 2. *If $\sigma \in \mathbb{M}_{\phi_1}^\infty$ for some $\phi_1 \in \mathcal{A}$, then $\sigma \in \mathbb{M}_\phi^\infty$ for all $\phi \in \mathcal{A}$.*

Proof: Fix arbitrary ϕ, $\phi_1 \in \mathcal{A}$, and $\sigma \in \mathbb{M}_{\phi_1}^\infty$. Since $\phi \preceq \phi_1 + \|\phi - \phi_1\| I_n$, we have by Proposition 1,

$$\Sigma_t^\sigma(\phi) \preceq \Sigma_t^\sigma(\phi_1) + g_t^\sigma(\phi_1) \cdot \|\phi - \phi_1\|.$$

The first term on the right-hand side is bounded because $\sigma \in \mathbb{M}_{\phi_1}^\infty$, while the second term is bounded due to Proposition 1. Thus, $\sigma \in \mathbb{M}_\phi^\infty$. ∎

Therefore, if an infinite-horizon schedule is feasible for some initial covariance matrix, it is also feasible for all initial covariances. This allows us to drop the dependence on the initial covariance and simply define

$$\begin{aligned} \mathbb{M}_f^\infty \quad = \quad & \{\sigma \in \mathbb{M}^\infty : \forall \phi \in \mathcal{A},\ \exists \beta < \infty,\ \text{s.t.} \\ & \Sigma_t^\sigma(\phi) \preceq \beta I_n, \forall t \in \mathbb{Z}_+\}. \end{aligned}$$

We next show that under a fixed schedule $\sigma \in \mathbb{M}_f^\infty$, all the trajectories starting from different initial covariances will eventually converge to the same trajectory.

Theorem 1. *For any feasible schedule $\sigma \in \mathbb{M}_f^\infty$, we have that*

$$\|\Sigma_t^\sigma(\phi_1) - \Sigma_t^\sigma(\phi_2)\| \to 0 \text{ exponentially as } t \to \infty,$$

for all $\phi_1, \phi_2 \in \mathcal{A}$.

Proof: Fix arbitrary $\phi_1 \in \mathcal{A}$ and $\phi_2 \in \mathcal{A}$. Define $\epsilon = \|\phi_1 - \phi_2\|$. Without loss of generality, let $\beta < \infty$ be the bound such that $\Sigma_t^\sigma(\phi_i) \preceq \beta I_n$ for all $t \in \mathbb{Z}_+$ and $i = 1, 2$. By Proposition 1, we have

$$\begin{aligned} \Sigma_t^\sigma(\phi_2) \quad & \preceq \quad \Sigma_t^\sigma(\phi_1 + \|\phi_2 - \phi_1\| I_n) \\ & \preceq \quad \Sigma_t^\sigma(\phi_1) + g_t^\sigma(\phi_1) \cdot \epsilon \\ & \preceq \quad \Sigma_t^\sigma(\phi_1) + \left(\frac{n\beta\epsilon}{\lambda_w^-}\eta^t\right) \cdot I_n. \end{aligned}$$

Similarly, we can obtain $\Sigma_t^\sigma(\phi_1) \preceq \Sigma_t^\sigma(\phi_2) + \left(\frac{n\beta\epsilon}{\lambda_w^-}\eta^t\right) \cdot I_n$, for all $t \in \mathbb{Z}_+$. The result follows directly from the above inequalities as $t \to \infty$. ∎

An immediate consequence of the above theorem is that the infinite-horizon average-per-stage cost of any feasible schedule is independent of the initial covariance matrix.

Corollary 1. *For any $\sigma \in \mathbb{M}_f^\infty$, $\bar{J}_\infty(\sigma; \phi_1) = \bar{J}_\infty(\sigma; \phi_2)$ for all $\phi_1, \phi_2 \in \mathcal{A}$.*

Proof: By Theorem 1, $\Sigma_t^\sigma(\phi_1) \to \Sigma_t^\sigma(\phi_2)$ as $t \to \infty$. Thus, the two average-per-stage cost sequences $\{\frac{1}{N}\sum_{t=1}^N \Sigma_t^\sigma(\phi_i)\}_N$, $N \in \mathbb{Z}_+$, $i = 1, 2$, must have the same limsup. ∎

By the above corollary, it is easy to see that if a feasible schedule σ is optimal for some initial covariance ϕ_1, then it must also be optimal for any other initial covariance ϕ_2. In addition, the optimal infinite-horizon average-per-stage costs corresponding to these two initial covariances must also be the same.

Corollary 2. *For any $\phi_1, \phi_2 \in \mathcal{A}$, if σ^* is optimal for ϕ_1, then it must also be optimal for ϕ_2; and in addition, $\bar{V}^*(\phi_1) = \bar{V}^*(\phi_2)$.*

Therefore, to solve Problem 1, we can start from any initial covariance matrix at our convenience. The obtained optimal solution would also be optimal for all the other initial covariances.

B. Properties of the Accumulation Set

For any $\sigma \in \mathbb{M}_f^\infty$, let $\mathcal{L}^\sigma$ be the *accumulation set* of the closed-loop trajectory of the nonlinear system (7) under schedule σ. In other words, the set $\mathcal{L}^\sigma$ contains all the points whose arbitrary neighborhoods will be visited infinitely often by the trajectory $\{\Sigma_t^\sigma(\phi)\}_{t \in \mathbb{Z}_+}$ for some initial condition $\phi \in \mathcal{A}$. Under Assumption 1, the sequence $\{\Sigma_t^\sigma(\phi)\}_{t \in \mathbb{Z}_+}$ is bounded if σ is feasible. Therefore, there exists a convergent subsequence and $\mathcal{L}^\sigma$ is not empty. Moreover, $\mathcal{L}^\sigma$ is closed since the subsequential limits of a bounded sequence in a metric space X form a closed subset of X. It follows that $\mathcal{L}^\sigma$ is bounded and closed in $\mathcal{A}$, and is thus compact.

According to Theorem 1, a trajectory $\{\Sigma_t^\sigma(\phi)\}_{t \in \mathbb{Z}_+}$ under schedule σ starting from any initial covariance $\phi \in \mathcal{A}$ has the same accumulation set $\mathcal{L}^\sigma$. This implies that $\mathcal{L}^\sigma$ is globally attractive, i.e. $\lim_{t \to \infty} d(\Sigma_t^\sigma(\phi), \mathcal{L}^\sigma) = 0, \forall \phi \in \mathcal{A}$, where $d(\phi, \mathcal{L}^\sigma) = \inf_{z \in \mathcal{L}^\sigma} \|\phi - z\|$ represents the distance from the point ϕ to the set $\mathcal{L}^\sigma$.

We summarize the above results in the following proposition:

Proposition 2. *The accumulation set of any feasible schedule is nonempty, compact and globally attractive.*

C. Universal Approximation Property of Periodic Schedules

The goal of this subsection is to show that the optimal infinite-horizon cost can be approximated with an arbitrary accuracy by periodic schedules. Actually a more general result is proved for approximating infinite-horizon costs of any feasible schedule. First, we derive the following result which will facilitate our main proof.

Lemma 3. *(Uniform Bound) Given $\sigma \in \mathbb{M}_f^\infty$, for any bounded set $E \subset \mathcal{A}$, there exists finite constants β_E, α_E, and $\eta_E \in (0, 1)$, such* *that $\Sigma_t^\sigma(\phi) \preceq \beta_E I_n$ and $tr(g_t^\sigma(\phi)) \leq \alpha_E \eta_E^t$, for all $t \in \mathbb{Z}_+$ and $\phi \in E$.*

Proof: Fix an arbitrary $\phi_1 \in E$. Define the covariance trajectory under σ with initial covariance ϕ_1 as $\psi_t = \Sigma_t^\sigma(\phi_1)$, $t \in \mathbb{Z}_+$. Since σ is feasible, there must exist a finite constant β_1 such that $\psi_t \leq \beta_1 I_n$ for all $t \in \mathbb{Z}_+$. By Proposition 1, there exist constants $\alpha_1 < \infty$ and $\eta_1 \in (0,1)$ such that $\mathrm{tr}(g_t^\sigma(\phi_1)) \leq \alpha_1 \eta_1^t$, for all $t \in \mathbb{Z}_+$. It follows that

$$\begin{aligned}\Sigma_t^\sigma(\phi) &\preceq \Sigma_t^\sigma(\phi_1 + \|\phi - \phi_1\| I_n) \preceq \Sigma_t^\sigma(\phi_1) + g_t^\sigma(\phi_1)\|\phi - \phi_1\| \\ &\preceq \psi_t + \alpha_1 \eta_1^t (\kappa_E + \beta_1) I_n,\end{aligned}$$

for all $\phi \in E$, where $\kappa_E \triangleq \sup_{\phi \in E} \|\phi\|$. This implies the existence of the desired constant β_E, which in turn guarantees the existence of the desired constants α_E and η_E according to Proposition 1. ■

The above lemma indicates that the covariance trajectories starting from any initial covariance in a bounded set E are bounded uniformly by $\beta_E I_n$. The bound β_E depends only on the underlying set E instead of the particular value of the initial covariance.

The following theorem presents the main contribution of this paper.

Theorem 2. *(Universal Approximation) For any feasible schedule $\sigma \in \mathbb{M}_f^\infty$ and any $\delta > 0$, there exists a periodic schedule $\tilde{\sigma}$ with a finite period $N \in \mathbb{Z}_+$, such that the infinite-horizon cost of σ is approximated by $\tilde{\sigma}$ with the error bound $\left|\bar{J}_\infty(\tilde{\sigma}) - \bar{J}_\infty(\sigma)\right| < \delta$.*

Proof: Pick an arbitrary feasible schedule $\sigma \in \mathbb{M}_f^\infty$ and an accumulation point $\hat{\phi} \in \mathcal{L}^\sigma$. Suppose that $\Sigma_t^\sigma(\hat{\phi}) < \beta I_n$. By Proposition 1 and Lemma 3, we have

$$\Sigma_t^\sigma(\phi) \preceq \Sigma_t^\sigma(\hat{\phi}) + r\alpha_r \eta_r^t I_n \preceq (\beta + r\alpha_r\eta_r^t) I_n, \quad \forall \phi \in \mathcal{B}(\hat{\phi}; r)$$

where $\alpha_r > 0$ and $0 < \eta_r < 1$ are constants depending on r. Denote $\hat{\beta} = \beta + r\alpha_r$. It is clear that $\Sigma_t^\sigma(\phi)$ is bounded by $\hat{\beta} I_n$, $\forall \phi \in \mathcal{B}(\hat{\phi}; r)$. Define $E = \{\phi : \ \phi \preceq \hat{\beta} I_n\}$. Clearly $\mathcal{L}^\sigma \subset E$, and $\mathcal{B}(\hat{\phi}; r) \subset E$. These sets are illustrated in Fig. 1.

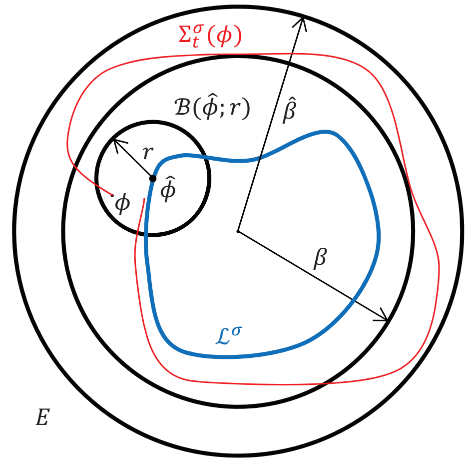

Fig. 1. Domain of consideration: bounded set E which contains all the trajectories starting from $\mathcal{B}(\hat{\phi}; r)$

The rest of the proof consists of three major steps. (i) Firstly, we show that there exists a common l-horizon schedule σ_l that can drive the covariance trajectory to $\mathcal{B}(\hat{\phi}; r)$ at the end of the l horizon for any initial covariance in E; (ii) Secondly, we show that there exists a subschedule σ_{N_k} whose average-per-stage cost converges to the infinite-horizon cost of σ uniformly for all initial condition in E; (iii) Lastly, we will construct a periodic schedule $\tilde{\sigma}$ based on σ_l and σ_{N_k}, which satisfies the desired error bound δ for all large enough k.

Step (i): By Proposition 1 and Lemma 3, $\forall \phi \in E$,

$$\left\|\Sigma_t^\sigma(\phi) - \Sigma_t^\sigma(\hat{\phi})\right\| \quad \leq \quad \hat{\beta}\alpha_E \eta_E^t, \tag{10}$$

where $\alpha_E > 0$ and $0 < \eta_E < 1$ are constants associated with E. Therefore $\exists l_0 > 0$ such that

$$\left\|\Sigma_t^\sigma(\phi) - \Sigma_t^\sigma(\hat{\phi})\right\| \quad \leq \quad \frac{r}{2}, \ \forall t > l_0. \tag{11}$$

Since $\hat{\phi} \in \mathcal{L}^\sigma$ and $\mathcal{L}^\sigma$ is attractive, $\exists l \geq l_0$ such that

$$\left\|\Sigma_l^\sigma(\hat{\phi}) - \hat{\phi}\right\| \leq \frac{r}{2}. \tag{12}$$

By (11) and (12), we have $\forall \phi \in E$

$$\left\|\Sigma_l^\sigma(\phi) - \hat{\phi}\right\| \leq \left\|\Sigma_l^\sigma(\phi) - \Sigma_l^\sigma(\hat{\phi})\right\| + \left\|\Sigma_l^\sigma(\hat{\phi}) - \hat{\phi}\right\| \leq r. \tag{13}$$

Denote the first l steps of σ by σ_l. Equation (13) shows that under σ_l, the final covariance $\Sigma_l^{\sigma_l}(\phi) \in \mathcal{B}(\hat{\phi}; r)$, $\forall \phi \in E$.

Step (ii): Now we construct another finite length sub-schedule σ_{N_k}, under which the performance obtained can be arbitrarily close to $\bar{J}_\infty(\sigma)$ when k is large enough.

Suppose $\bar{J}_N(\sigma; \phi_0)$ is the average-per-stage cost of the first N steps of σ for some initial condition $\phi_0 \in E$. For brevity, define $b_N = \bar{J}_N(\sigma; \phi_0)$. Since σ is feasible, b_N is bounded. Therefore, there exists a subsequence $\{b_{N_k}\}_k$, such that $\bar{J}_\infty(\sigma; \phi_0) = \lim_{k \to \infty} b_{N_k}$. It follows $\forall \delta > 0$, $\exists K_1 \in \mathbb{Z}_+$, such that

$$\left|\bar{J}_\infty(\sigma; \phi_0) - b_{N_k}\right| < \frac{\delta}{3}, \quad \forall k > K_1. \tag{14}$$

Let σ_{N_k} be the first N_k steps of σ. Note that $\{b_{N_k}\}_k$ and the associated sub-schedule $\{\sigma_{N_k}\}_k$ are constructed based on the initial condition ϕ_0. We need to be shown that the cost convergence is uniform with respect to all $\phi \in E$. To this end, consider arbitrary $\phi_1, \phi_2 \in E$. We know that $\|\phi_1 - \phi_2\| \leq \hat{\beta}$. It follows from Proposition 1 and Lemma 3 that

$$\begin{aligned}\Delta &\triangleq \left|\bar{J}_{N_k}(\sigma_{N_k}; \phi_1) - \bar{J}_{N_k}(\sigma_{N_k}; \phi_2)\right| \\ &\leq \frac{1}{N_k}\sum_{t=1}^{N_k} \left|\mathrm{tr}(\Sigma_t^\sigma(\phi_1)) - \mathrm{tr}(\Sigma_t^\sigma(\phi_2))\right| \\ &\leq \frac{1}{N_k}\sum_{t=1}^{N_k} n\hat{\beta}\alpha_E\eta_E^t \leq \frac{n\hat{\beta}\alpha_E\eta_E}{N_k(1-\eta_E)}.\end{aligned}$$

Note that n, $\hat{\beta}$, α_E, η_E are all constants. Hence, $\exists K_2 \in \mathbb{Z}_+$, such that $\Delta < \frac{\delta}{3}$, $\forall k > K_2$. Choose $K = \max\{K_1, K_2\}$, we have

$$\begin{aligned}\left|\bar{J}_\infty(\sigma; \phi_0) - \bar{J}_{N_k}(\sigma_{N_k}; \phi)\right| &\leq \left|\bar{J}_\infty(\sigma; \phi_0) - \bar{J}_{N_k}(\sigma_{N_k}; \phi_0)\right| \\ &\quad + \left|\bar{J}_{N_k}(\sigma_{N_k}; \phi_0) - \bar{J}_{N_k}(\sigma_{N_k}; \phi)\right| \\ &< \frac{2\delta}{3}, \quad \forall \phi \in E, \quad \forall k > K\end{aligned}$$

Step (iii): Now construct a periodic schedule $\tilde{\sigma} = \{\sigma_N, \sigma_N, \cdots\cdots\}$, where $\sigma_N = \{\sigma_{N_k}, \sigma_l\}$, $N = N_k + l$. Recall that N_k is constructed in *Step (ii)* for some large k to be determined later; and $l \geq l_0$ is from *Step (i)* so that equations (12) and (13) hold. Note that $\Sigma_{N_k}^{\sigma_{N_k}}(\phi) \in E$, $\forall \phi \in \mathcal{B}(\hat{\phi}; r)$. It follows from (13) that $\Sigma_N^{\sigma_N}(\phi) \in \mathcal{B}(\hat{\phi}; r)$, $\forall \phi \in \mathcal{B}(\hat{\phi}; r)$. Therefore $\Sigma_N^{\sigma_N}(\phi)$ is an invariant mapping with respect to $\mathcal{B}(\hat{\phi}; r)$. Further note that (10) and (11) implies $\forall \phi_1, \phi_2 \in \mathcal{B}(\hat{\phi}; r)$,

$$\begin{aligned}\|\Sigma_t^\sigma(\phi_1) - \Sigma_t^\sigma(\phi_2)\| &\leq \alpha_E \eta_E^t \|\phi_1 - \phi_2\| \\ &\leq \frac{r}{2\hat{\beta}} \|\phi_1 - \phi_2\|, \quad \forall t > l_0\end{aligned}$$

Recall that $\mathcal{B}(\hat{\phi}; r) \subset E$ as discussed in Step 1, and therefore $r < 2\hat{\beta}$. It follows that $\Sigma_N^{\sigma_N}$ is a contraction mapping on $\mathcal{B}(\hat{\phi}; r)$. Let P be the unique fixed point of $\Sigma_N^{\sigma_N}$ on $\mathcal{B}(\hat{\phi};r)$. Since $P \in \mathcal{L}^{\tilde{\sigma}}$ and $\mathcal{L}^{\tilde{\sigma}}$ is an N-cycle [1], the performance of $\tilde{\sigma}$ can be obtained as

$$\bar{J}_\infty(\tilde{\sigma};P) \quad = \quad \frac{N_k \bar{J}_{N_k}(\sigma_{N_k};P) + l\bar{J}_l(\sigma_l;\phi_P)}{N_k + l},$$

where ϕ_P denotes the point $\Sigma_{N_k}^{\sigma_{N_k}}(P)$.

Since l is independent of N_k and $\bar{J}_l(\sigma_l;\phi_P)$ is bounded, $\bar{J}_{N_k}(\sigma_{N_k};P) \to \bar{J}_\infty(\tilde{\sigma};P)$ as $k \to \infty$. That is $\forall \delta > 0$, $\exists K_3 \in \mathbb{Z}_+$, such that $\left|\bar{J}_\infty(\tilde{\sigma};P) - \bar{J}_{N_k}(\sigma_{N_k};P)\right| < \frac{\delta}{3}$, $\forall k > K_3$. Let $\bar{K} = \max\{K_3, K\}$, and it follows $\left|\bar{J}_\infty(\tilde{\sigma};P) - \bar{J}_\infty(\sigma;P)\right| < \delta$ when $k > \bar{K}$ and the length of the period of $\tilde{\sigma}$ is $N = N_{\bar{K}} + l$. By Theorem 1, we know that the infinite-horizon cost is independent of the initial condition, and thus the desired result follows. ■

Remark 3. The proof of Theorem 2 can also be applied to prove that the $\liminf$ (or any subsequential limits) of the sequence of average-per-stage costs can be approximated by periodic schedules arbitrarily closely by choosing σ_{N_k} corresponding to the $\liminf$ subsequences (or any convergent subsequences). Then the following corollary follows immediately from Theorem 2:

Corollary 3. *Suppose σ^* is an optimal sensor schedule to Problem 1, and $\{b_N^*\}$ is the corresponding sequence of the N-horizon average-per-stage costs. Then $\{b_N^*\}$ converges and the optimal cost is $\bar{V}^* = \lim_{N\to\infty} b_N^*$.*

Proof: Suppose $\{b_N^*\}$ does not converge. Let $\varepsilon = \limsup_{N\to\infty} b_N^* - \liminf_{N\to\infty} b_N^* > 0$. By Theorem 2 and Remark 3, there exists a periodic schedule $\tilde{\sigma}$ with finite period such that $\left|\bar{J}_\infty(\tilde{\sigma}) - \liminf_{N\to\infty} b_N^*\right| < \frac{\varepsilon}{2}$. It follows $\bar{J}_\infty(\tilde{\sigma}) < \liminf_{N\to\infty} b_N^* + \frac{\varepsilon}{2} < \limsup_{N\to\infty} b_N^*$. Thus $\tilde{\sigma}$ has a smaller infinite cost than the optimal schedule σ^*, which is a contradiction. ■

In addition, the proof of Theorem 2 also implies the stability of the covariance trajectory under a feasible periodic sensor schedule.

Corollary 4. *For any feasible periodic schedule $\tilde{\sigma}$, the discrete nonlinear system $\phi_{k+1} = \Sigma_N^{\tilde{\sigma}}(\phi_k)$, $k \in \mathbb{Z}_+$, $\forall \phi_0 \in \mathcal{A}$ is globally asymptotically and locally exponentially stable.*

Proof: From the proof of Theorem 2, the fixed point P of the contraction mapping $\Sigma_N^{\tilde{\sigma}}$ on $\mathcal{B}(\hat{\phi};r)$ is also the equilibrium of the system $\phi_{k+1} = \Sigma_N^{\tilde{\sigma}}(\phi_k)$. The result follows by futher noting that $P \in \mathcal{L}^{\tilde{\sigma}}$ is globally attractive. ■

V. Discussion and Conclusions

Under a mild feasibility assumption, we have proven that both the optimal infinite-horizon cost and the corresponding optimal schedule are independent of the initial error covariance. Furthermore, we have proven that the accumulation set of the composite Riccati mapping under a feasible schedule is nonempty, compact, and globally attractive. The most important result is the universal approximation theorem (Theorem 2) which states that the performance of any feasible schedule can be approximated arbitrarily closely by a periodic schedule with a finite period. Interestingly, this result leads to the conclusion that the average-per-stage cost of an optimal schedule must converge (Corollary 3).

These theoretical results provide us valuable insights into the infinite-horizon sensor scheduling problem. Theorem 2 motivates us to focus on the periodic schedules in solving Problem 1. A straightforward way to approach this problem is to first find the best N-periodic schedule by enumeration, and gradually increase the length of the period until the performance no longer improves. Although the complexity of this approach grows exponentially as N increases, it is still a practically reasonable solution procedure because the infinite-horizon performance of the periodic schedule can be computed efficiently and large periods are not preferred in practice. In addition, Corollary 3 can be used to simplify the objective function (cost function) in optimization-based approaches for finding the optimal/suboptimal periodic solutions. Our future research will focus on developing efficient infinite-horizon sensor scheduling algorithms with guaranteed suboptimal performance.

[1]By N-cycle, we mean a sequence $(\phi_0, \phi_1, \cdots, \phi_{N-1})$ where $\rho_{\tilde{\sigma}(0)}(\phi_0) = \phi_1$, $\rho_{\tilde{\sigma}(1)}(\phi_1) = \phi_2, \cdots$, and $\rho_{\tilde{\sigma}(N-1)}(\phi_{N-1}) = \phi_0$.

References

[1] S. C. Bengea and R. A. DeCarlo, "Optimal control of switching systems," *Automatica*, vol. 41, no. 1, pp. 11–27, 2005.

[2] D. Bertsekas, *Dynamic Programming and Optimal Control*, 2nd ed. Athena Scientific, 2001, vol. 2.

[3] Z. G. Feng, K. L. Teo, and V. Rehbock, "Hybrid method for a general optimal sensor scheduling problem in discrete time," *Automatica*, vol. 44, no. 5, pp. 1295–1303, 2008.

[4] V. Gupta, T. H. Chung, B. Hassibi, and R. M. Murray, "On a stochastic sensor selection algorithm with applications in sensor scheduling and sensor coverage." *Automatica*, vol. 42, no. 2, pp. 251–260, 2006.

[5] P. Hovareshti, V. Gupta, and J. S. Baras, "Sensor scheduling using smart sensors," in *IEEE Conference on Decision and Control*, New Orleans, LA, Dec. 2007.

[6] S. Joshi and S. Boyd, "Sensor selection via convex optimization," *IEEE Transactions on Signal Processing*, vol. 57, no. 2, pp. 451–462, 2009.

[7] P. R. Kumar and P. Varaiya, *Stochastic systems: estimation, identification and adaptive control*. Upper Saddle River, NJ, USA: Prentice-Hall, Inc., 1986.

[8] Y. Li, L. W. Krakow, E. K. P. Chong, and K. N. Groom, "Approximate stochastic dynamic programming for sensor scheduling to track multiple targets," *Digital Signal Processing*, vol. 19, no. 6, pp. 978–989, 2009.

[9] A. Logothetis and A. Isaksson, "On sensor scheduling via information theoretic criteria," in *Proceedings of the American Control Conference*, San Diego, CA, Jun. 1999.

[10] L. Meier, J. Peschon, and R. Dressler, "Optimal control of measurement subsystems," *IEEE Transactions on Automatic Control*, vol. 12, no. 5, pp. 528–536, 1967.

[11] A. I. Mourikis and S. I. Roumeliotis, "Optimal sensor scheduling for resource-constrained localization of mobile robot formations," *IEEE Transactions on Robotics*, vol. 22, no. 5, pp. 917 – 931, 2006.

[12] J. L. Ny, E. Feron, and M. A. Dahleh, "Scheduling Kalman filters in continuous time," in *Proceedings of the American Control Conference*, St Louis, MO, Jun. 2009.

[13] Y. Oshman, "Optimal sensor selection strategy for discrete-time state estimators," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 30, no. 2, pp. 307–314, 1994.

[14] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. I. Jordan, and S. S. Sastry, "Kalman filtering with intermittent observations," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1453–1464, Sep. 2004.

[15] M. P. Vitus, W. Zhang, A. Abate, J. Hu, and C. J. Tomlin, "On efficient sensor scheduling for linear dynamical systems," in *Proceedings of the American Control Conference*, Baltimore, MD, Jul. 2010.

[16] ——, "On sensor scheduling of linear dynamical systems with error bounds," in *IEEE Conference on Decision and Control*, Atlanta, GA, Dec. 2010.

[17] ——, "On efficient sensor scheduling for linear dynamical systems," *Automatica*, vol. 48, no. 10, pp. 2482–2493, Oct. 2012.