An ODE Comparison Theorem with Application in the Optimal Exit Time Control Problem

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***Abstract*— The optimal exit time control (OETC) problem tries to find the feedback control law with a reasonable cost that can keep the system state inside a certain subset of the state space, called the safe set, for the longest time under random perturbations. The symmetry property of its solutions has been proved previously when the state dimension is higher than one. However, the proof does not apply in the one dimensional (1-D) case. In this note, a comparison theorem is established that compares the solutions to two ODEs arising in the 1-D OETC problem, one with an arbitrary control and the other with a symmetric control. The symmetry of solutions to the 1-D OETC problem is proved using this comparison theorem. An example is presented to show how the symmetry result can help to solve the OETC problem analytically in certain cases.**

I. INTRODUCTION

Safety is a critical issue in many engineering problems, such as intelligent transportation systems [1], [2], control of Unmanned Aerial Vehicles (UAV) [3], chemical processes, etc. In these applications, the system state is typically required to stay inside a certain subset of the state space called the *safe set*. Whenever the system state gets outside the safe set, some costly procedure must be invoked to bring the system back to safety. Safety of deterministic systems is relatively easy to maintain by properly designed feedback control strategies. However, practical systems are often perturbed by random noises, which may steer the state outside the safe set despite the control efforts. A common approach to the probabilistic safety problem is through the probabilistic reachability analysis [4]. This approach analyzes and/or designs the control based on the safety probability, i.e., the probability that the system state stays inside the safe set over a certain time horizon. However, the computation of the safety probability for general stochastic systems is rather challenging, especially in high-dimensional state spaces. An alternative approach is to measure the system safety by certain aggregate quantities that are relatively easy to compute, such as the expectation of the first exit time of the system state from the safe set. The problem of designing a control law to maximize the expectation of the exit time is called the *optimal exit time control* (OETC) problem, and is the focus of this paper.

The general stochastic optimal control problems have been well studied in the literature [5], [6]. In particular, the stochastic optimal exit time control problems [7], [8], [9] try to minimize the expected integral of a running cost functional $L(t, X_t, u)$ along a controlled Markov process X_t with control u constrained by a feasible set $\mathcal{C}$, up to the first time τ that X_t exits from a given domain Ω. Using the dynamic programming approach, the value functions are shown to be the viscosity solution to the Hamilton-Jacobi-Bellman equations [10], [11], [12]. The OETC problem studied in this paper, however, is different from these studies in that, although the simplified running cost $L \equiv 1$ is considered, the control u is constrained through its aggregation $\int_\Omega \|u\|^2\, dx$ instead of pointwise on Ω. As a result, optimal solution can no longer be characterized by dynamic programming method. More importantly, our objective is not the local properties of optimal solutions (e.g., the characterizing PDEs), but the more challenging global properties such as their symmetry on symmetric domains, which can significantly reduce solution complexity. The OETC problem was first proposed in [13] and later extended to the general setting of stochastic hybrid systems in [14]. The symmetry property of optimal solutions has been proved in [15], [16] when the state dimension $n \geq 2$. However, when $n = 1$, certain multi-dimensional concepts such as surface and volume, and some key results used in the proof of [15], [16] such as the isoperimetric inequality, are not applicable.

In this paper, we propose a proof of the symmetry property of the solutions to the 1-D OETC problem. Compared with the direct method in the multidimensional proof in [15], [16], our proof of the 1-D case is based upon a key comparison theorem comparing the solutions to two second-order ODEs arising in the 1-D OETC problem, which has its own value in the general study of ODEs. Using the comparison theorem, we show that the solutions to the 1-D OETC problem are symmetric. This, together with the results in [15], [16], completely establishes the symmetry property of the solutions to the OETC problem with an arbitrary state dimension. Our proof makes essential use of the symmetrization operation [17], which can transform an arbitrary function to a symmetric one while preserving certain properties. The symmetrization method has been widely used in the literature for proving the symmetry of solutions to certain PDEs or PDE-constrained variational problems [18], [19]. Its applications to control problems, however, are still rare. Therefore, this paper also represents an important application of the symmetrization method in the study the stochastic optimal control problems.

The rest of the paper is organized as follows. In section II, the OETC problem is formulated. The symmetrization method is briefly reviewed in Section III and then used to prove the comparison theorem and the symmetry property in Section IV. An example is worked out in details in Section V to illustrate the application of the symmetry property in solving the OETC problem. Finally, some concluding remarks are given in Section VI.

II. OPTIMAL EXIT TIME CONTROL PROBLEM

A. General OETC Problem

Consider a general dynamical system whose state $X_t \in \mathbb{R}^n$ is required to reside inside a bounded open connected domain Ω of $\mathbb{R}^n$ called the safe set. To keep X_t inside Ω under the presence of random perturbations, a bounded measurable state feedback control law $u : \Omega \to \mathbb{R}^n$ is typically adopted.

This work was partially supported by the National Science Foundation under Grant CNS-0643805.

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The corresponding state dynamics can then be modeled by the following stochastic differential equation (SDE):

$$dX_t = u(X_t)dt + \sigma(X_t)dB_t. \tag{1}$$

Here, B_t is an n-dimensional Brownian motion, and the diffusion term $\sigma : \Omega \to \mathbb{R}^{n\times n}$ satisfies the *uniformly elliptic condition* on Ω: $\sigma(x)\sigma(x)^T \succeq \sigma_0^2 I_n$, $\forall x \in \Omega$, for some $\sigma_0 > 0$.

Define $\tau \triangleq \inf\{t \geq 0 : X_t \notin \Omega\}$ as the (first) exit time of X_t from Ω. Its expected value, under the initial condition $X_0 = x \in \Omega$, is denoted by

$$V(x) \triangleq E[\tau | X_0 = x].$$

Thus, $V(x)$ is the expected exit time of X_t from the safe set Ω starting from x at time 0. By a standard result of stochastic calculus [20], $V(x)$ is a solution to the following second order elliptic PDE on Ω with boundary value:

$$\begin{cases} \frac{1}{2}\sum_{i,j=1}^n \left(\sigma(x)\sigma(x)^T\right)_{ij} \frac{\partial^2 V}{\partial x_i \partial x_j}(x) \\ \qquad + \sum_{i=1}^n u_i(x)\frac{\partial V}{\partial x_i}(x) + 1 = 0, x \in \Omega \\ V(x) = 0, \quad x \in \partial\Omega. \end{cases} \tag{2}$$

Denote by $\mathcal{L}^\infty(\Omega; \mathbb{R}^n)$ the set of all bounded measurable vector fields on Ω. A feedback control $u(x)$ is called *feasible* if $u(x) \in \mathcal{L}^\infty(\Omega; \mathbb{R}^n)$. Our aim is to find a feasible control $u(x)$ with a fixed cost that can lead to the largest average exit time of X_t from Ω. For a given feedback control law $u(x)$ on Ω, its total cost is defined as

$$J(u) \triangleq \int_\Omega \|u(x)\|^2 dx,$$

while its effectiveness is measured by the performance index

$$W(u) \triangleq \int_\Omega \lambda(x) w(V(x)) dx,$$

where λ is a nonnegative function on Ω; and $w : \mathbb{R}_+ \to \mathbb{R}_+$ is monotonically increasing on $\mathbb{R}_+ = [0, \infty)$ with $w(0) = 0$. For example, $w(x) = x^k, x \geq 0$, for some positive integer k.

Problem 1 (OETC Problem): Find the feedback control law $u(x) \in \mathcal{L}^\infty(\Omega; \mathbb{R}^n)$ that achieves the largest $W(u)$ with a cost $J(u) \leq J_0$ for some positive constant J_0.

From the observation that controls with larger cost have the potential of achieving larger $W(u)$, an equivalent version of the above OETC problem can be formulated as follows.

Problem 2 (Dual OETC Problem): Find the feedback control law $u(x) \in \mathcal{L}^\infty(\Omega; \mathbb{R}^n)$ with the least cost $J(u)$ subject to $W(u) \geq W_0$ for some positive constant $W_0 \geq W(0)$.

Note that even under zero control $u \equiv 0$, it will still take the state X_t positive time to exit Ω from inside Ω, i.e., $W(0) > 0$. Hence, the requirement $W_0 \geq W(0)$ in Problem 2 is to avoid the trivial solution $u \equiv 0$. Solving any one of the above two problems is equivalent to solving the other [15]. In the rest of this paper, we may switch freely between these two formulations.

A necessary condition for the optimal solutions to the OETC problem is given in the following lemma.

Lemma 1 ([16]): Let $u(x)$ be an optimal solution to the OETC problem, and let $V(x)$ be the corresponding expected exit time, i.e., the solution to equation (2). Then V is positive on Ω and vanishes on $\partial\Omega$. Furthermore, it satisfies $1 + \frac{1}{2}\sum_{i,j=1}^n \left(\sigma(x)\sigma(x)^T\right)_{ij} \frac{\partial^2 V}{\partial x_i \partial x_j}(x) \leq 0$ almost everywhere on Ω.

Remark 1: The positiveness of V on Ω follows immediately from its definition. By (2), the condition that $1 + \frac{1}{2}\sum_{i,j=1}^n \left(\sigma(x)\sigma(x)^T\right)_{ij} \frac{\partial^2 V}{\partial x_i \partial x_j}(x) \leq 0$ is equivalent to $u \cdot \nabla V \geq 0$, i.e., the optimal control should always point toward the direction along which the expected exit time V increases the fastest, an intuitively obvious conclusion.

It has been proved in [15] that the function $V(x)$ corresponding to the optimal control on a radially symmetric domain is radially symmetric when the state dimension $n \geq 2$. The proof makes use of the isoperimetric inequality which is invalid when $n = 1$. In the rest of this paper, we shall focus on the one-dimensional OETC problem and adopt a different approach to prove the symmetry property of its solutions.

B. One-Dimensional OETC Problem

When $n = 1$, the safe set Ω becomes an open interval $(-a, a)$. The PDE (2) governing the expected exit time V is reduced to the following ODE with boundary condition:

$$\begin{cases} \sigma^2(x)V''(x) + 2u(x)V'(x) + 2 = 0, \quad x \in (-a, a) \\ V(-a) = V(a) = 0. \end{cases} \tag{3}$$

The uniformly elliptic assumption is simply

$$\sigma(x) \geq \sigma_0, \quad \forall x \in (-a, a).$$

Remark 2: In the case of constant diffusion term $\sigma(x) \equiv \sigma$, the ODE (3) has the explicit analytical solution:

$$\begin{aligned} V(x) =& 2\beta(x)\int_x^a e^{2\alpha(x)}\, dx + 2\int_{-a}^x e^{2\alpha(x)}\beta(x)\, dx \\ &- \frac{2\beta(x)}{\beta(a)}\int_{-a}^a e^{2\alpha(x)}\beta(x)\, dx, \end{aligned}$$

where

$$\alpha(x) \triangleq \tfrac{1}{\sigma^2}\int_{-a}^x u(x)\, dx \text{ and } \beta(x) \triangleq \tfrac{1}{\sigma^2}\int_{-a}^x e^{-2\alpha(x)}\, dx,$$

provided that u is bounded on $(-a, a)$.

Therefore, the 1-D OETC problem is equivalent to the following variational problem.

Problem 3 (1D OETC Problem): Find the optimal control $u : (-a, a) \to \mathbb{R}$ that

$$\begin{aligned} &\text{maximize} \quad W(u) = \int_{-a}^a \lambda w(V)\, dx \\ &\text{subject to} \quad J(u) = \int_{-a}^a u^2\, dx \leq J_0 \text{ and the ODE (3).} \end{aligned}$$

By Lemma 1, the optimal expected exit time satisfies $1 + \frac{1}{2}\sigma^2(x)V''(x) \leq 0$, hence the following result.

Lemma 2: Let $u \in \mathcal{L}^\infty(\Omega) \triangleq \mathcal{L}_\infty(\Omega; \mathbb{R})$ be an optimal solution to Problem 3, and let V be the solution to the ODE (3) corresponding to u. Then V is strictly concave on Ω.

Denote by $\mathcal{V}$ the set of all functions $V : (-a, a) \to \mathbb{R}^+$ that are twice differentiable, strictly concave, and vanish at $x = -a$ and $x = a$. Let $\mathcal{U}$ be the set of feasible controls

$u \in \mathcal{L}_\infty(\Omega)$ whose corresponding V functions belong to $\mathcal{V}$. By Lemma 2, the optimal solutions of Problem 3 must be in $\mathcal{U}$. Thus, it suffices to only consider the controls in $\mathcal{U}$ (or equivalently the expected exit time functions in $\mathcal{V}$) in studying the 1-D OETC problem. Solving Problem 3 is nontrivial even when the analytic solution to the ODE (3) is available.

In the rest of this paper, we will prove the following symmetry property of the 1-D OETC problem.

Theorem 1: Suppose that $\lambda(x)$ is an even function, i.e., $\lambda(-x) = \lambda(x)$, for $x \in (-a, a)$. Let $V(x)$ be the expected exit time of the process X_t in equation (1) with the diffusion term $\sigma(x) \geq \sigma_0$ under a control $u(x)$. Then there exists a skew symmetric control $\hat{u}(x)$, i.e., $\hat{u}(-x) = -\hat{u}(x)$, $x \in (-a, a)$, with the same control cost as u, $J(\hat{u}) = J(u)$, such that for the solution process $\hat{X}_t$ to

$$d\hat{X}_t = u(\hat{X}_t)dt + \sigma_0 dB_t, \tag{4}$$

the expected exit time $\hat{V}(x)$ of $\hat{X}_t$ from $(-a, a)$ starting from x satisfies $W(\hat{u}) \geq W(u)$.

If $\sigma(x) \equiv \sigma_0$ is constant, then equation (4) is the same as (1). Hence we have the following main result.

Corollary 1: Suppose $\lambda(x)$ is even and $\sigma(x) \equiv \sigma_0$ is constant. Then if solutions to the 1D OETC Problem 3 exist, at least one solution is skew symmetric on $(-a, a)$.

Remark 3: In [16, Corollary 9.4], it is shown that under the additional assumption that $\|u\| \leq M$ for a finite constant M, solutions to the OETC Problem exist. In general, however, solutions may not exist in $L^\infty(\Omega; \mathbb{R})$.

III. Symmetrization of $V(x)$

Symmetrization is an operation that can transform an arbitrary function to a symmetric one. Symmetrization of general functions on multi-dimensional domains is discussed in details in [18], [19]. In this section, we shall use this method to transform the expected exit time function $V : (-a, a) \to \mathbb{R}$ to a symmetric function $V^* : (-a, a) \to \mathbb{R}$ and derive some important connections between $V(x)$ and $V^*(x)$ that are useful in proving Theorem 1.

Let V be an arbitrary function in $\mathcal{V}$. Then, V must be continuous and strictly concave on $\Omega = (-a, a)$, and vanish on the boundary of Ω. Let $M = \max\{V(x) : x \in \Omega\}$. For each value ρ in the range $[0, M]$ of V, denote by $\{V > \rho\} \triangleq \{x \in \Omega : V(x) \geq \rho\}$ the *ρ-superlevel set* of V.

Definition 1 (Distribution Function [17]): Let $f : \Omega \to [0, M]$ be a Lebesgue measurable function. The distribution function of f, denoted by $\mu_f : [0, M] \to \mathbb{R}^+$, is defined as

$$\mu_f(\rho) \triangleq |\{f \geq \rho\}|, \quad \forall \rho \in [0, M],$$

where $|\{f \geq \rho\}|$ denotes the Lebesgue measure of the set $\{f \geq \rho\}$.

Definition 2 (Symmetrization of V): The function $V^* : (-a, a) \to [0, M]$ is called the *(Schwarz) symmetrization* of V if $V^*(x) = V^*(-x)$, $\forall x \in (-a, a)$, and $\mu_V(\rho) = \mu_{V^*}(\rho)$, $\forall \rho \in [0, M]$.

We now use this definition to derive the symmetrized function V^*. Since V is strictly concave on $(-a, a)$ and $V(a) = V(-a) = 0$, V achieves its maximum value M

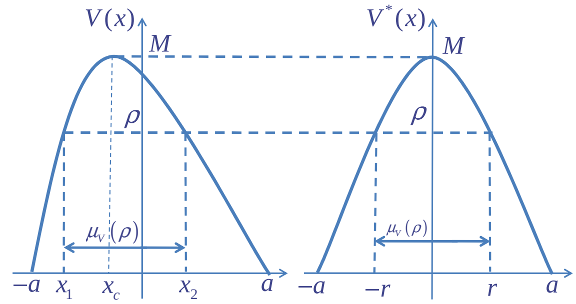

Fig. 1. Symmetrization of V

at a unique point $x_c \in (-a, a)$ (see Figure 1). Denote the restrictions of V on the two subintervals $[-a, x_c]$ and $[x_c, a]$ as

$$V_1 = V\big|_{[-a, x_c]}, \quad V_2 = V\big|_{[x_c, a]}.$$

Then V_1 and V_2 are monotone functions with inverse

$$x_1(\rho) = V_1^{-1}(\rho) \in [-a, x_c],$$
$$x_2(\rho) = V_2^{-1}(\rho) \in [x_c, a], \forall \rho \in [0, M].$$

For each $\rho \in [0, M]$, the superlevel set $\{V \geq \rho\}$ is the interval $[x_1(\rho), x_2(\rho)]$ with the length (distribution function)

$$\mu_V(\rho) = x_2(\rho) - x_1(\rho). \tag{5}$$

We can shift $\{V \geq \rho\}$ to obtain a symmetric interval $[-\frac{\mu_V(\rho)}{2}, \frac{\mu_V(\rho)}{2}]$ centered at the origin. Define V^* such that its superlevel set $\{V^* \geq \rho\}$ is exactly $[-\frac{\mu_V(\rho)}{2}, \frac{\mu_V(\rho)}{2}]$:

$$V^*\left[-\mu_V(\rho)/2\right] = V^*\left[\mu_V(\rho)/2\right] = \rho, \quad \forall \rho \in [0, M]. \tag{6}$$

The definition of V^* in (6) is in terms of ρ. To find its expression in terms of x, we note that since V is continuous and concave, its distribution function $\mu_V(\rho)$ must be continuous and strictly decreasing in ρ, and has the property

$$\mu_V(0) = |\Omega| = 2a, \quad \mu_V(M) = 0.$$

Denote by μ_V^{-1} the inverse function of μ_V, which is a strictly decreasing function that maps $[0, 2a]$ to $[0, M]$. Let $r = \mu_V(\rho)/2$, or equivalently, $\rho = \mu_V^{-1}(2r)$. Then r takes values in the range $[0, a]$; and the definition (6) becomes

$$V^*(-r) = V^*(r) = \mu_V^{-1}(2r), \quad \forall r \in [0, a]. \tag{7}$$

Obviously, V^* thus constructed is the symmetrization of V. For simplicity, in the rest of this paper, we shall drop the subscript and use $\mu(\cdot)$ to denote the distribution function of V (or equivalently the distribution function of V^*).

Some useful properties of symmetrization are summarized in the following lemma.

Lemma 3 ([19]): For an arbitrary concave continuous function V defined on the domain $\Omega = (-a, a)$, let V^* be its symmetrization defined on Ω. Then

1) V and V^* are equi-measurable, namely, $|\{V \geq \rho\}| = |\{V^* \geq \rho\}|$ for any ρ.
2) For any Lebesgue measurable function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\int_\Omega \phi[V]dx = \int_\Omega \phi[V^*]dx.$$

3) **(Hardy-Littlewood)** Let W be another function on Ω and W^* be its symmetrization. Then

$$\int_\Omega VW\,dx \le \int_\Omega V^*W^*\,dx. \tag{8}$$

IV. Proof of Main Result

This section is devoted to proving Theorem 1. Our strategy is to show that for an arbitrary control $u \in \mathcal{U}$, there always exists a (skew) symmetric control $\hat{u} \in \mathcal{L}^\infty(\Omega)$ that can achieve a no worse performance with the same cost, i.e., $W(\hat{u}) \ge W(u)$ and $J(u) = J(\hat{u})$. The proof is divided into two parts. In the first part, we shall use the symmetrization of V, namely, V^*, to construct a symmetric $\hat{u} \in \mathcal{L}_\infty(\Omega)$ for which $J(u) = J(\hat{u})$. Then in the second part we will prove that this $\hat{u}$ also satisfies that $W(\hat{u}) \ge W(u)$.

A. Construction of the Symmetric Control $\hat{u}$

Let V be a solution to the ODE (3) corresponding to an arbitrary control $u \in \mathcal{U}$. Our goal is to find a skew symmetric function $\hat{u} \in \mathcal{L}^\infty(\Omega)$ such that $J(u) = J(\hat{u})$ and $W(\hat{u}) \ge W(u)$. Notice that J and W are respectively functionals of the functions u and V related by the ODE (3). It is convenient to write both of them in terms of V. From (3) and Lemma 1, we have $u = |1 + \frac{1}{2}\sigma^2 V''|/V'$ whenever $V' \ne 0$. Using the same notations as in the last section, we know that $V'(x) = 0$ only when $x = x_c$. Therefore, the cost functional J can also be written as:

$$J(u) = \int_{-a}^{x_c} \frac{|1+\frac{1}{2}\sigma^2 V''|^2}{|V'|^2}dx + \int_{x_c}^{a} \frac{|1+\frac{1}{2}\sigma^2 V''|^2}{|V'|^2}dx.$$

To define $\hat{u}$, recall that V and V^* are related through their superlevel sets. For each $\rho \in [0, M]$, let $x_1(\rho)$, $x_2(\rho)$ and $\mu(\rho)$ be the same as in the last section. Since V is strictly concave, $x_1'(\rho) > 0$ and $x_2'(\rho) < 0$ for $\rho \in [0, M)$. The derivative of $\mu(\rho)$ with respect to ρ is

$$\mu'(\rho) = x_2'(\rho) - x_1'(\rho) = -(|x_1'(\rho)| + |x_2'(\rho)|). \tag{9}$$

Define

$$Q(\rho) \triangleq -\frac{d}{d\rho}\int_{\{V\ge\rho\}} u^2\,dx = -\frac{d}{d\rho}\int_{x_1(\rho)}^{x_2(\rho)} u^2\,dx. \tag{10}$$

Intuitively, $Q(\rho)\,d\rho$ is the part of the control cost $J(u)$ concentrated within the infinitesimal bands $\{\rho \le V \le \rho + d\rho\}$ sandwiched between the level sets $\{V = \rho\}$ and $\{V = \rho + d\rho\}$. Summing over all such bands, we have

Lemma 4: The function Q defined in (10) satisfies

$$\int_0^M Q(\rho)\,d\rho = \int_\Omega u^2\,dx.$$

Proof: By the definition of $Q(\rho)$, we have

$$\int_0^M Q(\rho)\,d\rho = -\int_0^M \frac{d}{d\rho}\left(\int_{\{V\ge\rho\}} u^2\,dx\right)d\rho$$
$$= \int_{\{V\ge 0\}} u^2\,dx - \int_{\{V \ge M\}} u^2\,dx = \int_\Omega u^2\,dx,$$

which is the desired result. ■

According to the definition of V^* in (7), each $\rho \in [0, M]$ corresponds to an $r = \mu(\rho)/2$ such that $V^*(r) = V^*(-r) = \rho$ (see Figure 1). Thus, the infinitesimal bands $\{\rho \le V \le \rho + d\rho\}$ corresponds to the union of two intervals $[r+dr, r] \cup [-r, -r-dr]$. We define the symmetric control $\hat{u}$ so that its cost on this union is the same as the cost of u on the band $\{\rho \le V \le \rho + d\rho\}$, namely, $Q(\rho)d\rho$. Specifically,

$$\begin{aligned}\hat{u}(r) = -\hat{u}(-r) &\triangleq -\sqrt{\frac{1}{2}Q(V^*(r))\cdot\frac{d\rho}{dr}}\\ &= -\frac{Q(V^*(r))^{1/2}}{|\mu'(V^*(r))|^{1/2}}, \quad r \in [0, a].\end{aligned} \tag{11}$$

Lemma 5: The symmetric control $\hat{u}$ defined above has the same cost as the original control u, namely, $J(\hat{u}) = J(u)$. Moreover, for any $r \in [0, a]$,

$$\int_0^r \hat{u}(x)\,dx = -\frac{1}{2}\int_{V^*(r)}^M Q(\eta)^{1/2}|\mu'(\eta)|^{1/2}\,d\eta. \tag{12}$$

Proof: For the first claim, by the symmetry of $\hat{u}$, we have

$$\int_\Omega \hat{u}^2\,dx = 2\int_0^a \hat{u}^2(r)\,dr = 2\int_0^a \frac{Q(V^*(r))}{|\mu'(V^*(r))|}\,dr.$$

Note that by (7), $V^*(r) = \mu^{-1}(2r)$ for $r \in [0, a]$; hence $\mu(V^*(r)) = 2r$. Taking the derivative yields

$$\mu'(V^*(r))\cdot\frac{d}{dr}V^*(r) = 2. \tag{13}$$

Thus, by a change of variable $\rho = V^*(r)$, we have

$$\begin{aligned}\int_\Omega \hat{u}^2\,dx &= -\int_0^a Q(V^*(r))\cdot\frac{d}{dr}V^*(r)\,dr\\ &= -\int_{V^*(0)}^{V^*(a)} Q(\rho)\,d\rho = \int_0^M Q(\rho)\,d\rho = \int_\Omega u^2\,dx,\end{aligned}$$

where the last step follows from Lemma 4.

To prove the second claim, perform a change of variables $\eta = V^*(x)$, or equivalently, $x = \mu(\eta)/2$, to obtain

$$\begin{aligned}\int_0^r \hat{u}(x)\,dx &= -\int_0^r \frac{Q(V^*(x))^{1/2}}{|\mu'(V^*(x))|^{1/2}}\,dx\\ &= \int_{V^*(r)}^M \frac{Q(\eta)^{1/2}}{|\mu'(\eta)|^{1/2}}\frac{\mu'(\eta)}{2}\,d\eta\\ &= -\frac{1}{2}\int_{V^*(r)}^M Q(\eta)^{1/2}|\mu'(\eta)|^{1/2}\,d\eta.\end{aligned}$$

■

The following lemma shows that the symmetric control $\hat{u}$ is feasible, namely, bounded.

Lemma 6: The symmetric control $\hat{u}$ defined in (11) is feasible, i.e., $\hat{u} \in \mathcal{L}^\infty(\Omega)$.

Proof: By (11), it suffices to show that $Q(\rho)/|\mu'(\rho)|$ is bounded on $(0, M]$. Since $u \in \mathcal{L}^\infty(\Omega)$, there exists a $K < \infty$ such that $\sup_{x\in\Omega} u(x) = K$. By (10) and (9), for each $\rho \in (0, M)$, we have

$$\begin{aligned}\frac{Q(\rho)}{|\mu'(\rho)|} &= \frac{u^2(x_1(\rho))x_1'(\rho) - u^2(x_2(\rho))x_2'(\rho)}{|x_1'(\rho)| + |x_2'(\rho)|}\\ &= \frac{u^2(x_1(\rho))|x_1'(\rho)| + u^2(x_2(\rho))|x_2'(\rho)|}{|x_1'(\rho)| + |x_2'(\rho)|} \le K^2.\end{aligned}$$

Taking the limit, the above is also true at $\rho = M$. Hence $\hat{u} \le K$ on Ω. ■

To sum up, for each candidate control $u \in \mathcal{U}$, we can find a feasible symmetric control $\hat{u} \in \mathcal{L}^\infty(\Omega)$ for which $J(u) = J(\hat{u})$. In the next subsection, we will show that this $\hat{u}$ also satisfies $W(\hat{u}) \ge W(u)$.

B. Comparison Theorem

With the symmetric control $\hat{u}$, let $\hat{V}$ be the corresponding expected exit time of the solution process to equation (1) with the diffusion term given by $\sigma(\cdot) \equiv \sigma_0$. Then $\hat{V}$ is the symmetric (i.e., even) solution to the ODE

$$\begin{cases} \sigma_0^2 \hat{V}''(x) + 2\hat{u}(x)\hat{V}'(x) + 2 = 0, & x \in (-a,a) \\ \hat{V}(a) = \hat{V}(-a) = 0. \end{cases} \tag{14}$$

Our goal is show that $W(\hat{u}) \ge W(u)$, namely, $\int_\Omega \lambda w(\hat{V})dx \ge \int_\Omega \lambda w(V)$. This is a direct consequence of the following more general *comparison theorem* of ODEs.

Theorem 2 (Comparison Theorem): Let V be the solutions to the ODE (3) corresponding to an arbitrary control $u \in \mathcal{U}$, and $\hat{V}$ be the solution to the ODE (14) corresponding to the symmetric control $\hat{u}$ in (11). Then

$$\hat{V}(x) \ge V^*(x), \quad \forall x \in [-a,a],$$

where V^* is the symmetrization of V.

We first demonstrate how to use this comparison theorem to prove our main result Theorem 1. The proof of the comparison theorem will be given afterward.

Proof: [Theorem 1] By Theorem 2, given any control $u \in \mathcal{U}$ and its corresponding expected exit time V on Ω, we can find a symmetric control $\hat{u} \in \mathcal{L}^\infty(\Omega)$ with the same cost as u whose corresponding expected exit time $\hat{V}$ for the process (1) with a constant diffusion term σ_0 satisfies $\hat{V}(x) \ge V^*(x)$ for all $x \in \Omega$. Thus,

$$\begin{aligned} W(\hat{u}) &= \int_\Omega \lambda(x) w(\hat{V}(x))\,dx \ge \int_\Omega \lambda(x) w(V^*(x))\,dx \\ &\ge \int_\Omega \lambda(x) w(V(x))\,dx = W(u). \end{aligned} \tag{15}$$

Note that the first inequality above follows from the comparison theorem and the monotonicity of $w(\cdot)$; and the second inequality follows from the Hardy-Littlewood inequality as $\lambda^* = \lambda$ and $[w(V)]^* = w(V^*)$. As a result, the feasible symmetric $\hat{u}$ has the same cost and no worse performance than u, which proves Theorem 1. ■

In the rest of this section, we will prove the comparison theorem through a series of lemmas regarding the solution $V(x)$ to the ODE (3), i.e., the expected exit time of process (1) with the diffusion term $\sigma(x)$.

Lemma 7: For each $\rho \in (0, M)$, the function $P(\rho)$ defined as

$$P(\rho) \triangleq -\frac{d}{d\rho}\int_{x_1(\rho)}^{x_2(\rho)} |V'|^2\,dx \tag{16}$$

satisfies that: (i) $P(\rho)|\mu'(\rho)| \ge 4$; (ii) $P(\rho) \to 0$ as $\rho \to M$; (iii) $Q(\rho)^{\frac{1}{2}} P(\rho)^{\frac{1}{2}} \ge \mu'(\rho) - \frac{\sigma_0^2}{2}P'(\rho)$.

Proof: (i) By (16), for $\rho \in (0, M)$, we have

$$\begin{aligned} P(\rho) &= |V'(x_1(\rho))|^2 x_1'(\rho) - |V'(x_2(\rho))|^2 x_2'(\rho) \\ &= \frac{1}{|x_1'(\rho)|} + \frac{1}{|x_2'(\rho)|}. \end{aligned} \tag{17}$$

Here the last step follows since $V(x_i(\rho)) = \rho$ implies that $V'(x_i(\rho))x_i'(\rho) = 1$ for $i = 1, 2$. Equivalently,

$$P(\rho) = V'(x_1(\rho)) - V'(x_2(\rho)). \tag{18}$$

Thus, by the Cauchy-Schwarz inequality,

$$\begin{aligned} P(\rho)|\mu'(\rho)| &= \left(\frac{1}{|x_1'(\rho)|} + \frac{1}{|x_2'(\rho)|}\right)(|x_1'(\rho)| + |x_2'(\rho)|) \\ &\ge (1+1)^2 = 4. \end{aligned}$$

(ii) As $\rho \to M$, $x_i'(\rho) = 1/V'(x_i(\rho)) \to \infty$ for $i = 1, 2$; thus $P(\rho) \to 0$.

(iii) The ODE (3) implies that $uV' = -(1 + \frac{1}{2}\sigma^2 V'')$. Since $\sigma \ge \sigma_0$ by the uniformly elliptic assumption and $V'' < 0$ as $V \in \mathcal{V}$ is strictly concave, we have $uV' \ge -(1 + \frac{1}{2}\sigma_0^2 V'')$. Hence, for $\rho \in (0, M)$ and small $\Delta\rho > 0$, denote the set $\{\rho \le V \le \rho + \Delta\rho\} \triangleq \{x \in \Omega : \rho \le V(x) \le \rho + \Delta\rho\}$. Then,

$$\begin{aligned} &\int_{\{\rho\le V\le\rho+\Delta\rho\}} uV'\,dx \ge -\int_{\{\rho\le V\le\rho+\Delta\rho\}} \left(1+\frac{1}{2}\sigma_0^2 V''\right)dx \\ &= \int_{x_1(\rho+\Delta\rho)}^{x_2(\rho+\Delta\rho)} \left(1+\frac{1}{2}\sigma_0^2 V''\right)dx - \int_{x_1(\rho)}^{x_2(\rho)} \left(1+\frac{1}{2}\sigma_0^2 V''\right)dx \\ &= \mu(\rho+\Delta\rho) - \frac{1}{2}\sigma_0^2 P(\rho+\Delta\rho) - \mu(\rho) + \frac{1}{2}\sigma_0^2 P(\rho), \end{aligned}$$

where in the last step we have used (18). On the other hand, by the Cauchy-Schwarz inequality,

$$\begin{aligned} &\left(\int_{\{\rho\le V\le\rho+\Delta\rho\}} u^2\,dx\right)^{1/2} \left(\int_{\{\rho\le V\le\rho+\Delta\rho\}} |V'|^2\,dx\right)^{1/2} \\ &\ge \int_{\{\rho\le V\le\rho+\Delta\rho\}} uV'\,dx. \end{aligned}$$

Combining the above two inequalities, we have

$$\begin{aligned} &\left(\int_{\{\rho\le V\le\rho+\Delta\rho\}} u^2\,dx\right)^{1/2} \left(\int_{\{\rho\le V\le\rho+\Delta\rho\}} |V'|^2\,dx\right)^{1/2} \\ &\ge [\mu(\rho+\Delta\rho) - \mu(\rho)] - \frac{1}{2}\sigma_0^2[P(\rho+\Delta\rho) - P(\rho)]. \end{aligned}$$

The desired property (iii) is then obtained by dividing both sides by $\Delta\rho$ and taking the limit $\Delta\rho \to 0$. ■

The above lemma can be used to prove an important inequality as described in the following lemma.

Lemma 8: For each $\rho \in (0, M)$, the following inequality holds:

$$\begin{aligned} &\sigma_0^{-2} e^{\int_\rho^M \sigma_0^{-2} Q(\eta)^{1/2}|\mu'(\eta)|^{1/2}\,d\eta} \\ &\cdot \int_\rho^M \mu'(\rho) e^{-\int_\rho^M \sigma_0^{-2} Q(\eta)^{1/2}|\mu'(\eta)|^{1/2}\,d\eta}\,d\rho \le \frac{2}{\mu'(\rho)}. \end{aligned} \tag{19}$$

Proof: By properties (i) and (iii) of Lemma 7, we have

$$\begin{aligned} \mu'(\rho) &\le \frac{\sigma_0^2}{2}P'(\rho) + Q(\rho)^{1/2}P(\rho)^{1/2} \cdot \frac{1}{2}P(\rho)^{1/2}|\mu'(\rho)|^{1/2} \\ &= \frac{\sigma_0^2}{2}P'(\rho) + \frac{1}{2}Q(\rho)^{1/2}|\mu'(\rho)|^{1/2}P(\rho). \end{aligned}$$

By the Gronwall inequality and the fact that $P(M)=0$, the above differential inequality implies that for $\rho\in(0,M)$,

$$P(\rho)\le -2\sigma_0^{-2}e^{\int_\rho^M \sigma_0^{-2}Q(\eta)^{1/2}|\mu'(\eta)|^{1/2}\,d\eta}\cdot\int_\rho^M \mu'(\rho)e^{-\int_\rho^M \sigma_0^{-2}Q(\eta)^{1/2}|\mu'(\eta)|^{1/2}\,d\eta}\,d\rho. \tag{20}$$

Note that $\int_\rho^M \sigma_0^{-2}Q(\eta)^{1/2}|\mu'(\eta)|^{1/2}\,d\eta$ in (20) is well defined as both $Q(\rho)$ and $|\mu'(\rho)|$ are integrable. Since by Lemma 7, $P(\rho)\ge 4/|\mu'(\rho)| = -4/\mu'(\rho)$ for $\rho\in[0,M)$, (20) then implies (19). ∎

We are now ready to prove the comparison theorem.

Proof: [Comparison Theorem] Since $\hat V$ is symmetric (even) and differentiable, we know that $\hat V'(0)=0$. By focusing on the positive half of the domain Ω, the ODE (14) is equivalent to

$$\begin{aligned}&\sigma_0^2\hat V''(r)+2\hat u(r)\hat V'(r)+2=0,\quad r\in[0,a],\\ &V'(0)=0,\ V(a)=0.\end{aligned} \tag{21}$$

Multiplying (21) by $\sigma_0^{-2}e^{2\int_0^r \sigma_0^{-2}\hat u(x)\,dx}$ and integrating from 0 to r, we have

$$e^{2\int_0^r \sigma_0^{-2}\hat u(x)\,dx}\hat V'(r) = -2\sigma_0^{-2}\int_0^r e^{2\int_0^r \sigma_0^{-2}\hat u(x)\,dx}\,dr,$$

for $r\in[0,a]$. Hence, by (12),

$$\hat V'(r) = -2\sigma_0^{-2}e^{\int_{V^*(r)}^M \sigma_0^{-2}Q(\eta)^{1/2}|\mu'(\eta)|^{1/2}\,d\eta}\cdot\int_0^r e^{-\int_{V^*(r)}^M \sigma_0^{-2}Q(\eta)^{1/2}|\mu'(\eta)|^{1/2}\,d\eta}\,dr.$$

A change of variable $\rho=V^*(r)$, i.e., $r=\mu(\rho)/2$, in the integral on the right hand side leads to

$$\begin{aligned}\hat V'(r) &= \sigma_0^{-2}e^{\int_{V^*(r)}^M \sigma_0^{-2}Q(\eta)^{1/2}|\mu'(\eta)|^{1/2}\,d\eta}\cdot\int_{V^*(r)}^M \mu'(\rho)e^{-\int_\rho^M \sigma_0^{-2}Q(\eta)^{1/2}|\mu'(\eta)|^{1/2}\,d\eta}\,d\rho\\ &\le \frac{2}{\mu'(V^*(r))} = \frac{d}{dr}V^*(r),\end{aligned} \tag{22}$$

for $r\in(0,a)$. Here we have used (19) with ρ replaced by $V^*(r)$, and (13). As $\hat V(a)=V^*(a)=0$, (22) then implies that

$$\begin{aligned}\hat V(x) &= \hat V(a)-\int_x^a \hat V'(r)\,dr \ge V^*(a)-\int_x^a \frac{d}{dr}V^*(r)\,dr\\ &= V^*(x),\quad \forall x\in[0,a].\end{aligned}$$

By the symmetry of $\hat V(x)$ and $V^*(x)$, it follows that $\hat V(x)\ge V^*(x)$ for all $x\in[-a,a]$. ∎

V. Example

We next present an example for which the result in Corollary 1 can help to find the analytic solution to the 1-D OETC problem. In this example, we assume that $\sigma(\cdot)\equiv 1$ is constant, $\lambda(\cdot)$ is the Kronecker delta function $\delta(\cdot)$, and $w(\cdot)$ is the identity function. Then, by the property of the delta function, the performance index becomes

$$W(u)=\int_\Omega \lambda(x)w(V(x))\,dx = \int_\Omega \delta(x)V(x)\,dx = V(0).$$

Here, we adopt a dual formulation (see Problem 2) of the OETC problem to minimize the control energy $J(u)=\int_{-a}^a u^2\,dx$ subject to the constraint $W(u)=V(0)\ge T$ for some constant $T\ge W(0)$. In other words, we want to find the control with the least energy so that the expected exit time, starting from the center of $\Omega=(-a,a)$, is no smaller than T. When $u\equiv 0$, the solution to the ODE (3) can be easily found to be $V(x)=-x^2+a^2$. Therefore, the constant T should be chosen larger than $W(0)=a^2$ to avoid a trivial solution.

By Theorem 1, the optimal control u is odd on Ω, and its corresponding V is even on Ω. Hence, $V'(0)=0$, and $J(u)=2\int_{-a}^0 u^2\,dx$. Since $\sigma(\cdot)\equiv 1$, by focusing on the subinterval $[-a,0]$, V satisfies the ODE

$$V''(x)+2u(x)V'(x)+2=0,\quad V'(0)=0,\ V(-a)=0.$$

Define the state variables $y_1(x)=V(x)$ and $y_2(x)=V'(x)$, for $x\in[-a,0]$. Then the above problem is equivalent to the following optimal control problem:

$$\begin{aligned}&\text{minimize}\int_{-a}^0 u^2\,dx\\ &\text{subject to}\begin{cases}y_1'=y_2,\\ y_2'=-2uy_2-2,\end{cases}\quad \forall x\in[-a,0],\\ &\qquad y_1(-a)=0,\ y_1(0)=T,\ y_2(0)=0.\end{aligned} \tag{23}$$

Define the Hamiltonian

$$H=u^2+\lambda_1y_2+\lambda_2(-2uy_2-2).$$

By the Maximum Principle, the following costate equations can be obtained:

$$\lambda_1' = -\frac{\partial H}{\partial y_1}=0,\quad \lambda_2'=-\frac{\partial H}{\partial y_2}=2u\lambda_2-\lambda_1, \tag{24}$$

with the boundary conditions $\lambda_2(-a)=0$, and the optimal control u is determined by

$$\frac{\partial H}{\partial u}=2u-2\lambda_2y_2=0\quad\Rightarrow\quad u=\lambda_2y_2. \tag{25}$$

Note that λ_1 is constant, and with u given in (25), the resulting Hamiltonian H should also be constant:

$$2\lambda_2-\lambda_1y_2+\lambda_2^2y_2^2\equiv C.$$

Solving for λ_2, and plugging the resulting λ_2 and u in (25) into the state equation (23), we have

$$\begin{aligned}&y_2'=-2\sqrt{1+y_2^2(\lambda_1y_2+C)}=-2\sqrt{1+\lambda_1y_2^2[y_2-y_2(-a)]},\\ &\text{with}\quad y_2(0)=0.\end{aligned} \tag{26}$$

Hence, the dynamics of y_2 depends on the unknown parameters λ_1 and $y_2(-a)$, but is decoupled from the dynamics of λ_2. Integrating from $x=-a$ to $x=0$, we obtain the first constraint on λ_1 and $y_2(-a)$:

$$\Psi(\lambda_1,y_2(-a))\triangleq\int_0^{y_2(-a)}\frac{dy_2}{\sqrt{1+\lambda_1y_2^2[y_2-y_2(-a)]}}=2a.$$

Another constraint is that $y_1(0)=T$, i.e., $\int_{-a}^0 y_2\,dx=T$. It can be verified that these two constraints uniquely determine the pair of parameters $\lambda_1\le 0$ and $y_2(-a)\ge 2a$ if $T\ge a^2$.

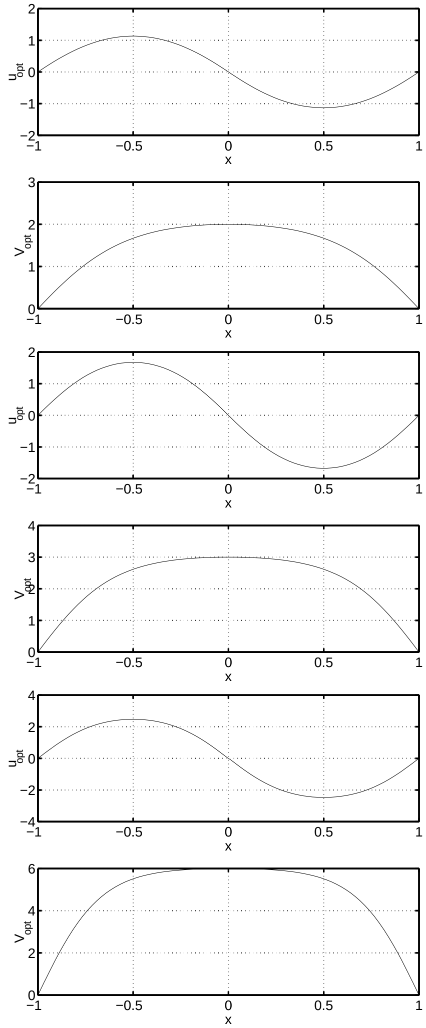

Fig. 2. Plots of optimal $u(x)$ and $V(x)$ when $a = 1$, with $T = 2$ (left), $T = 3$ (middle), and $T = 6$ (right).

Having determined λ_1 and $y_2(-a)$, the state $y_2(x)$, and hence $V(x) = y_1(x)$, can be found by integrating (26) over $[-a, 0]$, which are properly defined elliptic integral functions and their inverses.

Figure 2 plots the optimal $u(x)$ and $V(x)$ computed as above when $a = 1$ for $T = 1,\ 3,\ 6$ respectively. The left column is the optimal $u(x)$ and $V(x)$ for $T = 1$, the middle column for $T = 3$, and the right column for $T = 6$. As expected, as T increases, the optimal feedback control law $u(x)$ becomes more aggressive. An interesting fact is that the optimal u is nearly zero around the center of the interval, while much of the effort is spent roughly midway between the center and the boundary.

VI. CONCLUSION

This paper studies the 1-D OETC problem. A comparison theorem is established that compares the solutions to two ODEs arising in the 1-D OETC problem, one with an arbitrary control and the other with a symmetric control. The symmetry of solutions to the 1-D OETC problem is proved using the comparison theorem. This together with our previous results completely establishes the radial symmetry property of the solutions to the OETC problem with an arbitrary state dimension.

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