Application of reachability analysis for stochastic hybrid systems to aircraft conflict prediction

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***Abstract*—Aircraft conflict prediction can be naturally formulated as a reachability analysis problem in a stochastic hybrid system framework. In this paper, a switching diffusion model is adopted to predict the future positions of an aircraft following a given flight plan, and the probability that the aircraft will enter an unsafe region of the airspace is estimated through a numerical algorithm for reachability computations. Simulation results are reported to illustrate the approach.**

I. INTRODUCTION

Demand for air travel has been increasing rapidly in the last few decades and this poses great challenges to the current, mostly human-operated, Air Traffic Management (ATM) system, where Air Traffic Controllers (ATCs) are in charge of guaranteeing safety of air travel. The introduction of a methodology to predict *aircraft conflicts*, i.e., situations where an aircraft comes too close to another aircraft or enters a forbidden zone, could reduce the ATCs' workload, thus increasing capacity without compromising safety.

Several characteristics of aircraft dynamics need to be taken into consideration when developing methodologies for aircraft conflict prediction. First, aircraft typically try to follow nominal piecewise linear trajectories consisting of a sequence of way-points specified by the ATCs, and the onboard Flight Management System (FMS) will try to bring the aircraft back to the nominal path once deviated. Second, aircraft motions are subject to various random perturbations such as wind, air turbulence, etc. Last, but most importantly, aircraft dynamics may exhibit several distinct modes, for example, keeping a constant heading, turning, ascending, descending, and may switch modes at proper times when following the nominal paths.

The random perturbations and the mode-switching behavior exhibited in the aircraft motions make Stochastic Hybrid Systems (SHSs) the natural choice for predicting the aircraft future positions, [1]-[3]. The prediction model introduced in Section II tries to capture the relevant features determining the aircraft motion, while being sufficiently simple to allow for computations. It can be viewed as a simplified version of the SHS model introduced in [4] for simulation purposes and is in essence a switching diffusion with mode switchings used to describe the aircraft turns when reaching way-points.

In the adopted framework, the problem of aircraft conflict prediction becomes an instance of the general *reachability analysis* problem for a SHS, where the goal is to calculate the probability that the state of the system will enter a certain subset of the state space, starting from some given initial condition. Reachability analysis of SHSs is considerably more difficult than its counterpart for deterministic hybrid systems. In [5] and [6], an upper bound on the probability of reachability events is derived based on the theory of Dirichlet forms and on the introduction of barrier certificates. In [7] reachability for SHSs is studied in a discrete time setting. In [1] and [8], randomized algorithms are suggested for estimating the probability of conflict in a stochastic hybrid framework.

In Section III, we illustrate a numerical approximation scheme to compute an asymptotically convergent estimate of the probability that an aircraft will enter a forbidden area of the airspace such as special use airspace areas, and bad weather or congested zones that could make the flight uncomfortable or even unsafe. This scheme rests on the weak approximation of the switching diffusion for predicting the aircraft future positions by a Markov chain, and is based on a methodology for reachability computations that was first introduced in [2] for systems described by stochastic differential equations whose coefficients change value at prescribed time instants. The method was then extended to SHSs in [9], thus proving that the Markov chain weak approximation result in [10] is still valid in a hybrid setting. The results obtained by applying the proposed conflict prediction algorithm to a numerical example are shown in Section IV.

II. A SWITCHING DIFFUSION MODEL TO PREDICT THE AIRCRAFT POSITION

Consider an aircraft that is following a flight plan consisting of an ordered sequence of way-points $\{O_i, i = 1, 2, \dots, M+1\}$ at some fixed altitude during some time horizon $T = [0, t_f]$. Ideally, the aircraft should fly at some constant velocity $v \in \mathbb{R}^+$ along the path given by the concatenation of the ordered sequence $\{I_i, i = 1, 2, \dots, M\}$ of line segments I_i with starting point O_i and ending point O_{i+1}, $i = 1, 2, \dots, M$. Deviations from this path are caused by the wind affecting the aircraft position and by limitations in the aircraft dynamics in performing sharp turns, resulting in cross-track error. The onboard 3D FMS tries to reduce the cross-track error by issuing corrective actions based on the current geometric deviation of the aircraft from the nominal path (without taking into account timing specifications, however).

The model to predict the aircraft future positions during the look-ahead time horizon T is hybrid with the continuous state component $\mathbf{x}(t) \in \mathbb{R}^2$ representing the aircraft position with respect to some global coordinate system $(0, x_1, x_2)$ on the horizontal plane at the assigned fixed altitude, and the discrete state component $\mathbf{q}(t) \in \mathscr{Q} := \{1, 2, \dots, M\}$ representing the line segment that the aircraft is tracking at time $t \in T$. The aircraft position $\mathbf{x}(t) \in \mathbb{R}^2$ is governed by a stochastic differential equation (SDE):

$$d\mathbf{x}(t) = v\,[\cos(\theta(t))\ \sin(\theta(t))]^T dt + f(\mathbf{x}(t))dt + \sigma d\mathbf{w}(t), \qquad (1)$$

where the 2D standard Brownian motion $\mathbf{w}(t)$ accounts for the different sources of uncertainty affecting the aircraft motion. The variance of $\mathbf{w}(t)$ is modulated by the constant $\sigma \in \mathbb{R}^+$. As for the other terms in equation (1), $\theta(t)$ is the heading angle at time $t \in T$, whereas $f : \mathbb{R}^2 \to \mathbb{R}^2$ represents the nominal contribution of the wind disturbance to the aircraft velocity.

The corrective action of the 3D FMS is modeled by setting the heading angle $\theta(t)$ as an appropriate function of $\mathbf{x}(t)$. For each segment I_i of the reference path $\{I_i, i \in \mathscr{Q}\}$, we define as reference heading the angle $\Psi_i = \arg(x_{1,i+1} - x_{1,i} + j(x_{2,i+1} - x_{2,i}))$ that segment I_i joining $O_i = (x_{1,i}, x_{2,i})$ and $O_{i+1} = (x_{1,i+1}, x_{2,i+1})$ makes with the positive x_1-axis of the reference coordinate frame (see Fig. 1). Suppose that the aircraft is tracking the line segment I_i, for some $i \in \mathscr{Q}$, and is currently at a position x not on I_i. For the aircraft to get on the reference segment I_i as quickly as possible, it should assume a heading, called correction heading, that is orthogonal to and points towards I_i: $\Psi_c(x,i) = \Psi_i - \mathrm{sgn}(d(x,i))\frac{\pi}{2}$, where $d : \mathbb{R}^2 \times \mathscr{Q} \to \mathbb{R}$ denotes the cross-track error function $d((x_1,x_2),i) = -\sin(\Psi_i)(x_1 - x_{1,i}) + \cos(\Psi_i)(x_2 - x_{2,i})$, and $\mathrm{sgn} : \mathbb{R} \to \{-1, 0, +1\}$ denotes the sign

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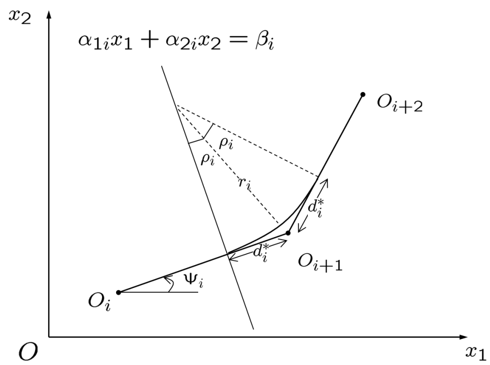

Fig. 1. Reference frame for the "fly-by" turning method.

function with $\mathrm{sgn}(0) = 0$. On the other hand, the aircraft should also head towards its next destination way-point O_{i+1}. In order to balance between these two objectives, the heading θ as specified by the FMS is modeled by a convex combination of the reference heading Ψ_i and the correction heading Ψ_c:

$$\theta = u(x,i) = \gamma(x,i)\Psi_c(x,i) + (1-\gamma(x,i))\Psi_i \tag{2}$$

with the coefficient of the convex combination taken to be a growing function of the absolute value of the cross-track error $\gamma(x,i) = \min\left(1, \frac{|d(x,i)|}{d_{\mathrm{m}}}\right)$. Here, $d_{\mathrm{m}} > 0$ is a threshold value for the cross-track error: the more closely the cross-track error approaches d_{m}, the more the aircraft will follow the correction heading $\Psi_c(x,i)$ rather than its reference heading Ψ_i. Note that the resulting function $u(\cdot,i)$ is continuous because $\gamma(\cdot,i)$ and $d(\cdot,i)$ are continuous.

Let $i \in \mathscr{Q}$ be the index of the reference line segment that the aircraft is tracking during some time interval $[t, t+\Delta t)$. Then, by plugging (2) into (1), we obtain that the dynamics of the aircraft during $[t, t+\Delta t)$ is governed by the SDE:

$$d\mathbf{x}(t) = a(\mathbf{x}(t),i)dt + \sigma d\mathbf{w}(t), \tag{3}$$

where we set $a(x,i) := v[\cos(u(x,i)) \sin(u(x,i))]^T + f(x),\ x \in \mathbb{R}^2$. While the aircraft is tracking line segment I_i, its predicted position is then described as a diffusion process with local characteristics determined by the drift term $a(\cdot,i)$ and the diffusion matrix σI.

The switching law from line segment I_i to the next one I_{i+1} is determined according to the commonly used "fly-by" method of performing turns, where the aircraft turns from I_i to I_{i+1} not by precisely passing through the way-point O_{i+1} but by "cutting the corner." In the higher-order aircraft model in [4], the turn starts when the aircraft enters the half-plane $\{(x_1,x_2) \in \mathbb{R}^2 : \alpha_{1i}x_1 + \alpha_{2i}x_2 \geq \beta_i\}$, whose boundary line $\alpha_{1i}x_1 + \alpha_{2i}x_2 = \beta_i$ is chosen so that an aircraft tracking the reference line segment I_i can fly with constant velocity v along an arc of circle joining I_i with I_{i+1} (see Fig. 1). If we denote by d_i^* the distance from the way-point O_{i+1} at which an aircraft flying exactly on line segment I_i should start turning, then, $\alpha_{1i} = \frac{x_{1,i+1}-x_{1,i}}{\|x_{i+1}-x_i\|}$, $\alpha_{2i} = \frac{x_{2,i+1}-x_{2,i}}{\|x_{i+1}-x_i\|}$, $\beta_i = \frac{x_{1,i+1}(x_{1,i+1}-x_{1,i})+x_{2,i+1}(x_{2,i+1}-x_{2,i})}{\|x_{i+1}-x_i\|} - d_i^*$. The expression $d_i^* = \frac{v^2}{g\tan(\bar{\phi})}\tan\left(\frac{|\Psi_{i+1}-\Psi_i|}{2}\right)$ for d_i^* is derived in [4] from $d_i^* = r_i \tan(\rho_i)$, where $\rho_i = \frac{|\Psi_{i+1}-\Psi_i|}{2}$ and r_i is computed as the speed v divided by the (constant) angular velocity $\frac{g}{v}\tan(\bar{\phi})$, which is obtained from a higher order aircraft model by assuming that the bank angle is kept constant and equal to $\bar{\phi}$.

Ideally, crossing the switching boundary $\alpha_{1i}x_1 + \alpha_{2i}x_2 = \beta_i$ while tracking I_i should cause a jump in the state component $\mathbf{q}(t)$ from i to $i+1$. In practice, however, the switching time instant is uncertain. To model this, we suppose that $\mathbf{q}(t), t \in T$, is a continuous time stochastic process with piecewise constant right continuous realizations, whose evolution at time t is conditionally independent on the past given $\mathbf{s}(t^-) = (x,i) \in \mathscr{S}$, and is governed by the transition probabilities

$$\mathrm{P}\{\mathbf{q}(t+\Delta) = h | \mathbf{s}(t^-) = (x,i)\} = \lambda_{i,h}(x)\Delta + o(\Delta), \quad h \neq i \in \mathscr{Q}, \tag{4}$$

with the transition rate function $\lambda_{i,h} : \mathbb{R}^2 \to [0,+\infty)$ given by

$$\lambda_{i,h}(x_1,x_2) = \begin{cases} \lambda_{\max}\, g(\alpha_{1i}x_1 + \alpha_{2i}x_2 - \beta_i), & h = i+1, i < M \\ 0, & \text{otherwise,} \end{cases} \tag{5}$$

where $\lambda_{\max} \in \mathbb{R}^+$ and $g : \mathbb{R} \to [0,1]$ is a continuous function increasing monotonically from 0 to 1. In this way, the switching rate from I_i to I_{i+1} grows from 0 to $\lambda_{\max}$ while crossing the switching boundary. If $g(\cdot)$ approaches a step function as $\lambda_{\max}$ tends to $+\infty$, one would recover in the limit instantaneous transitions in $\mathbf{q}(t)$ at boundary crossing. The transition rate functions determine switching intensity and reset map of the discrete state $\mathbf{q}(t)$. More precisely, during the infinitesimal time interval $[t, t+\Delta]$, $\mathbf{q}(t)$ will jump once with probability $\lambda(s)\Delta + o(\Delta)$, and two or more times with probability $o(\Delta)$, starting from $\mathbf{s}(t^-) = s$, where for any $s = (x,i) \in \mathscr{S}$ the jump intensity function $\lambda : \mathscr{S} \to [0,+\infty)$ is given by

$$\lambda(s) = \sum_{h \in \mathscr{Q}, h \neq i} \lambda_{i,h}(x) = \begin{cases} \lambda_{i,i+1}(x), & i < M \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

If $s \in \mathscr{S}$ is such that $\lambda(s) = 0$, then no instantaneous jump can occur from s. If $s = (x,i) \in \mathscr{S}$ is such that $\lambda(s) \neq 0$, then the distribution of $\mathbf{q}(t)$ over $\mathscr{Q}$, after a jump indeed occurs at time t from $\mathbf{s}(t^-) = (x,i)$, is given by the reset function $R : \mathscr{S} \times \mathscr{Q} \to [0,1]$:

$$R((x,i),h) = \begin{cases} \frac{\lambda_{i,h}(x)}{\lambda(s)}, & h \neq i \\ 0, & h = i \end{cases} = \begin{cases} 1, & h = i+1, i < M \\ 0, & \text{otherwise,} \end{cases} \tag{7}$$

where the last equality follows from equation (6).

If the drift term $a(\cdot,i)$ in (3) and the transition rate function $\lambda_{i,h}(\cdot)$ in (4) are bounded and Lipschitz continuous for each $i,h \in \mathscr{Q}$, $h \neq i$, then, for any initial condition $s_0 = (x_0,i_0) \in \mathscr{S}$, the prediction model admits a unique strong solution $\mathbf{s}(t) = (\mathbf{x}(t),\mathbf{q}(t))$, $t \geq 0$, known as switching diffusion process. Moreover, $\mathbf{s}(t)$ is a Markov process and the trajectories of $\mathbf{x}(t)$ are continuous.

III. Reachability analysis for aircraft conflict prediction

Given an unsafe region $\mathscr{D} \subset \mathbb{R}^2$ of the airspace, we address the problem of estimating the probability that the aircraft will enter $\mathscr{D}$ within the look-ahead time horizon $T = [0, t_f]$

$$\mathrm{P}_{s_0}\{\mathbf{x}(t) \in \mathscr{D} \text{ for some } t \in T\}, \tag{8}$$

starting at time $t = 0$ from $x_0 \in \mathbb{R}^2 \setminus \mathscr{D}$. Here, P_{s_0} denotes the probability measure induced by the switching diffusion process $\mathbf{s}(t)$ associated with the initial condition $s_0 = (x_0, 1)$. If $\mathscr{D}$ is measurable and closed, the quantity (8) is well-defined since the reachability event "$\mathbf{x}(t) \in \mathscr{D}$ for some $t \in T$" is measurable given that the process $\mathbf{x}$ has continuous trajectories, [5].

To evaluate the probability (8) numerically, we introduce a bounded open set $\mathscr{U} \subset \mathbb{R}^2$ containing $\mathscr{D}$ that is chosen large enough so that the situation can be declared safe once $\mathbf{x}(t)$ wanders outside $\mathscr{U}$. Let $\mathscr{U}^c$ denote the complement of $\mathscr{U}$ in $\mathbb{R}^2$. Then, with reference to the domain $\mathscr{U}$, the probability of entering $\mathscr{D}$ can be approximated by

$$\mathrm{P}_{s_0} := \mathrm{P}_{s_0}\{\mathbf{x}(t) \text{ hits } \mathscr{D} \text{ before hitting } \mathscr{U}^c \text{ within } T\}. \tag{9}$$

Hence, for the purpose of computing P_{s_0}, we can assume that $\mathbf{x}(t)$ in (3) is defined on the open domain $\mathscr{U} \setminus \mathscr{D}$ with initial condition $x_0 \in \mathscr{U} \setminus \mathscr{D}$, and that $\mathbf{x}(t)$ is stopped as soon as it hits the boundary $\partial\mathscr{U}^c \cup \partial\mathscr{D}$ of $\mathscr{U} \setminus \mathscr{D}$. We now describe a method to estimate P_{s_0} in (9) through reachability computations on a Markov chain. This method rests on the introduction of a discrete time Markov chain $\{\mathbf{v}_k, k \geq 0\}$ whose piecewise constant interpolation approximates weakly the switching diffusion process $\mathbf{s}(t)$, $t \geq 0$, describing the aircraft motion. The state $\mathbf{v}_k$ of the approximating Markov chain has two components $\mathbf{v}_k = (\mathbf{z}_k, \mathbf{m}_k)$, $k \geq 0$, which mimic the behavior of the two components of the switching diffusion process $\mathbf{s}(t) = (\mathbf{x}(t), \mathbf{q}(t))$, $t \geq 0$. $\mathbf{m}_k$ takes values in $\mathscr{Q}$, whereas $\mathbf{z}_k$ takes values in the set $\mathscr{Z}_\delta$ obtained by uniformly gridding the region $\mathscr{U} \setminus \mathscr{D}$ where the evolution of $\mathbf{x}(t)$ is confined, i.e., $\mathscr{Z}_\delta := (\mathscr{U} \setminus \mathscr{D}) \cap \mathbb{Z}^2_\delta$ with $\mathbb{Z}^2_\delta = \{(m_1\delta, m_2\delta) | \, m_i \in \mathbb{Z}, i = 1,2\}$, where $\delta > 0$ is a gridding scale parameter. The interpolation time interval Δ_δ is a function of δ that goes to zero faster than δ: $\Delta_\delta = o(\delta)$. To define the transition probabilities of the approximating Markov chain $\{\mathbf{v}_k, k \geq 0\}$, it is convenient to introduce an enlarged Markov chain process $\{(\mathbf{v}_k, \mathbf{j}_k), k \geq 0\}$, where $\{\mathbf{j}_k, k \geq 0\}$ is an i.i.d. Bernoulli process that determines the jump times in the $\mathbf{m}_k$ component: if $\mathbf{j}_k = 1$, then a jump, possibly of zero entity, occurs at time k in $\mathbf{m}_k$; whereas if $\mathbf{j}_k = 0$, then no jump occurs at time k in $\mathbf{m}_k$. Under the assumption that $\mathbf{j}_k$ is independent of the past variables $\mathbf{v}_i$, $i = 0, 1, \dots, k$, $\forall k \geq 0$, $\{\mathbf{v}_k, k \geq 0\}$ is indeed a Markov chain with transition probability

$$
\begin{aligned}
p_\delta(v \to v') &:= \mathrm{P}_\delta\{\mathbf{v}_{k+1} = v' \mid \mathbf{v}_k = v\} \qquad (10)\\
&= \sum_{j \in \{0,1\}} \mathrm{P}_\delta\{\mathbf{v}_{k+1} = v' \mid \mathbf{v}_k = v, \mathbf{j}_k = j\} \mathrm{P}_\delta\{\mathbf{j}_k = j\},
\end{aligned}
$$

specified by $\mathrm{P}_\delta\{\mathbf{j}_k = 1\}$, $\mathrm{P}_\delta\{\mathbf{v}_{k+1} = v' \mid \mathbf{v}_k = v, \mathbf{j}_k = 1\}$, and $\mathrm{P}_\delta\{\mathbf{v}_{k+1} = v' \mid \mathbf{v}_k = v, \mathbf{j}_k = 0\}$.

Probability $\mathrm{P}_\delta\{\mathbf{j}_k = 1\}$ governs the jump occurrences in $\mathbf{m}_k$ and is defined as $\mathrm{P}_\delta\{\mathbf{j}_k = 1\} = 1 - e^{-\lambda_{\max}\Delta_\delta} = \lambda_{\max}\Delta_\delta + o(\Delta_\delta)$. Probability $\mathrm{P}_\delta\{\mathbf{v}_{k+1} = v' \mid \mathbf{v}_k = v, \mathbf{j}_k = 1\}$ corresponds to the evolution of the switching diffusion $\mathbf{s}(t)$ when a jump (possibly of zero entity) occurs in the discrete state $\mathbf{q}(t)$. If $\mathbf{j}_k = 1$ (k is a jump time for $\mathbf{m}_k$), then, $\mathbf{z}_{k+1} = \mathbf{z}_k$ since the continuous state $\mathbf{x}(t)$ is reinitialized with the same value prior to a jump occurrence in $\mathbf{q}(t)$ and

$$
\begin{aligned}
&\mathrm{P}_\delta\{(\mathbf{z}_{k+1}, \mathbf{m}_{k+1}) = (z', q') \mid \mathbf{v}_k = (z,q), \mathbf{j}_k = 1\}\\
&\qquad = \begin{cases} 0, & z' \neq z \\ p_\delta(q \to q' | z), & z' = z, \end{cases}
\end{aligned}
$$

whereas the value of $\mathbf{m}_{k+1}$ is determined based on that of $\mathbf{v}_k$ through

$$
\begin{aligned}
p_\delta(q \to q' | z) &:= \mathrm{P}_\delta\{\mathbf{m}_{k+1} = q' \mid \mathbf{v}_k = (z,q), \mathbf{j}_k = 1\}\\
&= \begin{cases} \frac{\lambda_{q,q'}(z)}{\lambda_{\max}}, & q' \neq q \\ 1 - \frac{1}{\lambda_{\max}} \sum_{q^* \in \mathscr{Q}, q^* \neq q} \lambda_{q,q^*}(z), & q' = q. \end{cases}
\end{aligned}
$$

In this way, the probability distribution of $\mathbf{m}_{k+1}$ when a jump of non-zero entity occurs at time k from (z,q) is $\mathrm{P}_\delta\{\mathbf{m}_{k+1} = q' \mid \mathbf{v}_k = (z,q), \mathbf{j}_k = 1, \mathbf{m}_{k+1} \neq \mathbf{m}_k\} = R((z,q), q')$, where $R(\cdot,\cdot)$ is the reset function defined in (7). Also, the probability that a jump of non-zero entity occurs at time k from (z,q) is given by $\mathrm{P}_\delta\{\mathbf{j}_k = 1, \mathbf{m}_{k+1} \neq q \mid \mathbf{v}_k = (z,q)\} = \lambda((z,q))\Delta_\delta + o(\Delta_\delta)$, where $\lambda(\cdot)$ is the jump intensity function defined in (6). Probability $\mathrm{P}_\delta\{\mathbf{v}_{k+1} = v' \mid \mathbf{v}_k = v, \mathbf{j}_k = 0\}$ corresponds to the evolution of $\mathbf{x}(t)$ within a discrete state. If $\mathbf{j}_k = 0$ (k is not a jump time for $\mathbf{m}_k$), then $\mathbf{m}_{k+1} = \mathbf{m}_k$, whereas the value of $\mathbf{z}_{k+1}$ is determined from that of $\mathbf{v}_k$ through the (conditional) transition probabilities $p_\delta(z \to z' | q) := \mathrm{P}_\delta\{\mathbf{z}_{k+1} = z' \mid \mathbf{v}_k = (z,q), \mathbf{j}_k = 0\}$ describing the evolution of $\mathbf{z}_k$ for a fixed $q \in \mathscr{Q}$:

$$
\begin{aligned}
&\mathrm{P}_\delta\{(\mathbf{z}_{k+1}, \mathbf{m}_{k+1}) = (z', q') \mid \mathbf{v}_k = (z,q), \mathbf{j}_k = 0\}\\
&\qquad = \begin{cases} 0, & q' \neq q \\ p_\delta(z \to z' | q), & q' = q. \end{cases}
\end{aligned}
$$

For the weak convergence result to hold, the probabilities $p_\delta(z \to z' | q)$ are selected so as to approximate locally the characteristics of $\mathbf{x}(t)$ as a diffusion process with absorption on the boundary $\partial\mathscr{U}^c \cup \partial\mathscr{D}$. For each grid point $z \in \mathbb{Z}^2_\delta$, define the immediate neighbors set $\mathscr{N}_\delta(z)$, $z \in \mathscr{Z}_\delta$, as the set of points along each one of the x_i axis whose distance from q is δ, $i = 1,2$: $z_{1_+} = z + (\delta, 0)$, $z_{1_-} = z - (\delta, 0)$, $z_{2_+} = z + (0, \delta)$, $z_{2_-} = z - (0, \delta)$. The interior $\mathscr{Z}^\circ_\delta$ of $\mathscr{Z}_\delta$ consists of all those points in $\mathscr{Z}_\delta$ which have all their neighbors in $\mathscr{Z}_\delta$. The boundary $\partial\mathscr{Z}_\delta = \mathscr{Z}_\delta \setminus \mathscr{Z}^\circ_\delta$ of $\mathscr{Z}_\delta$ is the union of the sets $\partial\mathscr{Z}_{\delta\mathscr{U}^c}$ and $\partial\mathscr{Z}_{\delta\mathscr{D}}$, where $\partial\mathscr{Z}_{\delta\mathscr{U}^c}$ is the set of points with at least one neighbor inside $\mathscr{U}^c$ and $\partial\mathscr{Z}_{\delta\mathscr{D}}$ is the set of points with at least one neighbor inside $\mathscr{D}$. The points that satisfy both these conditions, if any, are assigned to either $\partial\mathscr{Z}_{\delta\mathscr{D}}$ or $\partial\mathscr{Z}_{\delta\mathscr{U}^c}$, so as to make these two sets disjoint. This eventually introduces an error in the estimate of the probability of interest, which however becomes negligible if $\mathscr{U}$ is chosen sufficiently large. For each $q \in \mathscr{Q}$, we define $p_\delta(z \to z' | q)$ so that each state z in $\partial\mathscr{Z}_\delta$ is an absorbing state, whereas from any state z in $\mathscr{Z}^\circ_\delta$, $\mathbf{z}_k$ moves to one of its neighbors in $\mathscr{N}_\delta(z)$ or remains at z according to probabilities determined by its current location

$$
p_\delta(z \to z' | q) = \begin{cases} \pi_\delta(z' | (z,q)), & z' \in \mathscr{N}_\delta(z) \cup \{z\} \\ 0, & \text{otherwise}, \end{cases} \quad z \in \mathscr{Z}^\circ_\delta. \qquad (11)
$$

For the local consistency property to hold, the probability distributions $\pi_\delta(\cdot | (z,q)) : \mathscr{N}_\delta(z) \cup \{z\} \to [0,1]$, $z \in \mathscr{Z}^\circ_\delta$ are chosen so that $\frac{1}{\Delta_\delta} \mathrm{E}_\delta[\mathbf{z}_{k+1} - \mathbf{z}_k \mid \mathbf{v}_k = (z,q), \mathbf{j}_k = 0] \to a(x,q)$ and $\frac{1}{\Delta_\delta} \mathrm{E}_\delta[(\mathbf{z}_{k+1} - \mathbf{z}_k)(\mathbf{z}_{k+1} - \mathbf{z}_k)^T | \, \mathbf{v}_k = (z,q), \mathbf{j}_k = 0] \to \sigma^2 I$ as $\delta \to 0$, for all $x \in \mathscr{U} \setminus \mathscr{D}$, where, for any δ, z is a point in $\mathscr{Z}^\circ_\delta$ closest to x.

Different choices are possible that satisfy the local consistency property, [10]. For instance, if the interpolation time interval Δ_δ is set equal to $\eta\delta^2$, the transition probability function $\pi_\delta(\cdot | v)$ over $\mathscr{N}_\delta(z) \cup \{z\}$ from $v = (z,q) \in \mathscr{Z}^\circ_\delta \times \mathscr{Q}$ can be chosen as follows:

$$
\pi_\delta(z' | v) = \begin{cases} c(v)\, \xi_0(v), & z' = z \\ c(v)\, e^{+\delta\xi_i(v)}, & z' = z_{i_+}, i = 1,2 \\ c(v)\, e^{-\delta\xi_i(v)}, & z' = z_{i_-}, i = 1,2, \end{cases} \qquad (12)
$$

with $\xi_0(v) = \frac{2}{\eta\sigma^2} - 4$, $\xi_i(v) = \frac{[a(v)]_i}{\sigma^2}$, $i = 1,2$, $c(v) = \frac{1}{2\sum_{i=1}^2 \mathrm{csh}(\delta\xi_i(v)) + \xi_0(v)}$, where for any $y \in \mathbb{R}^2$, $[y]_i$ denotes the component of y along the x_i direction, $i = 1,2$. $\eta \in \mathbb{R}^+$ has to be chosen small enough such that $\xi_0(v)$ defined above is positive for all $v \in \mathscr{Z}^\circ_\delta \times \mathscr{Q}$. In particular, $0 < \eta \leq \frac{1}{2\sigma^2}$.

Consider now the look-ahead time horizon $T = [0, t_f]$. Fix $\delta > 0$ so that $k_f := \frac{t_f}{\Delta_\delta}$ is an integer, and construct the approximating Markov chain $\{\mathbf{v}_k = (\mathbf{z}_k, \mathbf{m}_k), k \geq 0\}$ initialized at $v_0 = (z_0, 1)$ where $z_0 \in \mathscr{Z}^\circ_\delta$ is a point closest to $x_0 \in (\mathscr{U} \setminus \mathscr{D})$. Let $\{\Delta\tau_k, k \geq 0\}$ be i.i.d. random variables exponentially distributed with mean value Δ_δ, independent of $\{\mathbf{v}_k, k \geq 0\}$ and $\{\mathbf{j}_k, k \geq 0\}$.

Proposition 1 ([9]) *The continuous time stochastic process $\{\mathbf{v}(t), t \geq 0\}$ that is equal to $\mathbf{v}_k$ on the time interval $[\tau_k, \tau_{k+1})$ for all k, where $\tau_0 = 0$ and $\tau_{k+1} = \tau_k + \Delta\tau_k$, $k \geq 0$, converges weakly as $\delta \to 0$ to the switching diffusion process $\{\mathbf{s}(t) = (\mathbf{x}(t), \mathbf{q}(t)), t \geq 0\}$ associated with the initial condition $s_0 = (x_0, 1)$, with $\mathbf{x}(t)$ defined on $\mathscr{U} \setminus \mathscr{D}$ and absorption on the boundary $\partial\mathscr{U}^c \cup \partial\mathscr{D}$.*

This weak convergence results implies that

$$
\hat{\mathrm{P}}_{s_0} := \mathrm{P}_\delta\{\mathbf{z}_k \text{ hits } \partial\mathscr{Z}_{\delta\mathscr{D}} \text{ before } \partial\mathscr{Z}_{\delta\mathscr{U}^c} \text{ within } [0, k_f]\} \qquad (13)
$$

converges with probability one to the probability of interest P_{s_0} in (9), under the assumption that the probability that process $\mathbf{x}(t)$ touches the boundary $\partial\mathscr{D} \cup \partial\mathscr{U}^c$ without leaving $\mathscr{U} \setminus \mathscr{D}$ is zero. This situation is known to be critical also in the simulation of SHSs by discrete time approximation schemes, [11]. In our case, given that the diffusion matrix $\sigma^2 I$ is uniformly positive definite over $\mathscr{U} \setminus \mathscr{D}$, appropriate regularity conditions on $\mathscr{D}$ and $\mathscr{U}^c$ guarantee with probability one that this pathological situation does not occur, [11] and [10, Chapter 10].

Automated tools for model checking of Markov chains such as PRISM [12] and MCMC [13] can be applied to efficiently compute the probability $\hat{\mathrm{P}}_{s_0}$ in (13), the main issue being the exponential growth of the state space of the Markov chain as $\delta \to 0$. For practical purpose, the grid size δ should be chosen to balance the two conflicting considerations that large δ's may not allow for the simulation of fast moving processes and may lead to larger estimation errors, but for small δ's the memory requirements may be overwhelming and the running time may be too long. Here, we describe an algorithm for computing $\hat{\mathrm{P}}_{s_0}$ with $s_0 = (x_0, 1)$ as a function of x_0 over the whole set $\mathscr{U} \setminus \mathscr{D}$.
Since both the boundaries $\partial \mathscr{Z}_{\delta \mathscr{U}^c}$ and $\partial \mathscr{Z}_{\delta \mathscr{D}}$ are absorbing, $\hat{\mathrm{P}}_{s_0}$ in (13) reduces to $\hat{\mathrm{P}}_{s_0} = \mathrm{P}_\delta\{\mathbf{z}_{k_f} \in \partial \mathscr{Z}_{\delta \mathscr{D}}\}$. For each $k = 0, 1, \ldots, k_f$, define the probability map $\hat{\mathrm{p}}^{(k)} : \mathscr{Z}_\delta \times \mathscr{Q} \to [0,1]$ as

$$\hat{\mathrm{p}}^{(k)}(v) := \mathrm{P}_\delta\{\mathbf{z}_{k_f} \in \partial \mathscr{Z}_{\delta \mathscr{D}} \mid \mathbf{v}_{k_f - k} = v\} \tag{14}$$

representing the probability of $\mathbf{z}_k$ hitting $\partial \mathscr{Z}_{\delta \mathscr{D}}$ before $\partial \mathscr{Z}_{\delta \mathscr{U}^c}$ within $[k_f - k, k_f]$ starting from v at time $k_f - k$. Then, it is easily seen that $\hat{\mathrm{p}}_\delta^{(k)}$, $0 \leq k < k_f$, satisfies the recursion

$$\hat{\mathrm{p}}^{(k+1)}(v) = \sum_{v' \in \mathscr{Z}_\delta \times \mathscr{Q}} p_\delta(v \to v') \hat{\mathrm{p}}^{(k)}(v'), \quad v \in \mathscr{Z}_\delta \times \mathscr{Q}, \tag{15}$$

with $p_\delta(v \to v')$ given in (10). Thus, $\hat{\mathrm{P}}_{s_0} = \hat{\mathrm{p}}^{(k_f)}(v_0)$ can be computed by iterating equation (15) k_f times starting from $k = 0$ with $\hat{\mathrm{p}}^{(0)}(v) = 1_{\partial \mathscr{Z}_{\delta \mathscr{D}} \times \mathscr{Q}}(v)$, $v \in \mathscr{Z}_\delta \times \mathscr{Q}$, by definition (14). Despite the computation intensity, this algorithm has the advantage over randomized methods, [8], [1], that, after its completion, an estimate of the probability of conflict over the residual time horizon $[t_f - t, t_f]$ of length t is readily available for any $t \in (0, t_f)$, and is given by the map $\hat{\mathrm{p}}^{(\lfloor (t_f - t)/\Delta_\delta \rfloor)}$ evaluated at the state value at time $t_f - t$. Besides conflict detection, this result can be useful also for conflict resolution by adaptively adjusting the aircraft heading based on the pre-computed map. For instance, the heading of the aircraft could be chosen as the (numerically evaluated) negative gradient direction to rapidly decrease the probability of conflict, compatibly with the feasibility of the maneuver. Note that, since ATCs tend to preserve the air traffic altitude segmentation, considering a planar corrective maneuver is not much restrictive. In [1], a gradient-based approach to multi-aircraft conflict resolution using analytical expressions for the probability of conflict between each aircraft pair was suggested. In that case a Brownian motion with drift was used to predict the aircraft future positions. For more complex models, analytical solutions are difficult if not even impossible to obtain and one should head for numerical solutions.

IV. Numerical Example

Consider the sequence of way-points $O_1 = (60, -40)$, $O_2 = (40, -20)$, $O_3 = (40, 0)$, and $O_4 = (60, 20)$ (all coordinates are in the unit of km), and a disk $\mathscr{D}$ of radius 5 km centered at the point $(60, 15)$. Our goal is to estimate the probability P_{s_0} that an aircraft with flight plan $\{O_i, i = 1, 2, 3, 4\}$, velocity v equal to Mach 0.8, and located at an arbitrary initial position x_0 will ever wander into the forbidden zone $\mathscr{D}$ within some look-ahead time horizon $T = [0, t_f]$. Note that we have chosen a case where the nominal flight path crosses the forbidden zone to allow a more prominent visualization of the influence of the FMS correction action on P_{s_0}.

We perform two experiments with the same set of parameters ($\bar{\phi} = 0.2°$, $g = 9.81$ m s^{-2}, $d_\mathrm{m} = 200$ km, $\sigma = 0.3$ km$^{1/2}$s^{-1}, $f(\cdot) = 0$, $\lambda_\mathrm{max} = 0.03$ s^{-1}, t_f= 200 s, η= 0.5 km^{-1}s^2), except that in the second

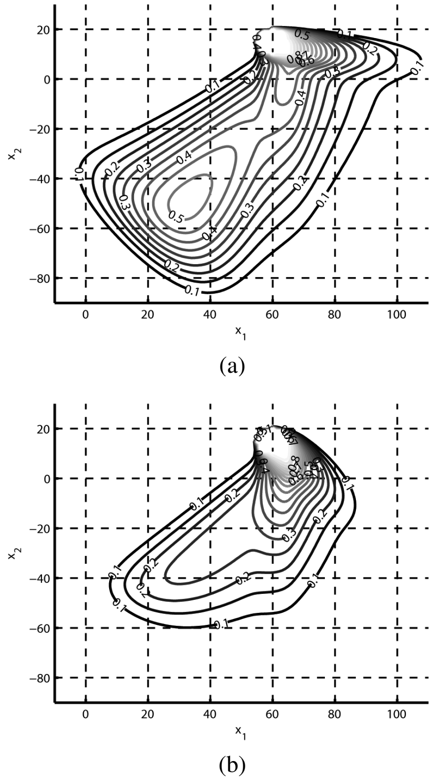

Fig. 2. 2-D contour plot of the estimated P_{s_0}: (a) with cross-track error correction; (b) with no cross-track error correction.

experiment we set $\gamma(x, i) = 1$ in (2) so that there is no cross-track error FMS correction effort in the aircraft dynamics. In both experiments, we choose the gridding parameter $\delta = 1$ km and the region $\mathscr{U}$ as the rectangle $\mathscr{U} = (-10, 110) \times (-90, 30)$. In addition, we assume that the function $g(\cdot)$ in equation (5) is given by $g(y) = 1/(1 + 0.1e^{-500y})$, $y \in \mathbb{R}$.

The estimated probability of conflict $\hat{\mathrm{P}}_{s_0}$ is plotted in Fig. 2 as a function of the initial position x_0. Note that the region with higher probability of conflict shrinks considerably in the case of no cross-track error correction (Fig. 2 (b)). This is because, regardless of the aircraft initial location, the cross-track error correction term tends to cause the aircraft to converge along the reference path, which in itself will pass through the forbidden zone. Without the cross-track error correction, the aircraft will deviate from the reference path with increased probability, thus reducing the likelihood of a conflict.

V. Conclusions

We studied aircraft conflict prediction as a reachability analysis problem for a switching diffusion. A stochastic approximation scheme to estimate the probability that a single aircraft will enter a forbidden area of the airspace within a finite time horizon was presented. The approach can be easily extended to more complex aircraft conflict prediction problems involving a time-varying forbidden area and multiple aircraft. The main issue is that the increased problem dimensionality causes an exponential growth in the Markov chain approximation. The research activity in system verification on the development of efficient model checkers for Markov chains is much relevant in this respect. Alternatively, the probability map could be approximated by a parameterized family of functions as suggested in the neuro-dynamic programming literature.

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