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SYMMETRY OF SOLUTIONS TO THE OPTIMAL EXIT TIME CONTROL PROBLEM*

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Abstract. In this paper, we study the solutions to the optimal exit time control problem. Such a problem tries to find the state feedback control law with a fixed cost that can keep the state of a randomly perturbed system inside a subset of the state space, called the safe set, for as long as possible on average. By formulating the problem as an optimization problem with PDE constraints and using symmetrization techniques, we show that, when the safe set is a ball, the optimal feedback control (if it exists) must be radially symmetric. Furthermore, we show that, among all safe sets with a fixed volume, the ball is the best in that it yields the most efficient optimal exit time control. The proofs make essential use of the general isoperimetric inequality.

Key words. optimal stochastic control, exit time, symmetrization

AMS subject classifications. 93E20, 93C20, 49K20, 35B37

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1. Introduction.

The optimal control problems for systems under uncertainty have many practical applications, such as aircraft conflict resolution [13], formation flight of unmanned aerial vehicles (UAVs) [28], automated highway systems [27], and robotics. In many such applications, the system under study can be modeled as a control system whose state dynamics is perturbed by random noises, and the primary control goal is to keep the system safe, i.e., to keep the state within a subset of the state space called the *safe set*. As a practical example, consider a platoon of vehicles driving on a highway, as shown in Figure 1.1. The goal is to design controllers for the vehicles so that a tight yet safe formation is maintained. In this case, the safe set is the set of vehicle locations with sufficient separations. Due to factors such as wind, road conditions, sensor and actuator errors, etc., the distances between vehicles may fluctuate randomly despite the effort of the controllers.

In this paper, we focus on an instance of the optimal control problems of systems under uncertainty called the *optimal exit time control problem* that is particularly relevant in safety-critical applications. For the stochastic control system $dX_t = u(X_t)dt+\sigma dB_t$, we aim at finding the optimal state feedback control law u over the safe set $\bar{\Omega}$ that can keep the state X_t within $\bar{\Omega}$ for as long as possible on average, with a fixed control cost $\int_{\bar{\Omega}} \|u\|^2\,dx$. This problem was first proposed in [10]. Under the assumption that the solutions are symmetric, the optimal control is characterized analytically in a one-dimensional state space in [10]. In [24], the problem is extended to the setting of stochastic hybrid systems, and a numerical solution is presented based on the adjoint method. In [30], a generalized version of the problem is studied in the one-dimensional state space. In these studies, the symmetry property of the optimal solutions plays a key role but has not been established rigorously in its full generality.

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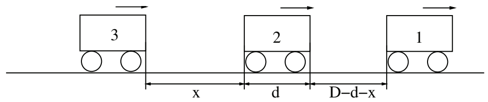


FIG. 1.1. *A platoon of vehicles in formation.*

A main contribution of this paper is the proof of the symmetry property of the solutions to the optimal exit time control problem in arbitrary dimensional state spaces. Namely, we prove that if the state space is a ball, then the solutions to the optimal exit time control problem, if they exist, are radially symmetric (see Theorem 3.1). Another contribution is our proof that, among all safe sets with a fixed volume, the ball is the best in that it yields the most efficient optimal exit time control scheme (see Theorem 3.2). With these results, the complexity of finding the solutions to multidimensional problems is greatly reduced.

The technique used in the proofs, the *symmetrization method* [3, 12], was originally employed in [21, 22] in the study of various mathematical physics problems. For example, by symmetrizing the scalar function f in the partial differential equation (PDE) $-\frac{1}{2}\Delta V = f$, $V|_{\partial\Omega} \equiv 0$, it was shown in [21] that a Brownian motion takes the longest expected time to exit a ball among all the domains with the same volume. This approach was later extended to more general elliptic and parabolic PDEs [7, 11, 15, 18]. In particular, a version of Talenti's theorem [2, 26] symmetrizes f in the PDE $\frac{\sigma^2}{2}\Delta V + u\cdot\nabla V = f$ to conclude that for a Brownian motion with drift u of bounded amplitude $\|u\| \le M$, the largest expected exit time is achieved when the domain is a ball and the drift u points to its center with *constant* maximal amplitude M. In comparison, the PDE studied in this paper is of the form $\frac{\sigma^2}{2}\Delta V + u\cdot\nabla V = 1$. Instead of symmetrizing a scalar function, we symmetrize the vector field u to obtain a symmetric vector field u^* of *varying* amplitude, and we show that for a maximal exit time of a Brownian motion with drift determined by a vector field with bounds on its L^∞ and L^2 norms, one should choose a symmetric domain and a symmetric u^*. Our proof is inspired by the original work [22]; a modern treatment, as in [12], may lead to a more succinct proof.

The rest of the paper is organized as follows. In section 2, the problem of optimal exit time control is formulated. To prove the main results outlined in section 3, an equivalent formulation of the problem is derived in section 4. Using the symmetrization techniques introduced in section 5 and the preliminary results in section 6, the main results are proved in section 7 and extended in section 8. Finally, sections 9 and 10 contain some numerical examples and concluding remarks, respectively.

2. Problem formulation.

2.1. Expected exit time. Let Ω be a bounded, connected, and simply connected, open subset of $\mathbb{R}^n$ with a C^2 boundary $\partial\Omega$. Its closure $\bar{\Omega} = \Omega \cup \partial\Omega$, which is compact, is called the *safe set*. Consider a stochastic process X_t given as the solution to the following stochastic differential equation (SDE) on $\bar{\Omega}$:

$$dX_t = u(X_t)dt + \sigma dB_t, \tag{2.1}$$

where σ is a positive constant, B_t is an n-dimensional standard Brownian motion, and $u \in \mathcal{U}_M(\bar{\Omega})$ is an *admissible control* vector field in the set

$$\mathcal{U}_M(\bar{\Omega}) := \{u : \bar{\Omega} \to \mathbb{R}^n \mid \|u(x)\| \le M\}.$$

Here $M \in (0, \infty]$ is a constant modeling the physical limitation on the control magnitude.

Define $\tau := \inf\{t \geq 0 : X_t \notin \Omega\}$ as the *(first) exit time*, or *escape time*, of X_t from Ω. Then τ is a stopping time and its expectation is denoted by

$$V(x) = E_x[\tau], \quad x \in \bar{\Omega}. \tag{2.2}$$

Here E_x indicates that the expectation is taken under the initial condition $X_0 = x$. Thus $V(x)$ is the expected time the state X_t will stay inside the safe set before its first exit, given that it starts from x at time $t = 0$. Obviously $V(x) \equiv 0$ for $x \in \partial\Omega$.

LEMMA 2.1. *The function $V(x)$ defined in* (2.2) *is the weak solution in the Soblev space $H^2(\bar{\Omega})$ to the following second order elliptic PDE:*

$$\begin{cases} \frac{\sigma^2}{2}\Delta V + u \cdot \nabla V + 1 = 0, & x \in \bar{\Omega}, \\ V(x) = 0, & x \in \partial\Omega. \end{cases} \tag{2.3}$$

Here $\Delta V := \sum_{i=1}^{n} \frac{\partial^2 V}{\partial x_i^2}$ is the Laplacian of V, and $\nabla V := (\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n})$ is the gradient vector of V.

Proof. This is a standard result in stochastic analysis. We include a brief proof here since some of the notions introduced will be used in later proofs. By [19], the infinitesimal generator L of the diffusion (2.1) is given by

$$Lg = \frac{\sigma^2}{2}\Delta g + u \cdot \nabla g \tag{2.4}$$

for functions $g : \bar{\Omega} \to \mathbb{R}$ with weak second order derivatives almost everywhere (a.e.) on $\bar{\Omega}$.

For an admissible control u, let $V \in H^2(\bar{\Omega})$ be a weak solution to the PDE (2.3). Such a V exists by [5, Theorem 6.4]. Thus, $LV \equiv -1$ on $\bar{\Omega}$, and $V \equiv 0$ on $\partial\Omega$. By Dynkin's formula [19], we have

$$E_x[V(X_\tau)] - E_x[V(X_0)] = E_x\left[\int_0^\tau LV(X_t)\, dt\right] = -E_x[\tau].$$

Note that $V(X_\tau) = 0$ as $X_\tau \in \partial\Omega$. Hence, $E_x[V(X_0)] = V(x)$ is indeed the expected exit time $E_x[\tau]$. ∎

Since weak solutions may have exotic behaviors (see [5, Example 5.4]), it is more convenient to work with classical solutions of (2.3). Therefore, we impose the following assumption.

ASSUMPTION 2.2. *For the admissible controls u considered in this paper, assume that the expected exit time V is a classical solution to the PDE* (2.3)*. In other words, at almost all $x \in \Omega$, $V(x)$ is second order differentiable and* (2.3) *holds.*

In the case when the state dimension $n = 1$, the safe set $\bar{\Omega}$ becomes an interval, say, $\bar{\Omega} = [-a, a]$, for some $a > 0$. Then (2.3) reduces to a second order ordinary differential equation (ODE),

$$\frac{\sigma^2}{2}V''(x) + u(x)V'(x) + 1 = 0, \quad x \in [-a, a], \tag{2.5}$$

with boundary condition $V(-a) = V(a) = 0$. The solution to (2.5) can be verified to be

$$V(x) = 2u_2(x)\int_x^a e^{2u_1(y)}\, dy + 2\int_{-a}^x e^{2u_1(y)}u_2(y)\, dy - \frac{2u_2(x)}{u_2(a)}\int_{-a}^a e^{2u_1(y)}u_2(y)\, dy,$$

where $u_1(x) := \frac{1}{\sigma^2} \int_{-a}^{x} u(y)\, dy$ and $u_2(x) := \frac{1}{\sigma^2} \int_{-a}^{x} e^{-2u_1(y)}\, dy$. Thus, Assumption 2.2 is trivially satisfied when $n = 1$.

2.2. Optimal exit time control problem and its dual. We now formulate the problems to be studied in this paper. The SDE (2.1) defines the dynamics of a control system whose state X_t is subject to the feedback control law u in the state space $\bar{\Omega}$. A natural problem is to find the least expensive control u that can keep X_t inside Ω for at least a certain amount of time on average. More precisely, define the *(aggregated) energy* or *cost* for an admissible control u as

$$J(u) := \int_{\bar{\Omega}} \|u\|^2\, dx \quad \forall u \in \mathcal{U}_M(\bar{\Omega}). \tag{2.6}$$

The performance of the control u is measured by the *aggregated expected exit time*

$$W(u) := \int_{\bar{\Omega}} w(V)\, dx, \tag{2.7}$$

where V is the expected exit time defined in (2.2), and $w : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing function on $\mathbb{R}_+ = [0, \infty)$ with $\lim_{x\to\infty} w(x) = \infty$, e.g., $w(x) = x^\alpha$ for some $\alpha > 0$. In particular, if $\alpha = 1$, $W(u)$ reflects the overall expected exit time under u when the initial position $X_0 = x$ is uniformly distributed in $\bar{\Omega}$.

PROBLEM 1 (optimal exit time control). *Find the admissible control $u \in \mathcal{U}_M(\bar{\Omega})$ to maximize $W(u)$ subject to $J(u) \le J_0$.*

PROBLEM 2 (dual optimal exit time control). *Find the admissible control $u \in \mathcal{U}_M(\bar{\Omega})$ to minimize $J(u)$ subject to $W(u) \ge W_0$.*

The choices of the parameters J_0 and W_0 in the above problems cannot be arbitrary. For example, in Problem 1, to prevent the constraint $J(u) \le J_0$ from being a trivial result of $u \in \mathcal{U}_M(\bar{\Omega})$, we need to assume $J_0 < M^2|\bar{\Omega}|$, where $|\bar{\Omega}|$ is the volume of $\bar{\Omega}$. On the other hand, since zero control $u \equiv 0$ has cost $J(0) = 0$ but performance $W(0) > 0$, we need to assume $W_0 > W(0)$ so that the solution to Problem 2 is not the trivial zero control. Also, as the performance $W(u)$ of admissible controls u in $\mathcal{U}_M(\bar{\Omega})$ may be bounded from above by a finite value (see Corollary 9.3), W_0 needs to be upper bounded by the same value to ensure the existence of solutions to Problem 2.

Remark 2.3. Let $W^*(J_0) = \sup\{W(u) |\, u \in \mathcal{U}_M(\bar{\Omega}), J(u) \le J_0\}$ be the optimal expected exit time achieved by the solutions to Problem 1, and let $J^*(W_0) = \inf\{J(u) |\, u \in \mathcal{U}_M(\bar{\Omega}), W(u) \ge W_0\}$ be the minimal cost achieved by the solutions to Problem 2. By Lemma 4.2 in section 4, we will show that $W^*(J_0)$, $J_0 \ge 0$, and $J^*(W_0)$, $W_0 \ge W(0)$, are strictly increasing functions. Indeed, they are the inverse functions of each other on their proper domains of definition.

3. Summary of main results. We will now outline the main results of this paper. Suppose first that $\bar{\Omega} = \bar{B}(a) := \{x \in \mathbb{R}^n : \|x\| \le a\}$ is the n-dimensional ball of radius a.[1] A function $g : \bar{\Omega} \to \mathbb{R}$ is called *(radially) symmetric* if $g(x) = \tilde{g}(\|x\|)$ for some function $\tilde{g} : [0, a] \to \mathbb{R}$; and a vector field $u : \bar{\Omega} \to \mathbb{R}^n$ is (radially) symmetric if it is of the form $u(x) = \tilde{u}(\|x\|)\frac{x}{\|x\|}$, $x \ne 0$, for some function $\tilde{u} : (0, a] \to \mathbb{R}$, i.e., $u(x)$ always points along the radial direction, with an amplitude dependent only on $\|x\|$.

By formulating Problems 1 and 2 as PDE-constrained optimization problems, Raffard, Hu, and Tomlin [24] obtained their numerical solutions by using the adjoint-based method [14]. It was observed that the solutions on a ball $\bar{\Omega}$ are always radially

[1] In this paper, balls are assumed to be centered at the origin by default.

symmetric. Motivated by this observation, in this paper we will prove the following result.

THEOREM 3.1. *Suppose that* $\bar{\Omega} = \bar{B}(a)$ *is a ball. Then solutions* u *to Problem* 1 *(or Problem* 2*), if they exist, can be chosen to be radially symmetric.*

Furthermore, if the shape of $\bar{\Omega}$ can be designed, we have the following result.

THEOREM 3.2. *Suppose that the volume of* $\bar{\Omega}$ *is fixed and that solutions to Problems* 1 *and* 2 *exist for each* $\bar{\Omega}$*.*

(i) *Denote by* $W_{\max}(\bar{\Omega}) = \sup\{W(u) \mid u \in \mathcal{U}_M(\bar{\Omega}), J(u) \leq J_0\}$ *the solution to Problem* 1 *on* $\bar{\Omega}$ *for a fixed* J_0*. Then among all* $\bar{\Omega}$ *with the same volume, the ball* $\bar{\Omega}^*$ *is the best in that* $W_{\max}(\bar{\Omega}^*) \geq W_{\max}(\bar{\Omega})$ *for all other* $\bar{\Omega}$*.*

(ii) *Denote by* $J_{\min}(\bar{\Omega}) = \inf\{J(u) \mid u \in \mathcal{U}_M(\bar{\Omega}), W(u) \geq W_0\}$ *the solution to Problem* 2 *on* $\bar{\Omega}$ *for a fixed* W_0*. Then among all* $\bar{\Omega}$ *with the same volume, the ball* $\bar{\Omega}^*$ *is the best in that* $J_{\min}(\bar{\Omega}^*) \leq J_{\min}(\bar{\Omega})$ *for all other* $\bar{\Omega}$*.*

Thus, the "best" shape of $\bar{\Omega}$ for designing a control u with large $W(u)$ and small $J(u)$ is a ball, and on such a domain, the optimal u and V are both radially symmetric.

4. Reformulation of the problems. To prove Theorems 3.1 and 3.2, we next reformulate the problems in terms of the function V. To this purpose, we shall first study the set of feasible V, i.e., the set of all V that are the expected exit times for the optimal solutions u to Problems 1 and 2.

Let V be the expected exit time corresponding to an admissible control $u \in \mathcal{U}_M(\bar{\Omega})$. According to Lemma 2.1, V satisfies the PDE (2.3). Therefore,

$$u \cdot \nabla V = -\left(1 + \frac{\sigma^2}{2}\Delta V\right) \quad \text{on } \bar{\Omega}. \tag{4.1}$$

It directly follows that V must satisfy $\Delta V = -2\sigma^{-2}$ at those critical points of V where $\nabla V = 0$. Another implication is that, for different controls u, as long as $u \cdot \nabla V$ remain the same at each $x \in \bar{\Omega}$ and satisfy (4.1), the same function V can be the solution to the PDE (2.3), and hence the expected exit time, associated with these u. Since among all such u, the one whose direction is aligned with ∇V at each $x \in \bar{\Omega}$ has the least energy $J(u)$, we have the following lemma.

LEMMA 4.1. *Let* $u \in \mathcal{U}_M(\bar{\Omega})$ *be a solution to Problem* 1 *or* 2*, and let* V *be the corresponding expected exit time. Then, for almost all* $x \in \bar{\Omega}$*,* $1 + \frac{\sigma^2}{2}\Delta V \leq 0$*, and*

$$u = \begin{cases} -(1 + \frac{\sigma^2}{2}\Delta V)\frac{\nabla V}{\|\nabla V\|^2} & \text{if } \nabla V \neq 0, \\ 0 & \text{if } \nabla V = 0. \end{cases} \tag{4.2}$$

Proof. Define a control $\hat{u}$ as follows. At each $x \in \bar{\Omega}$, project $u(x)$ onto $\nabla V(x)$ and, if necessary, reverse its direction to be of the same direction as $\nabla V(x)$:

$$\hat{u} = \begin{cases} |u \cdot \nabla V|\frac{\nabla V}{\|\nabla V\|^2} & \text{if } \nabla V \neq 0, \\ 0 & \text{if } \nabla V = 0. \end{cases}$$

It is easily verified that $\|\hat{u}\| \leq \|u\|$, and hence $J(\hat{u}) \leq J(u)$, with equality only if u and ∇V are collinear a.e. on $\bar{\Omega}$. It can also be verified that $\hat{u} \cdot \nabla V \geq u \cdot \nabla V$. Now consider the system under the new control $\hat{u}$: $d\hat{X}_t = \hat{u}(\hat{X}_t)dt + \sigma dB_t$. Its infinitesimal generator $\hat{L}$ is given by $\hat{L}g = \frac{\sigma^2}{2}\Delta g + \hat{u} \cdot \nabla g$. Thus, $\hat{L}V = \frac{\sigma^2}{2}\Delta V + \hat{u} \cdot \nabla V \geq \frac{\sigma^2}{2}\Delta V + u \cdot \nabla V = -1$. Denote by $\hat{\tau}$ the first exit time of $\hat{X}_t$ from Ω. Then by Dynkin's formula,

$$E_x[V(\hat{X}_{\hat{\tau}})] - E_x[V(\hat{X}_0)] = E_x\left[\int_0^{\hat{\tau}} \hat{L}V(\hat{X}_t)\,dt\right] \geq -E_x[\hat{\tau}], \tag{4.3}$$

or equivalently, $E_x(\hat{\tau}) \geq E_x(\tau)$, for all $x \in \Omega$. By the optimality of u, we must have both $J(\hat{u}) = J(u)$ and $E_x(\hat{\tau}) = E_x(\tau)$. From the construction of $\hat{u}$, this is possible only if u and ∇V are of the same direction a.e. on $\bar{\Omega}$. Together with (4.1), this implies that u is of the form (4.2) and that $1 + \frac{\sigma^2}{2}\Delta V \leq 0$ a.e. on $\bar{\Omega}$. ∎

Hence, the optimal control u always points towards the direction along which the corresponding expected exit time V increases the fastest. By following a similar procedure, we can show the following useful fact.

LEMMA 4.2. *Suppose that the expected exit time V corresponding to an admissible control u satisfies $1 + \frac{\sigma^2}{2}\Delta V \leq 0$ a.e. on $\bar{\Omega}$. Then, the expected exit time $\hat{V}$ corresponding to another control $\hat{u}$ satisfies $\hat{V}(x) > V(x)$ for $x \in \Omega$ if $\hat{u} = \lambda u$ for some function $\lambda(x) \geq 1$ with strict inequality on a subset of $\bar{\Omega}$ of nonzero measure.*

Based on the above results, we can focus our attention on a restricted family of the functions V defined below when looking for optimal solutions.

DEFINITION 4.3 (feasible V for optimal solutions). *The set $\mathcal{V}(\bar{\Omega})$ is defined to be the family of all functions $V : \bar{\Omega} \to \mathbb{R}_+$ satisfying the following properties:*

1. *V is positive in the interior Ω and zero on the boundary $\partial\Omega$.*
2. *$V \in C^1(\bar{\Omega})$ and has second order derivatives a.e. on $\bar{\Omega}$.*
3. *The corresponding control u as defined in (4.2) is admissible: $u \in \mathcal{U}_M(\bar{\Omega})$.*
4. *$1 + \frac{\sigma^2}{2}\Delta V \leq 0$ a.e. on $\bar{\Omega}$.*

To exclude certain pathological cases, we make an additional technical assumption, which will greatly simplify our proofs later on.

ASSUMPTION 4.4. *The set $\{x \in \bar{\Omega} : \nabla V(x) = 0\}$ of the critical points of $V \in \mathcal{V}(\bar{\Omega})$ and the set $\{V(x) : \nabla V(x) = 0\}$ of the corresponding critical values both have measure zero. Moreover, all maximizers of V in Ω are nondegenerate; i.e., the Hessian matrices of V are negative definite at those x with $V(x) = \max_{x\in\bar{\Omega}} V(x)$.*

Remark 4.5. Since V must satisfy the PDE (2.5) for some admissible control u, ∇V cannot be identically zero in any open subset of Ω. We also note that the functions V satisfying Assumption 4.4 are dense in $C^1(\Omega)$ in its strong topology [8]: a slight perturbation of an arbitrary $V \in C^1(\Omega)$, if necessary, will result in another satisfying Assumption 4.4. This is because the set of Morse functions (which trivially satisfy Assumption 4.4) is dense in the set of smooth functions on Ω (see [17, Corollary 6.8]), and the latter is in turn dense in $C^1(\Omega)$ (see [8, Theorem 2.4]).

In view of Assumption 4.4, for $V \in \mathcal{V}(\bar{\Omega})$, the control u specified by (4.2) satisfies the PDE $\frac{\sigma^2}{2}\Delta V + u \cdot \nabla V + 1 = 0$ and has the cost

$$J(u) = \int_{\bar{\Omega}} \frac{(1 + \frac{\sigma^2}{2}\Delta V)^2}{\|\nabla V\|^2}\, dx < \infty. \tag{4.4}$$

Problem 1 can now be equivalently formulated in terms of $V \in \mathcal{V}(\bar{\Omega})$ as follows.

PROBLEM 3. *Among all functions $V \in \mathcal{V}(\bar{\Omega})$, find the ones that*

$$\textit{maximize } \; W(u) = \int_{\bar{\Omega}} w(V)\, dx \;\textit{ subject to }\; J(u) = \int_{\bar{\Omega}} \frac{(1 + \frac{\sigma^2}{2}\Delta V)^2}{\|\nabla V\|^2}\, dx \; \leq J_0.$$

Problem 2 can also be similarly reformulated in terms of V, which is omitted here. The main results, Theorems 3.1 and 3.2, can be proved for the reformulated problems instead. To this purpose, we will prove the following key proposition in section 7.

PROPOSITION 4.6. *For each $V \in \mathcal{V}(\bar{\Omega})$, there exists a radially symmetric function $V^* \in \mathcal{V}(\bar{\Omega}^*)$ defined on the ball $\bar{\Omega}^*$ in $\mathbb{R}^n$ with the same volume as $\bar{\Omega}$ such that*

$$\int_{\bar{\Omega}^*} \frac{(1+\frac{\sigma^2}{2}\Delta V^*)^2}{\|\nabla V^*\|^2}\,dx = \int_{\bar{\Omega}} \frac{(1+\frac{\sigma^2}{2}\Delta V)^2}{\|\nabla V\|^2}\,dx \tag{4.5}$$

$$\text{and} \quad \int_{\bar{\Omega}^*} w(V^*)\,dx \geq \int_{\bar{\Omega}} w(V)\,dx. \tag{4.6}$$

It is easy to see that Proposition 4.6 implies Theorem 3.1. Indeed, suppose that u is an optimal control on a ball $\bar{\Omega}$ that solves Problem 3 (hence Problem 1) and V is the corresponding expected exit time function. Then $V \in \mathcal{V}(\bar{\Omega})$, and, according to Proposition 4.6, there exists a radially symmetric expected exit time function $V^* \in \mathcal{V}(\bar{\Omega})$ on the same domain $\bar{\Omega}$ whose corresponding control u^* satisfies $J(u^*) = J(u)$ and $W(u^*) \geq W(u)$. In other words, u^* is no worse a solution to Problem 3 than u. Similarly, Theorem 3.2 is also a direct result of Proposition 4.6.

5. Symmetrization. Proposition 4.6 will be proved using the method of *symmetrization*, which is an operation that transforms a domain and the functions defined on it into symmetric ones while preserving certain associated quantities. Symmetrization has proved to be a powerful tool in establishing the symmetry of solutions to PDE-constrained variational problems [3, 22]. Among the various types of symmetrization operations, the one employed in this paper for proving radial symmetry is called Schwarz symmetrization, henceforth referred to as symmetrization for simplicity. In the following, some basic notions and properties of symmetrization are reviewed. More details can be found in [3, 12].

DEFINITION 5.1 (symmetrization of sets). *The symmetrization of a bounded measurable set $\bar{\Omega} \subset \mathbb{R}^n$ is the unique ball in $\mathbb{R}^n$ with the same Lebesgue measure as $\bar{\Omega}$. The symmetrization of $\bar{\Omega}$ is denoted by $\bar{\Omega}^\sharp$.*

Since the volume of the unit ball in $\mathbb{R}^n$ is given by [1]

$$\omega_n := \left|\bar{B}(1)\right| = \frac{\pi^{n/2}}{\Gamma(1+\frac{n}{2})}, \tag{5.1}$$

where $\Gamma(\cdot)$ is the Gamma function and $|\cdot|$ denotes the Lebesgue measure, we must have $\bar{\Omega}^\sharp = \bar{B}(a)$, with the radius a satisfying $\omega_n a^n = |\bar{\Omega}|$. In particular, in the one-dimensional case, the symmetrization of $\bar{\Omega} \subset \mathbb{R}$ is the interval $[-\frac{|\bar{\Omega}|}{2}, \frac{|\bar{\Omega}|}{2}]$. Note that in the above definition, $\bar{\Omega}$ is not required to be simply connected or even connected.

Let $\bar{\Omega} \subset \mathbb{R}^n$ be a bounded measurable domain, and let $V : \bar{\Omega} \to \mathbb{R}_+$ be a measurable function with $0 \leq V \leq \rho_m$ for some $\rho_m := \sup_{x\in\bar{\Omega}} V(x) < \infty$. For each $\rho \in [0, \rho_m]$, define the ρ-level set and the ρ-superlevel set of V as

$$C_\rho := \{x \in \bar{\Omega} : V(x) = \rho\}, \quad D_\rho := \{x \in \bar{\Omega} : V(x) \geq \rho\}, \tag{5.2}$$

respectively. Note that as ρ increases from 0 to ρ_m, the set D_ρ shrinks monotonically from $D_0 = \bar{\Omega}$ to D_{ρ_m}, a set consisting of the maximizers of V in Ω.

DEFINITION 5.2 (symmetrization of functions). *The symmetrization of the function $V : \bar{\Omega} \to \mathbb{R}_+$ is the unique radially symmetric function $V^\sharp : \bar{\Omega}^\sharp \to \mathbb{R}$ in which each superlevel set is the symmetrization of the corresponding superlevel set of V, i.e., $\{x \in \bar{\Omega}^\sharp : V^\sharp(x) \geq \rho\} = (D_\rho)^\sharp$ for all $\rho \in [0, \rho_m]$. Or equivalently,*

$$V^\sharp(x) := \sup\{\rho : x \in (D_\rho)^\sharp\} \quad \forall x \in \bar{\Omega}^\sharp.$$

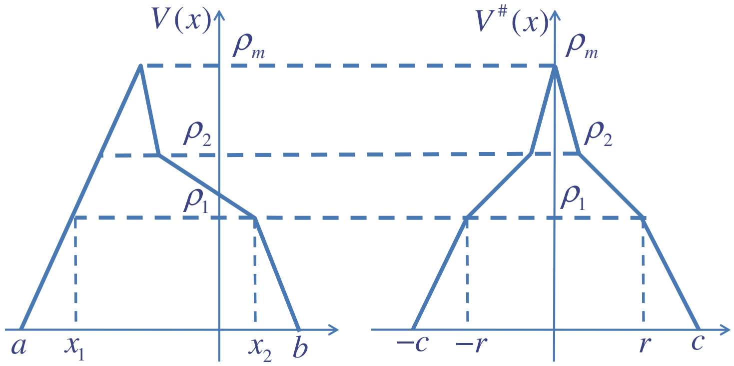


FIG. 5.1. *Symmetrization of a function in the one-dimensional case.*

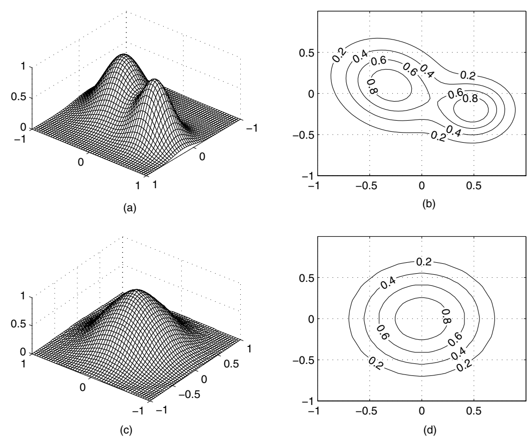


FIG. 5.2. *Symmetrization of a function on a two-dimensional domain.* (a) *Function V;* (b) *level sets of V;* (c) *symmetrization $V^\sharp$ of V;* (d) *level sets of $V^\sharp$.*

As ρ increases from 0 to ρ_m, the shrinking sets D_ρ after symmetrization will result in a set of concentric balls $(D_\rho)^\sharp$ with decreasing radii. Since these balls form the superlevel sets of $V^\sharp$, $V^\sharp$ must be radially symmetric. As an example, Figure 5.1 illustrates the symmetrization of a continuous function V defined on the one-dimensional domain $\bar{\Omega} = [a, b]$. In this case, for each $\rho \in [0, \rho_m)$, C_ρ consists of two points x_1 and x_2 with $x_1 < x_2$, and $D_\rho = [x_1, x_2]$. Let $r = (x_2 - x_1)/2$. Then the function $V^\sharp$ satisfies $V^\sharp(-r) = V^\sharp(r) = \rho$. This equality uniquely defines $V^\sharp$ as ρ varies continuously from 0 to ρ_m. The symmetrization of a function defined on a two-dimensional domain is shown in Figure 5.2.

Several useful properties of the symmetrization operation are listed below. Their proofs and more thorough discussions of the operation can be found in [3, 12].

LEMMA 5.3. *Let $U, V : \bar{\Omega} \to \mathbb{R}_+$ be two bounded, nonnegative measurable functions defined on the bounded measurable domain $\bar{\Omega}$.*

1. *If V is (Lipschitz) continuous, so is its symmetrization $V^\sharp$.*

2. *For any measurable function* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\int_{\bar{\Omega}^\sharp} \phi(V^\sharp)\, dx = \int_{\bar{\Omega}} \phi(V)\, dx.$$

In particular, the integral of V *is preserved:* $\int_{\bar{\Omega}^\sharp} V^\sharp\, dx = \int_{\bar{\Omega}} V\, dx$.

3. (Hardy–Littlewood inequality) *For* $U, V \in L^2(\bar{\Omega})$, *we have*

$$\int_{\bar{\Omega}} U(x)V(x)\, dx \le \int_{\bar{\Omega}^\sharp} U^\sharp(x)V^\sharp(x)\, dx.$$

As we shall see later on, to preserve certain quantities other than the integral of $V : \bar{\Omega} \to \mathbb{R}_+$, it is often necessary to follow the symmetrization operation by a scaling operation using a suitably defined monotonically increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$, resulting in the new function $V^* := f \circ V^\sharp = f(V^\sharp)$. In this case, V^* can be characterized as the unique radially symmetric function on $\bar{\Omega}^\sharp$ whose $f(\rho)$-superlevel set is the symmetrization of the ρ-superlevel set of V:

$$\{x \in \bar{\Omega}^\sharp : V^*(x) \ge f(\rho)\} = \{x \in \bar{\Omega} : V(x) \ge \rho\}^\sharp \quad \forall \rho \in [0, \rho_m].$$

6. Construction of the scaling function. The main idea for proving Proposition 4.6 is as follows. Given a function $V \in \mathcal{V}(\bar{\Omega})$ that is the expected exit time under some control u on $\bar{\Omega}$, we will find a suitable scaling function $f : \mathbb{R}_+ \to \mathbb{R}_+$ so that $V^* = f \circ V^\sharp$ defined on $\bar{\Omega}^\sharp$ satisfies the two properties (4.5) and (4.6) with $\bar{\Omega}^* := \bar{\Omega}^\sharp$; hence V^* corresponds to a radially symmetric control u^* on $\bar{\Omega}^*$ with no worse performance. The construction of f makes essential use of the famed isoperimetric inequality, which is valid only when the state dimension $n \ge 2$. Thus, in the rest of the paper, we shall assume $n \ge 2$. The proof of the main results for the case $n = 1$ has been established in [30].

Let $V \in \mathcal{V}(\bar{\Omega})$ be a function defined on $\bar{\Omega}$ with the range $[0, \rho_m]$, and let

$$\mathcal{S} := \{\rho : \nabla V \ne 0 \quad \forall x \in C_\rho\}$$

be the set of its regular values. Since $\mathcal{S}^c := [0, \rho_m] \setminus \mathcal{S} = V(\{x \in \bar{\Omega} : \nabla V = 0\})$ is the image of a compact set $\{x : \nabla V = 0\}$ under the continuous map V, $\mathcal{S}^c$ is also compact and hence closed, and $\mathcal{S}^c$ has measure zero by Assumption 4.4. Thus, $\mathcal{S}$ is a dense open subset of $[0, \rho_m]$.

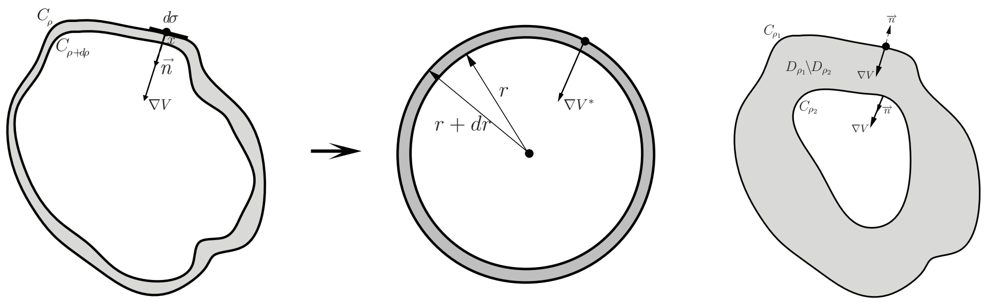
For each $\rho \in \mathcal{S}$, the level set C_ρ is a smooth $(n-1)$-dimensional submanifold (hypersurface) whose unit normal field pointing inward is denoted by $\overrightarrow{n}$. Let $x \in C_\rho$ be arbitrary. Then the volume element dx of $\mathbb{R}^n$ at x can be decomposed as $dx = d\sigma\, dn$, where $d\sigma$ is the area element of the $(n-1)$-dimensional hypersurface C_ρ and dn is the infinitesimal element along its normal $\overrightarrow{n}$ (see Figure 6.1(a)). Note that dx, $d\sigma$, and dn are all scalars. Furthermore, since the gradient ∇V at x is orthogonal to the level set C_ρ, and hence of the same direction as $\overrightarrow{n}$, we have $\|\nabla V\| = \frac{dV}{dn} = \frac{d\rho}{dn} \ne 0$, and thus $dn = \|\nabla V\|^{-1} d\rho$. As a result, at $x \in C_\rho$,

$$dx = d\sigma\, dn = \|\nabla V\|^{-1} d\sigma\, d\rho.$$

Using this decomposition, the following result can be easily obtained.

Lemma 6.1 ([22]). *For any integrable function* $\phi \in L^1(\bar{\Omega})$, *we have*

$$\int_{\bar{\Omega}} \phi\, dx = \int_0^{\rho_m} \int_{C_\rho} \phi\, \|\nabla V\|^{-1} d\sigma\, d\rho.$$



(a) Symmetrization of level sets of V (b) The set $D_{\rho_1} \setminus D_{\rho_2}$

FIG. 6.1. *Decomposition of $\bar{\Omega}$ into ring-shaped domains.*

More generally, for any constant $\rho \in [0, \rho_m]$,

$$\int_{D_\rho} \phi \, dx = \int_{\rho}^{\rho_m} \int_{C_\rho} \phi \, \|\nabla V\|^{-1} d\sigma \, d\rho.$$

See also the so-called co-area theorem in [6]. Intuitively, Lemma 6.1 implies that the integration of a function over the domain $\bar{\Omega}$ (or D_ρ) can be carried out alternatively through its decomposition into an infinite number of ring-shaped infinitesimal domains, each one sandwiched between the level sets C_ρ and $C_{\rho+d\rho}$ (see Figure 6.1(a)).

6.1. Some preliminary functions. Inspired by [22], we now define several functions that will be useful later on. By choosing $\phi \equiv 1$ in Lemma 6.1, we define

$$A(\rho) := |D_\rho| = \int_{D_\rho} dx = \int_{\rho}^{\rho_m} \int_{C_\rho} \|\nabla V\|^{-1} \, d\sigma \, d\rho, \tag{6.1}$$

i.e., $A(\rho)$ is the volume of D_ρ. In the case when D_ρ has more than one connected component, $A(\rho)$ is the total volume of all of them. By Assumption 4.4, as ρ increases from 0 to ρ_m, D_ρ shrinks from $D_0 = \bar{\Omega}$ to a set D_{ρ_m} of zero measure; hence $A(\rho)$ decreases monotonically from $|\bar{\Omega}|$ to 0. By Assumption 4.4, $|C_\rho| = 0$ for all ρ. Thus $A(\rho)$ is an absolutely continuous function and is differentiable a.e. on $[0, \rho_m]$. Indeed, since the inner integral $\int_{C_\rho} \|\nabla V\|^{-1} \, d\sigma$ in (6.1) is well defined for $\rho \in \mathcal{S}$, we have

$$A'(\rho) = -\int_{C_\rho} \|\nabla V\|^{-1} \, d\sigma \in (-\infty, 0) \quad \forall \rho \in \mathcal{S}. \tag{6.2}$$

In the following, we shall extend the definition of $A'(\rho)$ in (6.2) to all $\rho \in [0, \rho_m]$, with the understanding that it is possible that $A'(\rho) = -\infty$ for $\rho \notin \mathcal{S}$.

Define another function $P(\rho) \geq 0$ by

$$P(\rho) := \int_{C_\rho} \|\nabla V\| \, d\sigma, \quad \rho \in [0, \rho_m]. \tag{6.3}$$

Note that $P(\rho)$ is continuous (hence bounded) on $[0, \rho_m]$ as both C_ρ and ∇V vary continuously with ρ. It is also easy to see that $P(\rho_m) = 0$, and $P'(\rho)$ exists at $\rho \in \mathcal{S}$.

LEMMA 6.2 (lower bound on $P(\rho)$). *The function $P(\rho)$ defined in* (6.3) *satisfies*

$$P(\rho) \geq \frac{n^2 \omega_n^{2/n} A^{(2n-2)/n}(\rho)}{|A'(\rho)|} \quad \forall \rho \in [0, \rho_m]. \tag{6.4}$$

Here we recall that the constant ω_n is the volume of the unit ball in $\mathbb{R}^n$ given in (5.1).

Proof. First assume $\rho \in \mathcal{S}$. By applying the Cauchy–Schwarz inequality, we have

$$P(\rho)\,|A'(\rho)| = \int_{C_\rho} \|\nabla V\|\,d\sigma \cdot \int_{C_\rho} \|\nabla V\|^{-1}\,d\sigma \geq \left(\int_{C_\rho} d\sigma\right)^2 = \big[\text{area}(C_\rho)\big]^2.$$

Here $\text{area}(C_\rho)$ denotes the $(n-1)$-dimensional area of the hypersurface C_ρ. Applying the isoperimetric inequality in $\mathbb{R}^n$ (see [20] and the references therein) to C_ρ and the region D_ρ it encloses, we have

$$\text{area}\,(C_\rho) \geq n\omega_n^{1/n}|D_\rho|^{(n-1)/n} = n\omega_n^{1/n}A^{(n-1)/n}(\rho),$$

with equality if and only if C_ρ is a sphere. The above two inequalities together yield (6.4). This completes the proof for $\rho \in \mathcal{S}$. The same reasoning still applies if $\rho \notin \mathcal{S}$ but $|A'(\rho)| < \infty$. If $\rho \notin \mathcal{S}$ and $|A'(\rho)| = \infty$, then (6.4) is trivially satisfied. ∎

Define two more nonnegative functions $Q(\rho)$ and $G(\rho)$ for $\rho \in [0, \rho_m]$ by

$$Q(\rho) := \int_{C_\rho} \frac{(1+\frac{\sigma^2}{2}\Delta V)^2}{\|\nabla V\|^3}\,d\sigma, \qquad G(\rho) := \frac{2|A'(\rho)|^{1/2}}{n\omega_n^{1/n}A^{(n-1)/n}(\rho)}Q^{1/2}(\rho). \tag{6.5}$$

Since $\|\nabla V\| > 0$, and $1 + \frac{\sigma^2}{2}\Delta V = -u \cdot \nabla V$ is finite for $\rho \in \mathcal{S}$, the function $Q(\rho)$, and hence $G(\rho)$, is well defined and finite on $\mathcal{S}$. Even though $Q(\rho)$ and $G(\rho)$ may have infinite values for $\rho \notin \mathcal{S}$, the following lemma shows that they are integrable.

LEMMA 6.3. *Both $Q(\rho)$ and $G(\rho)$ are integrable on* $[0, \rho_m]$.

See Appendix A for the proof of Lemma 6.3.

LEMMA 6.4. *The functions $P(\rho)$ and $Q(\rho)$ satisfy*

$$P^{1/2}(\rho)\,Q^{1/2}(\rho) \geq A'(\rho) - \frac{\sigma^2}{2}P'(\rho) \quad \textit{for almost all } \rho \in [0, \rho_m].$$

Proof. Assume $\rho \in \mathcal{S}$. An application of the Cauchy–Schwarz inequality yields

$$P(\rho)\,Q(\rho) = \int_{C_\rho} \|\nabla V\|\,d\sigma \cdot \int_{C_\rho} \frac{(1+\frac{\sigma^2}{2}\Delta V)^2}{\|\nabla V\|^3}\,d\sigma \geq \left(\int_{C_\rho} \frac{|1+\frac{\sigma^2}{2}\Delta V|}{\|\nabla V\|}\,d\sigma\right)^2.$$

Since $1 + \frac{\sigma^2}{2}\Delta V \leq 0$ as $V \in \mathcal{V}(\bar{\Omega})$, taking the square root of both sides and integrating over a neighborhood $[\rho_1, \rho_2]$ of ρ contained entirely within $\mathcal{S}$, we obtain

$$\begin{aligned}
\int_{\rho_1}^{\rho_2} P^{1/2}(\rho)\,Q^{1/2}(\rho)\,d\rho &\geq -\int_{\rho_1}^{\rho_2}\int_{C_\rho} \frac{1+\frac{\sigma^2}{2}\Delta V}{\|\nabla V\|}\,d\sigma\,d\rho \\
&= -\int_{D_{\rho_1}\setminus D_{\rho_2}} \left(1+\frac{\sigma^2}{2}\Delta V\right)dx \qquad \text{(by Lemma 6.1)} \\
&= -[A(\rho_1) - A(\rho_2)] - \frac{\sigma^2}{2}\int_{D_{\rho_1}\setminus D_{\rho_2}} \Delta V\,dx. \qquad (6.6)
\end{aligned}$$

For the second term in (6.6), since the integration domain $D_{\rho_1} \setminus D_{\rho_2}$ is a ring-shaped domain sandwiched between the two hypersurfaces C_{ρ_1} and C_{ρ_2} (see Figure 6.1(b)), by the divergence theorem in multivariate calculus [4],

$$\int_{D_{\rho_1}\setminus D_{\rho_2}} \Delta V\,dx = \int_{D_{\rho_1}\setminus D_{\rho_2}} \text{div}\,(\nabla V)\,dx = \int_{\partial(D_{\rho_1}\setminus D_{\rho_2})} \nabla V \cdot \vec{n}\,d\sigma,$$

where $\overrightarrow{n}$ is the outer normal direction of $D_{\rho_1} \setminus D_{\rho_2}$ along its boundary. Note that the boundary $\partial(D_{\rho_1} \setminus D_{\rho_2}) = C_{\rho_1} \cup C_{\rho_2}$, and that the outer normal $\overrightarrow{n}$ and ∇V are of the opposite direction on C_{ρ_1} and the same direction on C_{ρ_2}. Therefore, we have

$$\int_{D_{\rho_1} \setminus D_{\rho_2}} \Delta V \, dx = \int_{C_{\rho_2}} \|\nabla V\| \, d\sigma - \int_{C_{\rho_1}} \|\nabla V\| \, d\sigma = P(\rho_2) - P(\rho_1).$$

Thus (6.6) becomes

$$\int_{\rho_1}^{\rho_2} P^{1/2}(\rho) \, Q^{1/2}(\rho) \, d\rho \geq A(\rho_2) - A(\rho_1) - \frac{\sigma^2}{2}\big[P(\rho_2) - P(\rho_1)\big].$$

Since both $A(\rho)$ and $P(\rho)$ are differentiable on $\mathcal{S}$, we rewrite the above inequality as

$$\int_{\rho_1}^{\rho_2} \left[P^{1/2}(\rho)Q^{1/2}(\rho) - A'(\rho) + \frac{\sigma^2}{2}P'(\rho)\right] d\rho \geq 0 \quad \forall\, [\rho_1, \rho_2] \subset \mathcal{S}.$$

Thus, the integrand is nonnegative a.e. on $\mathcal{S}$, which is the desired conclusion. ∎

Using Lemma 6.4 and a version of the Gronwall inequality, in Appendix B we prove the following upper bound estimate of $P(\rho)$.

LEMMA 6.5 (upper bound on $P(\rho)$). *The function $P(\rho)$ satisfies*

$$P(\rho) \leq 2\sigma^{-2} e^{-\int_0^\rho \sigma^{-2} G(\eta)\, d\eta} \int_\rho^{\rho_m} |A'(\xi)| \, e^{\int_0^\xi \sigma^{-2} G(\eta)\, d\eta} \, d\xi \quad \forall \rho \in [0, \rho_m]. \tag{6.7}$$

Note that by Lemma 6.3, $e^{\int_0^\rho \sigma^{-2} G(\eta)\, d\eta}$ is bounded for $\rho \in [0, \rho_m]$. Hence the integral in (6.7) is finite for all $\rho \in [0, \rho_m]$, leading to a nontrivial upper bound for $P(\rho)$.

6.2. Scaling function $f(\cdot)$. Combining the lower bound of $P(\rho)$ in Lemma 6.2 and its upper bound in Lemma 6.5, we have

$$\frac{n^2 \omega_n^{2/n} A^{(2n-2)/n}(\rho)}{|A'(\rho)|} \leq 2\sigma^{-2} e^{-\int_0^\rho \sigma^{-2} G(\eta)\, d\eta} \int_\rho^{\rho_m} |A'(\xi)| \, e^{\int_0^\xi \sigma^{-2} G(\eta)\, d\eta} \, d\xi \tag{6.8}$$

for $\rho \in [0, \rho_m]$. Define a nonnegative function $h(\rho)$, $\rho \in [0, \rho_m]$, as

$$h(\rho) := \frac{2\sigma^{-2}|A'(\rho)| e^{-\int_0^\rho \sigma^{-2} G(\eta)\, d\eta}}{n^2 \omega_n^{2/n} A^{(2n-2)/n}(\rho)} \int_\rho^{\rho_m} |A'(\xi)| e^{\int_0^\xi \sigma^{-2} G(\eta)\, d\eta} \, d\xi. \tag{6.9}$$

LEMMA 6.6. *The function $h(\rho)$ satisfies $h(\rho) \geq 1$ for all $\rho \in [0, \rho_m]$. Moreover, as $\rho \to \rho_m$, $h(\rho)$ will converge to a finite value. As a result, $h(\rho)$ is integrable on $[0, \rho_m]$.*

The conclusion that $h(\rho) \geq 1$ is immediate from (6.8) and the definition of $h(\rho)$ in (6.9). The rest of the conclusions can be proved by following the same steps as in the proof of Lemma A.1, which is omitted here.

Finally, we are able to define the scaling function $f(\rho)$ as

$$f(\rho) := \int_0^\rho h(\xi) \, d\xi \quad \forall \rho \in [0, \rho_m]. \tag{6.10}$$

Using Lemma 6.6, we obtain the following important properties of $f(\rho)$.

COROLLARY 6.7. *The function $f(\rho)$ defined in* (6.10) *is a bounded, continuous, and strictly increasing function on $[0, \rho_m]$, and it satisfies $f(0) = 0$ and $f(\rho) \geq \rho$, $\rho \in [0, \rho_m]$.*

7. Proof of Proposition 4.6. With the construction of the scaling function $f(\cdot)$, we now prove Proposition 4.6. Assume $V \in \mathcal{V}(\bar{\Omega})$ is a function defined on the domain $\bar{\Omega}$ with range $[0, \rho_m]$. Recall that the symmetrization of V results in a radially symmetric function $V^\sharp$ defined on the domain $\bar{\Omega}^\sharp = \bar{B}(a)$, a ball with the radius a such that

$$\omega_n a^n = |\bar{B}(a)| = |\bar{\Omega}|, \quad \text{i.e.,} \quad a = \omega_n^{-1/n} |\bar{\Omega}|^{1/n}.$$

By Lemma 5.3, $V^\sharp(x)$ is continuous, has the range $[0, \rho_m]$, and decreases monotonically from ρ_m to 0 as $\|x\|$ increases from 0 to a. Let $f(\rho)$, $\rho \in [0, \rho_m]$, be the scaling function given in (6.10). Define the function V^* on $\bar{\Omega}^* := \bar{\Omega}^\sharp$ as $V^* = f \circ V^\sharp$, i.e.,

$$V^*(x) = f[V^\sharp(x)] \quad \forall x \in \bar{\Omega}^*. \tag{7.1}$$

Then by Corollary 6.7, $V^*(x)$ is also radially symmetric and continuous, and it takes the maximum value $f(\rho_m)$ at $x = 0$ and decreases strictly to 0 as $\|x\|$ increases to a.

PROPOSITION 7.1. *The function V^* defined in* (7.1) *satisfies*

$$\int_{\bar{\Omega}^*} w(V^*)\, dx \geq \int_{\bar{\Omega}} w(V)\, dx, \tag{7.2}$$

with equality if and only if $\bar{\Omega} = \bar{\Omega}^*$ and $V = V^*$.

Proof. By Corollary 6.7, $V^* = f(V^\sharp) \geq V^\sharp$; hence $w(V^*) \geq w(V^\sharp)$ by the monotonicity of w. Therefore,

$$\int_{\bar{\Omega}^*} w(V^*)\, dx \geq \int_{\bar{\Omega}^\sharp} w(V^\sharp)\, dx = \int_{\bar{\Omega}} w(V)\, dx.$$

Here the last step follows from Lemma 5.3. In order to have equality in (7.2), we must have $f(\rho) = \rho$, i.e., $h \equiv 1$ a.e. on $[0, \rho_m]$. Thus equality holds a.e. in (6.8), and hence in (6.4) and (6.7) as well. This is possible if and only if the ρ-level set C_ρ of V is a sphere for almost all ρ, or equivalently, V is radially symmetric on $\bar{\Omega}$. ∎

Since $V^*(x)$ is symmetric, its representation in the polar coordinates is simply $V^*(r)$, where $r := \|x\| \in [0, a]$. The following lemma characterizes $V^*(r)$ precisely.

LEMMA 7.2. *For each $r \in [0, a]$, $V^*(r) = f(\rho)$, where $\rho \in [0, \rho_m]$ is given by*

$$\omega_n r^n = A(\rho). \tag{7.3}$$

Proof. By the discussion at the end of section 5, for each $\rho \in [0, \rho_m]$, the $f(\rho)$-superlevel set of $V^*(x)$, which is a ball $\bar{B}(r)$ of a certain radius r, is the symmetrization of the ρ-superlevel set D_ρ of $V(x)$. Thus, $\bar{B}(r)$ and D_ρ should have the same volume, $\omega_n r^n = |\bar{B}(r)| = |D_\rho| = A(\rho)$, which is exactly (7.3). ∎

Note that (7.3) can be thought of as a coordinate transform whose differential is

$$n\omega_n r^{n-1} dr = A'(\rho) d\rho \quad \Rightarrow \quad \frac{d\rho}{dr} = \frac{n\omega_n r^{n-1}}{A'(\rho)}. \tag{7.4}$$

We next compute the integral $\int_{\bar{\Omega}^*} \frac{(1 + \frac{\sigma^2}{2} \Delta V^*)^2}{\|\nabla V^*\|^2}\, dx$. The computation can be greatly simplified by noting the radial symmetry of $V^*(x)$. We first derive ∇V^* in polar coordinates. It is always of the inward radial direction (see Figure 6.1(a)), with a magnitude

$$\|\nabla V^*\| = \left| \frac{dV^*(r)}{dr} \right| = \left| \frac{df(\rho)}{dr} \right| = \left| \frac{df(\rho)}{d\rho} \frac{d\rho}{dr} \right| = \left| h(\rho) \frac{n\omega_n r^{n-1}}{A'(\rho)} \right|.$$

Note that (7.4) is used in deriving the last step. Furthermore, since by (7.3), $r^{n-1} = (r^n)^{(n-1)/n} = [\omega_n^{-1}A(\rho)]^{(n-1)/n}$, and $h(\rho)$ is given in (6.9), we have

(7.5)
$$\|\nabla V^*\| = \frac{n\omega_n^{1/n}A^{(n-1)/n}(\rho)\,h(\rho)}{|A'(\rho)|} = \frac{2\sigma^{-2}e^{-\int_0^\rho \sigma^{-2}G(\eta)\,d\eta}}{n\omega_n^{1/n}A^{(n-1)/n}(\rho)}\int_\rho^{\rho_m}|A'(\xi)|e^{\int_0^\xi \sigma^{-2}G(\eta)\,d\eta}\,d\xi.$$

Taking the square of (7.5) and plugging in $G(\rho)$ and $Q(\rho)$ as in (6.5), we get

$$\|\nabla V^*\|^2 = \frac{\sigma^{-4}G^2(\rho)}{Q(\rho)|A'(\rho)|}e^{-2\int_0^\rho \sigma^{-2}G(\eta)\,d\eta}\left[\int_\rho^{\rho_m}|A'(\xi)|e^{\int_0^\xi \sigma^{-2}G(\eta)\,d\eta}\,d\xi\right]^2. \tag{7.6}$$

An important implication of (7.5) is that $\|\nabla V^*\|$, and hence ∇V^*, is continuous and nonzero everywhere on $\bar{\Omega}$ except possibly at $x = 0$. Thus, it is meaningful to further compute higher derivatives, such as ΔV^*. In polar coordinates, we have

$$\begin{aligned}\Delta V^* &= \frac{1}{r^{n-1}}\frac{d}{dr}\left[r^{n-1}\frac{dV^*(r)}{dr}\right] = \frac{1}{r^{n-1}}\frac{d\rho}{dr}\frac{d}{d\rho}\left[r^{n-1}h(\rho)\frac{d\rho}{dr}\right]\\ &= \frac{n\omega_n}{A'(\rho)}\frac{d}{d\rho}\left[n\omega_n^{(2-n)/n}\frac{A^{(2n-2)/n}(\rho)}{A'(\rho)}h(\rho)\right]\\ &= \frac{2\sigma^{-2}}{|A'(\rho)|}\frac{d}{d\rho}\left[e^{-\int_0^\rho \sigma^{-2}G(\eta)\,d\eta}\int_\rho^{\rho_m}|A'(\xi)|e^{\int_0^\xi \sigma^{-2}G(\eta)\,d\eta}\,d\xi\right]\\ &= -2\sigma^{-2} - \frac{2\sigma^{-4}G(\rho)}{|A'(\rho)|}e^{-\int_0^\rho \sigma^{-2}G(\eta)\,d\eta}\int_\rho^{\rho_m}|A'(\xi)|e^{\int_0^\xi \sigma^{-2}G(\eta)\,d\eta}\,d\xi. \end{aligned} \tag{7.7}$$

Therefore,

$$1 + \frac{\sigma^2}{2}\Delta V^* = -\frac{\sigma^{-2}G(\rho)}{|A'(\rho)|}e^{-\int_0^\rho \sigma^{-2}G(\eta)\,d\eta}\int_\rho^{\rho_m}|A'(\xi)|e^{\int_0^\xi \sigma^{-2}G(\eta)\,d\eta}\,d\xi \le 0. \tag{7.8}$$

Combining (7.6) and (7.8) yields

$$\frac{(1+\frac{\sigma^2}{2}\Delta V^*)^2}{\|\nabla V^*\|^2} = \frac{Q(\rho)}{|A'(\rho)|}. \tag{7.9}$$

Integrating (7.9) over $\bar{\Omega}^*$ and using its radial symmetry, we have

$$\begin{aligned}\int_{\bar{\Omega}^*}\frac{(1+\frac{\sigma^2}{2}\Delta V^*)^2}{\|\nabla V^*\|^2}\,dx &= \int_0^a \frac{(1+\frac{\sigma^2}{2}\Delta V^*)^2}{\|\nabla V^*\|^2}n\omega_n r^{n-1}\,dr\\ &= \int_0^{\rho_m}\frac{Q(\rho)}{|A'(\rho)|}\cdot|A'(\rho)|\,d\rho\\ &= \int_0^{\rho_m}Q(\rho)\,d\rho < \infty. \end{aligned} \tag{7.10}$$

A comparison of the above equation with (A.1) leads to the following result.

PROPOSITION 7.3. *The function V^* defined in* (7.1) *satisfies*

$$\int_{\bar{\Omega}^*}\frac{(1+\frac{\sigma^2}{2}\Delta V^*)^2}{\|\nabla V^*\|^2}\,dx = \int_{\bar{\Omega}}\frac{(1+\frac{\sigma^2}{2}\Delta V)^2}{\|\nabla V\|^2}\,dx.$$

We now complete the proof of Proposition 4.6. The function V^* constructed in (7.1) is a radially symmetric function defined on the ball $\bar{\Omega}^*$, and $\bar{\Omega}^* = \bar{\Omega}^\sharp$ has the same volume as $\bar{\Omega}$. In view of Propositions 7.1 and 7.3, V^* also satisfies both the conditions (4.5) and (4.6). It remains to show that $V^* \in \mathcal{V}(\bar{\Omega}^*)$. Obviously, V^* satisfies the first property of Definition 4.3. From (7.5) and (7.7), it can be seen that $V^* \in C^1(\bar{\Omega}^*)$ and is second order differentiable a.e. on $\bar{\Omega}^*$. Let u^* be the control on $\bar{\Omega}^*$ corresponding to V^*. Then u^* must be radially symmetric and satisfy $-u^* \cdot \nabla V^* = 1 + \frac{\sigma^2}{2}\Delta V^*$. From the radial symmetry of both u^* and ∇V^*, we obtain

$$\|u^*\|^2 = \frac{(1 + \frac{\sigma^2}{2}\Delta V^*)^2}{\|\nabla V^*\|^2}.$$

Hence $u^* \in L^2(\bar{\Omega}^*; \mathbb{R}^n)$ by (7.10). Furthermore, by (7.9),

$$\|u^*\|^2 = \frac{Q(\rho)}{|A'(\rho)|} = \frac{\int_{C_\rho} \|u\|^2 \|\nabla V\|^{-1}\, d\sigma}{\int_{C_\rho} \|\nabla V\|^{-1}\, d\sigma} \leq \sup_{x \in \bar{\Omega}} \|u(x)\|^2 < \infty. \tag{7.11}$$

This implies that $\sup_{\bar{\Omega}} \|u^*\| \leq \sup_{\bar{\Omega}} \|u\|$ and thus that u^* is admissible. Finally, the fact that $1 + \frac{\sigma^2}{2}\Delta V^* \leq 0$ is established in (7.8). This completes the proof of Proposition 4.6.

Remark 7.4. From the above derivation, V^* and u^* satisfy $\|u^*\|_{L^\infty} \leq \|u\|_{L^\infty}$, $\|u^*\|_{L^2} = \|u\|_{L^2}$, $\|V^*\|_{L^\infty} \geq \|V\|_{L^\infty}$, and $\|V^*\|_{L^2} \geq \|V\|_{L^2}$.

8. Generalized $W(u)$.

The definition of $W(u)$ in (2.7) can be generalized to

$$W_\mu(u) := \int_{\bar{\Omega}} \mu\, w(V)\, dx,$$

where $\mu : \bar{\Omega} \to \mathbb{R}_+$ is an integrable weight function on $\bar{\Omega}$. $W(u)$ reduces to $W_\mu(u)$ if $\mu \equiv 1$. With this new definition, Problems 1 and 2 can be generalized accordingly.

COROLLARY 8.1. *Suppose that $\bar{\Omega}$ is a ball and $\mu : \bar{\Omega} \to \mathbb{R}_+$ is radially symmetric. Consider Problems* 1 *and* 2 *on $\bar{\Omega}$ with the generalized $W_\mu(u)$ replacing $W(u)$. Then the conclusions in Theorem* 3.1 *still hold.*

Proof. For any $V \in \mathcal{V}(\bar{\Omega})$ that is not radially symmetric, let V^* and u^* be as constructed in section 7. Then $J(u^*) = J(u)$ by Proposition 7.3, and

$$W_\mu(u^*) = \int_{\bar{\Omega}} \mu\, w(V^*)\, dx > \int_{\bar{\Omega}} \mu\, w(V^\sharp)\, dx = \int_{\bar{\Omega}} \mu^\sharp [w(V)]^\sharp\, dx \geq \int_{\bar{\Omega}} \mu\, w(V)\, dx = W_\mu(u)$$

by the Hardy–Littlewood inequality in Lemma 5.3. Thus, u^* is a better solution. ∎

9. Numerical solutions on radially symmetric domains.

Suppose $\bar{\Omega} = \bar{B}(a)$, $a > 0$, is a ball in $\mathbb{R}^n$. By Theorem 3.1, the optimal solutions u and V to Problem 1, if they exist, are symmetric. We can thus focus on such u and V,

$$u(x) = -\tilde{u}(r)\frac{x}{r}, \quad V(x) = \tilde{V}(r),$$

for some functions $\tilde{u}, \tilde{V} : [0, a] \to \mathbb{R}$, where $r = \|x\|$. By Lemma 4.1, $\tilde{u} \geq 0$. Due to the symmetry of u and V, the costs $J(u)$ and $W(u)$ can be written as

$$J(u) = \int_0^a n\omega_n r^{n-1} \tilde{u}^2\, dr, \quad W(u) = \int_0^a n\omega_n r^{n-1} \tilde{V}\, dr.$$

Here we assume $w(\cdot)$ is the identity function and $\sigma = 1$. Then, the PDE (2.3) becomes

$$\tilde{V}'' + \left(\frac{n-1}{r} - 2\tilde{u}\right)\tilde{V}' + 2 = 0 \quad \forall r \in (0,a], \quad \tilde{V}'(0) = 0,\ V(a) = 0. \tag{9.1}$$

Note that $\tilde{V}'(0) = 0$ as V achieves its maximum at $x = 0$ and is differentiable in Ω, and hence at $x = 0$, by Lemma 2.1. Thus, Problem 1 is equivalent to

$$\text{maximize} \int_0^a n\omega_n r^{n-1}\tilde{V}\,dr \text{ subject to (9.1)}, \quad \int_0^a n\omega_n r^{n-1}\tilde{u}^2\,dr \le J_0,\ \tilde{u} \in [0,M]. \tag{9.2}$$

Let M be finite. We first obtain an upper bound on the optimal $\tilde{V}$. Let $u_M = -M\frac{x}{r}$ (i.e., $\tilde{u}_M \equiv M$) be the most aggressive symmetric feasible control (which may have a cost larger than J_0), and let $\tilde{V}_M$ be the corresponding solution to (9.1),

$$\tilde{V}_M'' + \left(\frac{n-1}{r} - 2M\right)\tilde{V}_M' + 2 = 0 \quad \forall r \in (0,a], \quad \tilde{V}_M'(0) = 0,\ V_M(a) = 0. \tag{9.3}$$

LEMMA 9.1. *$\tilde{V}_M$ is bounded with $\tilde{V}_M(0) \le \frac{a}{nM}e^{2Ma} < \infty$.*

Proof. Fix a small $\epsilon > 0$. Multiplying (9.3) by $\epsilon^{n-1}e^{\int_\epsilon^r(\frac{n-1}{r}-2M)\,dr} = r^{n-1}e^{-2M(r-\epsilon)}$ and integrating from ϵ to r, we have $[r^{n-1}e^{-2M(r-\epsilon)}\tilde{V}_M']_\epsilon^r + 2\int_\epsilon^r s^{n-1}e^{-2M(s-\epsilon)}\,ds = 0$. Let $\epsilon \to 0$ and note that $\tilde{V}_M'(0) = 0$. Then the above implies

$$\tilde{V}_M'(r) = -2r^{1-n}e^{2Mr}\int_0^r s^{n-1}e^{-2Ms}\,ds.$$

Since $\tilde{V}_M(a) = 0$, we can integrate the above equation from r to a to obtain

$$\tilde{V}_M(r) = \int_r^a 2r^{1-n}e^{2Mr}\int_0^r s^{n-1}e^{-2Ms}\,ds\,dr. \tag{9.4}$$

Thus, $\tilde{V}_M(r) \le \tilde{V}_M(0) \le \int_0^a 2r^{1-n}e^{2Mr}\int_0^r s^{n-1}\,ds\,dr \le \frac{2}{n}\int_0^a re^{2Mr}\,dr \le \frac{a}{nM}e^{2Ma}$. ∎

Remark 9.2. Another implication of (9.4) is that

$$\begin{aligned}\tilde{V}_M(r) &\ge \int_r^a 2r^{1-n}e^{2Mr}\int_0^{r/2} s^{n-1}e^{-Mr}\,ds\,dr = \frac{2^{1-n}}{n}\int_r^a re^{Mr}\,dr \\ &= \frac{1-Mr}{n2^{n-1}M^2}e^{Mr} + \frac{Ma-1}{n2^{n-1}M^2}e^{Ma}.\end{aligned}$$

As $M \to \infty$, the last term as a function of r will diverge to $+\infty$ for any $r \in [0,a)$, implying that $W(u_M) \to \infty$. This shows that, in Problem 1, if M and J_0 are large enough, the aggregated expected exit time $W(u)$ can be made arbitrarily large.

By Lemma 4.2, $\tilde{V}_M$ in (9.4) is an upper bound for the optimal $\tilde{V}$ in problem (9.2).

COROLLARY 9.3. *The solution $\tilde{V}$ to problem* (9.2) *satisfies $\tilde{V}(r) \le \tilde{V}_M(r)$, $r \in [0,a]$, where $\tilde{V}_M$ is defined as in* (9.4). *In particular, $\tilde{V}(r) \le \frac{a}{nM}e^{2Ma}$ for all $r \in [0,a]$.*

We now show that solutions to the problem (9.2) exist for finite M.

COROLLARY 9.4. *Suppose M is finite. Then bounded solutions to the variational problem* (9.2) *exist.*

Proof. Let $\tilde{u}_n$ be a sequence satisfying $0 \le \tilde{u}_n \le M$, $\int_0^a n\omega_n r^{n-1}\tilde{u}_n^2\,dr \le J_0$, and that $\int_0^a n\omega_n r^{n-1}\tilde{V}_n\,dr$ converges to the maximum value achieved by the solution to (9.2) as $n \to \infty$. By the weak compactness of L^2-balls [25], a subsequence $\tilde{u}_{n_k}$ converges weakly to some $\tilde{u}^*$ in the same L^2-ball, i.e., $\int_0^a n\omega_n r^{n-1}(\tilde{u}^*)^2\,dr \le J_0$, and, as a result of weak convergence, we have $0 \le \tilde{u}^* \le M$ a.e. Let $\tilde{V}_{n_k}$ be the solutions to (9.1) for $\tilde{u}_{n_k}$, which are continuous nonincreasing functions bounded from above by Corollary 9.3. Then, by finding a subsequence if necessary, we can assume $\tilde{V}_{n_k}$ converges pointwise a.e. (hence in L^2 as well) to a continuous nonincreasing function $\tilde{V}^*$. Thus, $\int_0^a n\omega_n r^{n-1}\tilde{V}^*\,dr = \lim_{n\to\infty}\int_0^a n\omega_n r^{n-1}\tilde{V}_n^*\,dr$ achieves the maximum value in the problem (9.2). Since the homogeneous version of (9.1) does not admit nonzero solutions, by [5, Theorem 6.4], $\tilde{V}^*$ must be the unique solution to (9.1) for $\tilde{u}^*$. Hence, the pair $\tilde{u}^*$ and $\tilde{V}^*$ is indeed a solution to the problem (9.2). ∎

To solve (9.2), define $z_1 := \tilde{V}$, $z_2 := \tilde{V}'$, and $z_3 := \int_0^r r^{n-1}\tilde{u}^2\,dr$. Then

$$(9.5)\qquad \begin{cases} z_1' = z_2, & z_1(a) = 0,\\ z_2' = -\left(\frac{n-1}{r} - 2\tilde{u}\right) z_2 - 2, & z_2(0) = 0,\\ z_3' = r^{n-1}\tilde{u}^2, & z_3(0) = 0,\ z_3(a) = \frac{J_0}{n\omega_n}. \end{cases}$$

Problem (9.2) is reformulated as an optimal control problem for the one-dimensional dynamical system (9.5) with control $\tilde{u}$ and the cost function $\int_0^a r^{n-1} z_1\,dr$. This is a two-point boundary value problem on $r \in [0,a]$, with singularity at the left boundary 0.

Define the Hamiltonian $H := r^{n-1}z_1 + \lambda_1 z_2 - \lambda_2\left(\frac{n-1}{r} - 2\tilde{u}\right)z_2 - 2\lambda_2 + \lambda_3 r^{n-1}\tilde{u}^2$, where λ_i, $i = 1,2,3$, are costates with dynamics $\lambda_i' = -\frac{\partial H}{\partial z_i}$. Thus λ_3 is a constant, $\lambda_1' = -r^{n-1}$ with $\lambda_1(0) = 0$ (implying $\lambda_1 = -\frac{r^n}{n}$), and

$$(9.6)\qquad \lambda_2' = \lambda_2\left(\frac{n-1}{r} - 2\tilde{u}\right) + \frac{r^n}{n}, \quad \lambda_2(a) = 0.$$

By the maximum principle [23], the optimal control is given by

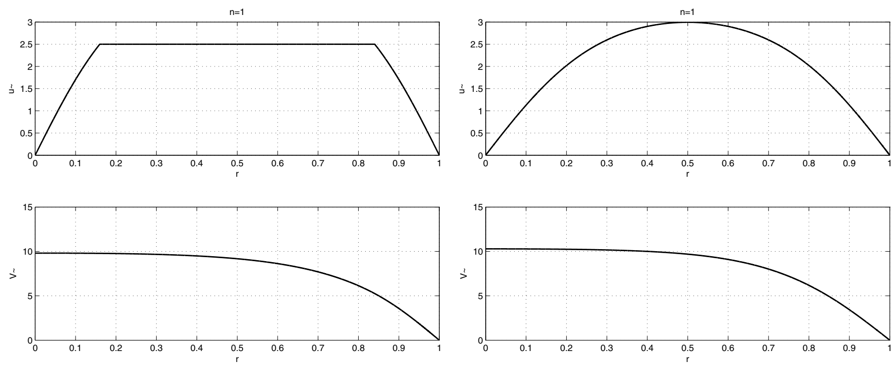
$$(9.7)\qquad \tilde{u} = \operatorname{argmin}_{\tilde{u}\in[0,M]} H = \max\left\{\min\left\{-\lambda_2\lambda_3^{-1}r^{1-n}z_2,\ M\right\}, 0\right\}.$$

Plugging (9.7) into (9.6) and the last two equations of (9.5), we obtain a system of three ODEs with two unknown parameters λ_3 and $z_2(a)$, which can be determined by the shooting method, i.e., integrating the ODEs backward from $r = a$ to $r = 0$ and iterating until $z_2(0) = z_3(0) = 0$. Due to the singularity at $r = 0$, numerical integration is carried out only from $r = a$ to $r = \epsilon$ for some small $\epsilon > 0$. The optimal $\tilde{u}$ and $\tilde{V} = z_1$ can be obtained from (9.7) and (9.5), respectively.

In Figure 9.1, we plot the computed optimal solutions to Problem 1 in the one-dimensional case ($n = 1$) on $\bar{\Omega} = [-1, 1]$, with the control cost $J_0 = 10$, for $M = 2.5$ (left) and a large $M > 3$ (right). In both cases, the optimal controls $\tilde{u}$, shown in the upper figures, are roughly symmetric around $r = 0.5$ and vanish at the center ($r = 0$) and the boundary ($r = 1$) of $\bar{\Omega}$. Moreover, the control magnitude is capped at M if $M < 3$ and at 3 for any large $M > 3$. This indicates that even for $M = \infty$, with $J_0 = 10$, Problem 1 still admits a bounded control, shown in the upper right of Figure 9.1, as its solution.

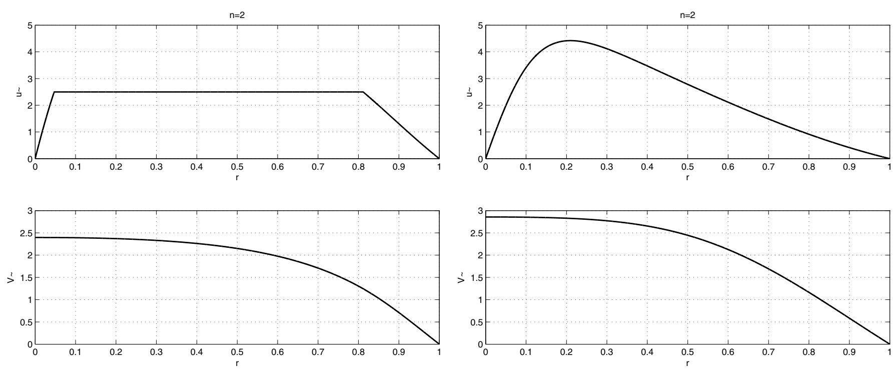
Figure 9.2 plots the solutions on the unit ball in the two-dimensional space ($n = 2$) with $J_0 = 15$ and $M = 2.5$ (left) and a large M (right). The optimal controls u vanish at the center and boundary as in the one-dimensional case, but is no longer symmetric around $r = 0.5$. Moreover, for very large M, the optimal u is capped at approximately 4.4, indicating a bounded optimal u even for the $M = \infty$ case.

For the three-dimensional unit ball ($n = 3$), the solutions for Problem 1 with $J_0 = 11$ are plotted in Figure 9.3 for $M = 10$ and $M = 100$, respectively. The



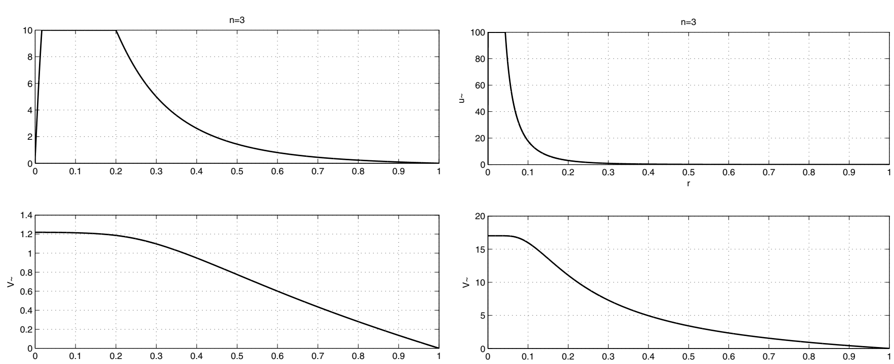
(a) $W^* = 15.83$ for $M = 2.5$. (b) $W^* = 16.505$ for a large M.

FIG. 9.1. *Solutions $\tilde{u}$ and $\tilde{V}$ for dimension $n = 1$ with control cost $J_0 = 10$ and different M.*



(a) $W^* = 4.7351$ for $M = 2.5$ (b) $W^* = 4.9977$ for large M.

FIG. 9.2. *Solutions $\tilde{u}$ and $\tilde{V}$ for dimension $n = 2$ with control cost $J_0 = 15$ and different M.*



(a) $W^* = 1.5524$ for $M = 10$. (b) $W^* = 6.8721$ for $M = 100$

FIG. 9.3. *Solutions $\tilde{u}$ and $\tilde{V}$ for dimension $n = 3$ with control cost $J_0 = 11$ and different M.*

observation that the optimal control vanishes at the center and boundary still holds true in both cases (which is hard to see in the upper right figure but will become clear if plotted in semilog axis). However, as M increases, unlike the one- and two-dimensional cases, the maximal magnitude of the optimal control u will grow unboundedly, and the corresponding $W(u)$ will also grow to infinity. As a result, Problem 1 with $J_0 = 11$ and $M = \infty$ does not admit a bounded solution u, and the optimal $W(u)$ can be made arbitrarily large. In this case, the limiting optimal control u as $M \to \infty$ has an impulse at $r = 0$, and the impulsive control method in [16, 29] may be used to characterize it. This will be investigated in our future work.

10. Conclusions. In this paper, the method of symmetrization is applied to study the optimal sojourn time control problem and its dual problem. It is found that the optimal solutions to both problems are radially symmetric when the domain under consideration is a ball. Furthermore, among all domains with the same volume, balls are the best in generating the most efficient control for sojourn time maximization. Several extensions of the results are given and numerical simulations are presented.

Appendix A. Proof of Lemma 6.3. We first characterize the asymptotic speed at which $G(\rho) \to \infty$ as ρ approaches ρ_m from below.

Define $\delta = \rho_m - \rho \geq 0$. Then $\delta \to 0$ as $\rho \to \rho_m$. For a function $p(\rho) \geq 0$ and a real number α, the notation $p(\rho) = \Theta(\delta^\alpha)$ means that $p(\rho)$ and δ^α are of the same asymptotic order, i.e., $0 < \liminf_{\rho\to\rho_m} p(\rho)/\delta^\alpha \leq \limsup_{\rho\to\rho_m} p(\rho)/\delta^\alpha < \infty$, while the notation $p(\rho) = O(\delta^\alpha)$ means that $0 \leq \liminf_{\rho\to\rho_m} p(\rho)/\delta^\alpha \leq \limsup_{\rho\to\rho_m} p(\rho)/\delta^\alpha < \infty$.

LEMMA A.1. *As* $\rho \to \rho_m$, $G(\rho) = O(\delta^{-1/2})$.

Proof of Lemma A.1. By Assumption 4.4, C_{ρ_m} consists of nondegenerate critical points only. Without loss of generality, we can assume C_{ρ_m} consists of a single point z. Then $\nabla V(z) = 0$, and the Hessian $\nabla^2 V(z)$ is negative definite. Choose a suitable orthonormal coordinate near z so that $\nabla^2 V(z) = -\Sigma = \operatorname{diag}(-\sigma_1, \ldots, -\sigma_n)$ is diagonal for some $\sigma_1, \ldots, \sigma_n > 0$. Note that $\sum_{i=1}^n \sigma_i = 2$ as $1 + \frac{1}{2}\Delta V = 0$ at z.

Since $V(x)$ can be expanded as $\rho_m - \frac{1}{2}(x-z)^T\Sigma(x-z) +$ higher order terms for x close to z, if $\delta = \rho_m - \rho$ is small, D_ρ can be approximated by an ellipsoid $\{x|\frac{1}{2}(x-z)^T\Sigma(x-z) \leq \delta\}$ centered at z, and C_ρ by the boundary of the ellipsoid. Thus the volume of D_ρ satisfies $A(\rho) = \Theta(\delta^{n/2})$ with a derivative satisfying $|A'(\rho)| = \Theta(\delta^{(n-2)/2})$, and the area of C_ρ is $\Theta(\delta^{(n-1)/2})$. For $x \in C_\rho$, $\|x - z\| = \Theta(\delta^{1/2})$. Hence the fact that $\nabla V(z) = 0$ and $\nabla^2 V(z) \neq 0$ implies that $\|\nabla V(x)\| = \Theta(\|x - z\|) = \Theta(\delta^{1/2})$. In addition, since $1 + \frac{1}{2}\Delta V = -u \cdot \nabla V$ and u is bounded, we have $\left|1 + \frac{1}{2}\Delta V\right| = |u \cdot \nabla V| = O(\|\nabla V\|) = O(\delta^{1/2})$ for $x \in C_\rho$.

To sum up, as $\rho \to \rho_m$, $Q(\rho)$ defined in (6.5) is of the order $\Theta(\delta^{(n-1)/2}) \cdot [O(\delta^{1/2})]^2/[\Theta(\delta^{1/2})]^3$, i.e., $O(\delta^{(n-2)/2})$, while $G(\rho)$ is of the order $[\Theta(\delta^{(n-2)/2})]^{1/2} \cdot [O(\delta^{(n-2)/2})]^{1/2}/[\Theta(\delta^{n/2})]^{(n-1)/n} = O(\delta^{-1/2})$. ◻

We now return to the proof of Lemma 6.3. The integrability of $Q(\rho)$ follows immediately from the following:

(A.1)
$$\int_0^{\rho_m} Q(\rho)\, d\rho = \int_0^{\rho_m}\int_{C_\rho} \frac{(1+\frac{\sigma^2}{2}\Delta V)^2}{\|\nabla V\|^2}\|\nabla V\|^{-1} d\sigma\, d\rho = \int_{\bar\Omega} \frac{(1+\frac{\sigma^2}{2}\Delta V)^2}{\|\nabla V\|^2} dx < \infty,$$

where we have used Lemma 6.1 and (4.4). For the integrability of $G(\rho)$, first note that, by Lemma A.1, $G(\rho) = O(\delta^{-1/2})$ as $\rho \to \rho_m$; hence $G(\rho)$ is integrable on $[\rho_m - \epsilon, \rho_m]$ for ϵ small enough. On the other hand, applying the Cauchy–Schwarz inequality and noting that $A(\rho)$ is strictly decreasing, we get

$$\int_0^{\rho_m-\epsilon} G(\rho)\,d\rho \le \frac{2}{n\omega_n^{1/n}A^{(n-1)/n}(\rho_m-\epsilon)}\int_0^{\rho_m-\epsilon}|A'(\rho)|^{1/2}\,Q^{1/2}(\rho)\,d\rho$$
$$\le \frac{2}{n\omega_n^{1/n}A^{(n-1)/n}(\rho_m-\epsilon)}\left(\int_0^{\rho_m-\epsilon}|A'(\rho)|\,d\rho\right)^{1/2}\cdot\left(\int_0^{\rho_m-\epsilon}Q(\rho)\,d\rho\right)^{1/2}$$
$$< \infty,$$

as both $|A'(\rho)|$ and $Q(\rho)$ are integrable. Therefore, $G(\rho)$ is integrable on $[0,\rho_m]$.

Appendix B. Proof of Lemma 6.5. Let $\rho \in \mathcal{S}$ be arbitrary. By Lemma 6.2,

$$\frac{1}{4}P(\rho)G^2(\rho) = P(\rho)\frac{|A'(\rho)|}{n^2\omega_n^{2/n}A^{(2n-2)/n}(\rho)}Q(\rho) \ge Q(\rho).$$

The square root of the above inequality yields

$$\frac{1}{2}P^{1/2}(\rho)G(\rho) \ge Q^{1/2}(\rho).$$

Applying Lemma 6.4, we have, for almost all $\rho \in \mathcal{S}$,

$$\frac{1}{2}P(\rho)G(\rho) = P^{1/2}(\rho)\cdot\frac{1}{2}P^{1/2}(\rho)G(\rho) \ge P^{1/2}(\rho)\cdot Q^{1/2}(\rho) \ge A'(\rho) - \frac{\sigma^2}{2}P'(\rho),$$

or equivalently,

$$-P'(\rho) - \sigma^{-2}P(\rho)G(\rho) \le 2\sigma^{-2}|A'(\rho)|.$$

A Gronwall-like inequality about $P(\rho)$ can be derived as follows. First note that

$$\frac{d}{d\xi}\left[-P(\xi)\,e^{\int_0^\xi \sigma^{-2}G(\eta)\,d\eta}\right] = \left[-P'(\xi) - \sigma^{-2}P(\xi)G(\xi)\right]e^{\int_0^\xi \sigma^{-2}G(\eta)\,d\eta}$$
$$\le 2\sigma^{-2}|A'(\xi)|e^{\int_0^\xi \sigma^{-2}G(\eta)\,d\eta}$$

a.e. on $\mathcal{S}$. Since $P(\rho)\,e^{\int_0^\rho \sigma^{-2}G(\eta)\,d\eta}$ is differentiable on $\mathcal{S}$ and the set $\mathcal{S}$ is open and dense in $[0,\rho_m]$, we can integrate the above inequality from ρ to ρ_m to obtain, noting that $P(\rho_m)=0$ and $e^{\int_0^{\rho_m} \sigma^{-2}G(\eta)\,d\eta}$ is finite,

$$P(\rho)\,e^{\int_0^\rho \sigma^{-2}G(\eta)\,d\eta} \le 2\int_\rho^{\rho_m}\sigma^{-2}|A'(\xi)|e^{\int_0^\xi \sigma^{-2}G(\eta)\,d\eta}\,d\xi.$$

By the continuity of both sides with respect to $\rho \in [0,\rho_m]$, the above inequality is valid for all $\rho \in [0,\rho_m]$. This completes the proof. $\square$

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