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Dual-mode robust fault estimation for switched linear systems with state jumps

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ABSTRACT

Fault detection and estimation is a critical part of modern control system design and ensures safety and reliability of expensive machinery. In this paper, a dual-mode state and fault estimation scheme is proposed for switched linear systems with a class of state jumps in the presence of simultaneous actuator and sensor faults. In the absence of sensor faults, a switched sliding mode observer is developed to estimate the state and actuator fault signal. A residual signal is computed to detect the inception of a sensor fault. Upon detecting the sensor fault, a robust state observer is triggered which guarantees ultimate boundedness of the state estimation error. The performance of the proposed dual-mode switched observer is illustrated using a numerical example.

1. Introduction

An increasing demand for safety and reliability has motivated research in the area of fault detection and isolation (FDI) [1]. Excellent surveys of the field are [2–4]. One method for fault detection is to employ a state observer to construct a residual signal and determine whether a fault has occurred based on the magnitude of the residual signal relative to a predefined threshold. In an industrial application, exogenous signals such as unknown disturbances or sensor errors could produce false alarms. Hence, there is a need to construct robust fault detection methods which function satisfactorily in the presence of exogenous inputs.

Recent methods involve modeling a system's fault dynamics to enable the analysis of fault detection. This type of modeling often uses switched system models (see [5] for an overview of switched systems). Results are available in the FDI literature for linear, stochastic, and uncertain systems, for example [6–15]; however, FDI for switched linear systems (SLS) is an open problem. In [16,17], methods are proposed to estimate discrete-time SLS with state delays in an $\mathcal{H}_\infty$ framework. For time varying delays, methods are proposed for fault detection in [18,19]. A recent paper [20] addresses the case of fault detection with intermittent measurements. A design method for fault-tolerant controllers for switched linear systems is discussed in [21]. A common limitation of these methods is that sensor faults are not considered explicitly. Generally, the fault detection problem is posed therein as a safe-to-failure mode transition detection problem. Some results in this field are discussed in [22,23].

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Design of observers for switched systems has proven to be a rich problem. In [24], sliding mode observers constructed for each mode are used to estimate the unknown switching signal and the state. Observer construction in [24] is extended in [25] to the case of partially unknown inputs. Estimating the switching signal and the discrete state for discrete-time switched systems is analyzed in [26] for the case of bounded uncertainties. Uniform convergence of the continuous and discrete state is investigated in [27], where a bank of observers is used to reconstruct the switching signal and continuous state based on a residual signal for determining the active mode. In contrast, this work uses the residual signal to detect the presence of abrupt sensor faults. When the switching signal is known, asymptotic observers can be constructed to guarantee estimator convergence in the presence of bounded disturbances via a common quadratic Lyapunov function as described in [28]. In this paper, a similar problem is addressed. A key difference is that the actuator faults are simultaneously reconstructed in addition to the state estimates.

An open problem in observer design of switched linear systems is constructing observers to handle state jumps. Observer design using a common Lyapunov framework for switched systems with state jumps is found in [29–31]. Some results using multiple Lyapunov functions are described in [32] to reduce conservativeness of the observer design. In this paper, we use a common Lyapunov approach which guarantees exponential stability of the observer error under known but arbitrary switching and a class of state jumps. We also incorporate state jumps within our dual-mode observer to provide ultimate bounds on the estimation error and reconstruct the actuator fault signal.

In this paper, we propose a systematic method to design dual-mode observers for state estimation of switched linear systems with state jumps. These systems are additionally disturbed by bounded sensor and actuator faults. In the presence of actuator faults only, we propose a sliding mode observer for state and fault signal estimation. We compute a residual signal which is utilized to detect a sensor fault. If a sensor fault is detected, a robust observer is triggered. This robust observer operates at a specified performance level, which is intrinsic to the design. Our contributions include: (i) formulating new linear matrix inequality conditions for sliding mode observer for switched systems; (ii) proposing linear matrix inequalities for $\mathcal{L}_\infty$-gain attenuating observer design for switched linear systems with guaranteed performance, and (iii) extending these observer designs to incorporate state and actuator fault estimation for switched systems with state jumps.

The rest of the paper is organized as follows. The overall fault detection and isolation problem is discussed in Section 2. The actuator fault detecting observer is analyzed in Section 3. In the event of a sensor fault a disturbance rejecting observer is developed in Section 4. Finally, simulation results are shown in Section 5 and conclusions are presented in Section 6.

2. Problem statement

2.1. System description

We consider a class of dynamical systems modeled as a switched linear system (SLS) with state jumps at switching times t_k modeled by

$$\begin{aligned} \dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + G_{\sigma(t)}f_a(t)\mathbf{1}^+(t-\Delta_a), \\ y(t) &= C_{\sigma(t)}x(t) + D_{\sigma(t)}f_s(t)\mathbf{1}^+(t-\Delta_s), \\ x(t_k^+) &= \Theta_{\sigma(t_k^-),\sigma(t_k^+)}x(t_k^-) + \Gamma_{\sigma(t_k^-),\sigma(t_k^+)}, \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector of the SLS, $u(t) \in \mathbb{R}^{m_1}$ is the vector of known control inputs and $y(t) \in \mathbb{R}^p$ represents the measured outputs at time t. The switching signal $\sigma(t) \in \{1, \ldots, N\}$ indicates the active mode amongst N subsystems at time t. The vectors $f_a(t) \in \mathbb{R}^{m_2}$ and $f_s(t) \in \mathbb{R}^q$ represent the actuator and sensor fault, respectively. The unit step function $\mathbf{1}^+(t-\tau)$ is a step signal which comes into effect after delay τ. In this paper, we consider that the actuator fault occurs after Δ_a seconds and that the sensor fault occurs after Δ_s seconds. Note that the times of fault occurrence Δ_a, Δ_s are *unknown*.

Remark 1. Suppose that $\Delta_a < \Delta_s$. This implies that the sensor fault occurs after the actuator fault. Note that this is not a restrictive assumption. If a sensor fault occurs before the actuator fault, that is, if $\Delta_s < \Delta_a$, then the residual will trigger the $\mathcal{O}^+$ observer after Δ_s second until the faulty sensor is replaced. If $\Delta_s = \Delta_a = 0$, then the robust observer $\mathcal{O}^+$ will run shortly after the initial time, as soon as the residual threshold condition $r \geq \bar{r}$ is triggered.

For $i = 1, \ldots, N$, the matrices $A_i \in \mathbb{R}^{n\times n}$ are state matrices, $B_i \in \mathbb{R}^{n\times m_1}$ are input matrices, $C_i \in \mathbb{R}^{p\times n}$ are output matrices and $G_i \in \mathbb{R}^{n\times m_2}$ and $D_i \in \mathbb{R}^{p\times q}$ represent how the *unknown* actuator and sensor faults enter the system in the ith mode, respectively.

Finally, the *known* matrices $\Theta_{r,s} \in \mathbb{R}^{n\times n}$ and $\Gamma_{r,s} \in \mathbb{R}^n$ for all $r, s \in \{1, 2, \ldots, N\}$ represent the linear state jumps considered in this paper. Note that such a linear form of state jumps has been explored previously in [33] and [34]. Finally, in our notation, we drop the argument t for brevity; for example $A_{\sigma(t)}$ is written as A_σ. We make the following technical assumptions to enable the development of our main results.

Assumption 1. The switching signal σ is known and has finite number of switches per unit time interval, that is, it does not exhibit Zeno behavior.

Assumption 2. For $i = 1, \ldots, N$, $\operatorname{rank}(C_iG_i) = \operatorname{rank}(G_i) = m_2$.

Assumption 3. There exist scalars $\rho_a > 0$ and $\rho_s > 0$ such that,

$$\|f_a(t)\| \le \rho_a \text{ and } \|f_s(t)\| \le \rho_s,$$

for all t. That is, the functions that model the actuator and sensor faults are bounded.

Assumption 4. The sensor fault is abrupt and non-return-to-zero (NRZ), that is, $f_s(t) \neq 0$ for all $t \ge \Delta_s$.

Assumptions 1 and 3 are standard assumptions for practical systems. Assumption 2 is fairly standard in the unknown input estimation literature, see for example [35–37]. Assumption 4 is not restrictive either. For example, in electrical circuits or machines, we suspect that a fault occurring in the sensor will result in the degradation of that sensor until replaced. This can be modeled as an NRZ sensor fault.

2.2. Proposed dual-mode observer

We propose the following dual-mode observer structure for pre- and post-sensor fault scenarios.

2.2.1. Pre sensor-fault observer

The observer labeled $\mathcal{O}^-$ is proposed for the time prior to a sensor fault, that is, for all $t \in [t_0, \Delta_s)$, we have

$$\boxed{\begin{aligned} \dot{\hat{x}} &= A_\sigma \hat{x} + L_\sigma^-(y-\hat{y}) + B_\sigma u + G_\sigma \hat{f}_a, \\ \hat{y} &= C_\sigma \hat{x}, \\ \hat{x}(t_k^+) &= \Theta_{\sigma(t_k^-),\sigma(t_k^+)}\hat{x}(t_k^-) + \Gamma_{\sigma(t_k^-),\sigma(t_k^+)} \\ \hat{f}_a &= \begin{cases} \rho \frac{F_\sigma(y-\hat{y})}{\|F_\sigma(y-\hat{y})\|} & \text{if } F_\sigma(y-\hat{y}) \neq 0 \\ 0 & \text{if } F_\sigma(y-\hat{y}) = 0 \end{cases} \\ x(0) &= x_0 \\ r &= \int_{t-\delta}^{t} \|y(\tau) - \hat{y}(\tau)\|^2 \, \mathrm{d}\tau \end{aligned}} \tag{2}$$

Here, the hatted variables indicate estimated quantities. The matrices $L_i^- \in \mathbb{R}^{n\times p}$ and $F_i \in \mathbb{R}^{m\times p}$ for $i = 1, \dots, N$ are switched Luenberger and sliding mode observer gain matrices, respectively. The scalar $\rho > 0$ represents the sliding mode observer gain. The observer design involves the design of gains L_i, F_i and ρ.

The residual signal is denoted by r, and $\delta > 0$ is the length of the horizon over which the residual is computed. The observer $\mathcal{O}^-$ will be designed to reconstruct the state and actuator fault signal asymptotically. Due to the finite time convergence to the sliding surface, the residual signal computed prior to a sensor fault converges to zero in finite time. The selection of the residual computation time δ is chosen to be larger than the time of convergence to the sliding surface. Thus, if the computed residual signal consistently exceeds a small user-defined threshold $\bar{r}$, we deem a sensor fault to have occurred and switch to the post-fault observer, which we will describe next.

2.2.2. Post sensor-fault observer

In the event of a sensor fault, the presence of uncertainties in the measurements warrants a less stringent performance criterion from the robust observer. Unlike the observer $\mathcal{O}^-$, we are not concerned with reconstructing the sensor fault signal. Instead, we design a robust observer that attenuates the effect of the sensor fault and drives the state estimation error to a disturbance invariant set about the origin. Herein, we present our notion of a disturbance invariant set.

Definition 1 (*Disturbance Invariant Set*)**.** A set $\Omega \subset \mathbb{R}^n$ is a disturbance invariant set for the SLS $\dot{x} = A_\sigma x + B_\sigma w$ if for every initial state $x(t_0) \in \Omega$ and disturbance $w \in \mathbb{W}$, where $\mathbb{W}$ is a compact set, the trajectories $x(t) \in \Omega$ for all $t \ge t_0$ and any arbitrary switching sequence σ.

The post-fault observer $\mathcal{O}^+$ is modeled as

$$\boxed{\begin{aligned} \dot{\hat{x}} &= A_\sigma \hat{x} + L_\sigma^+(y-\hat{y}) + B_\sigma u, \\ \hat{y} &= C_\sigma \hat{x}, \\ \hat{x}(t_k^+) &= \Theta_{\sigma(t_k^-),\sigma(t_k^+)}\hat{x}(t_k^-) + \Gamma_{\sigma(t_k^-),\sigma(t_k^+)}, \\ \hat{x}(0) &= \hat{x}_0. \end{aligned}} \tag{3}$$

where $\bar{L}_i^+$ for $i = 1, \dots, N$ are linear observer gain matrices.

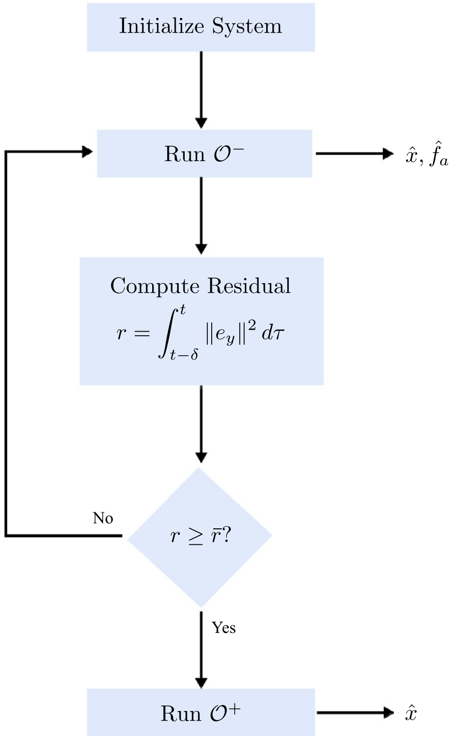

Fig. 1. Proposed dual-mode fault observer, where $e_y = y - \hat{y}$ is the output error.

The objective of the $\mathcal{O}^+$ observer is to restrict the state estimation error $e = x - \hat{x}$ to a disturbance invariant set Ω in the presence of both actuator and sensor faults in conjunction with state jumps. Since the sensor fault is NRZ by Assumption 4, we assume that this observer continues to run until the faulty sensor is replaced.

The overall dual-mode fault observer scheme is shown in Fig. 1.

3. Design of pre sensor-fault observer $\mathcal{O}^-$

Let the observer error be defined as $e = x - \hat{x}$. In the absence of sensor fault, the state estimation error dynamics of the observer $\mathcal{O}^-$ are given by,

$$\dot{e} = (A_\sigma - L_\sigma^- C_\sigma)e + G_\sigma(f_a - \hat{f}_a), \tag{4a}$$

$$e(t_k^+) = \Theta_{\sigma(t_k^-),\sigma(t_k^+)} e(t_k^-). \tag{4b}$$

Next we discuss the stability of the observer $\mathcal{O}^-$ and describe the actuator fault reconstruction method under ideal sliding mode. We also describe an implementable reconstruction technique for non-ideal (or boundary layer) sliding mode employing a low-pass filtering architecture.

3.1. Stability of observer $\mathcal{O}^-$

We present sufficient conditions guaranteeing that the state estimation error converges exponentially to the origin. The sufficient conditions are in the form of linear matrix inequalities (LMIs) which leverage efficient convex optimization solvers.

In order to provide exponential stability for the proposed $\mathcal{O}^-$ observer, we construct a Lyapunov-like function which is guaranteed to be decreasing over any interval of time (t_k, t_{k+1}) and non-increasing at the switching time t_k, for $k \in \mathbb{N}$. To this end, we require the following technical lemma.

Lemma 1. *Suppose Assumption 1 holds. Let $\mathcal{D}_s \subset \mathbb{R}$ be the set of switching times. Let $V : \mathbb{R} \to [0, \infty)$ be a piecewise differentiable function where the derivative $\dot{V}$ exists on $[0, \infty) \setminus \mathcal{D}_s$.*

Define $\bar{V} : \mathbb{R} \to [0, \infty)$ to be

$$\bar{V}(t) = \exp(-\alpha(t - t_0))\bar{V}(t_0)$$

for some fixed $\alpha > 0$, *and* $\bar{V}(t_0) = V(t_0)$. *If*

(i) $\forall\ t \in \mathcal{D}_s$, $V(t^-) \geq V(t^+)$, *and*
(ii) $\forall\ t \notin \mathcal{D}_s$, $\dot{V}(t) \leq -\alpha V(t)$,

then $V(t) \leq \bar{V}(t)$ *for all* $t \geq t_0$.

Proof. By construction, $V(t_0) = \bar{V}(t_0)$. Without loss of generality, we suppose $t_1 > t_0$ is the smallest time such that $t_1 \in \mathcal{D}_s$. For any $t \in (t_0, t_1)$, we know that V is differentiable. Using the comparison lemma [38], we obtain

$$\begin{aligned} V(t) &\leq \exp(-\alpha(t - t_0))V(t_0) \\ &= \exp(-\alpha(t - t_0))\bar{V}(t_0) \\ &= \bar{V}(t). \end{aligned}$$

At t_1, condition (i) guarantees $V(t_1^+) \leq V(t_1^-) \leq \bar{V}(t_1^-)$ as $t_1^- \in (t_0, t_1)$. Since $\bar{V}(t_1^+) = \bar{V}(t_1^-)$, this implies $V(t_1^+) \leq \bar{V}(t_1^+)$. The result follows by induction on t_k for $k \in \mathbb{N}$. □

Now, we provide sufficiency conditions for the exponentially stability of the $\mathcal{O}^-$ observer error dynamics.

Theorem 1. *Suppose Assumptions 1–4 hold and* $f_s = 0$. *If there exist matrices* $P_1 = P_1^\top, Y_1, \ldots, Y_N, F_1, \ldots, F_N$ *and positive scalars* $\alpha > 0$, $\rho \geq \rho_a$ *such that the following conditions are satisfied,*

$$A_i^\top P_1 + P_1 A_i - C_i^\top Y_i^\top - Y_i C_i + 2\alpha P_1 \preceq 0, \tag{5a}$$

$$G_i^\top P_1 = F_i C_i, \tag{5b}$$

$$P_1 \succ 0, \tag{5c}$$

$$\Theta_{i,j}^\top P_1 \Theta_{i,j} - P_1 \preceq 0 \tag{5d}$$

for $i, j \in \{1, \ldots, N\}$, *then the observer* $\mathcal{O}^-$ *described in (2) with gains* $L_i^- = P_1^{-1} Y_i$, F_i *and* ρ *has state estimation error (4) converging asymptotically to the origin, that is,* $e \to 0$ *as* $t \to \infty$. *Furthermore, the error convergence is exponential with decay rate* α, *that is*

$$\|e(t)\| \leq c_0 \exp(-\alpha(t - t_0))\|e(t_0)\|, \tag{6}$$

for any $t \geq t_0$, *and* $c_0^2 = \lambda_{\max}(P_1)/\lambda_{\min}(P_1)$.

Proof. We consider a quadratic function of the form,

$$V = e(t)^\top P_1 e(t),$$

which is positive definite due to (5c), for the error system (4). For any switching time t_1 from mode i to mode j, (5d) implies

$$\begin{aligned} V(t_1^+) &= e(t_1^+)^\top P_1 e(t_1^+) \\ &= e(t_1^-)^\top \Theta_{i,j}^\top P_1 \Theta_{i,j} e(t_1^-) \\ &\leq e(t_1^-)^\top P_1 e(t_1^-) \\ &= V(t_1^-). \end{aligned}$$

Since $V(t)$ is differentiable at any time t excluding switching times, $V(\cdot)$ satisfies condition (i) of Lemma 1.

The derivative of V evaluated on the trajectories of the observer error dynamics (4) between switching times for a fixed mode $i \in \{1, \ldots, N\}$ is given by,

$$\begin{aligned} \dot{V} &= 2e^\top P_1 (A_i - L_i^- C_i) e + 2e^\top P_1 G_i (f_a - \hat{f}_a) \\ &= e^\top \left((A_i - L_i^- C)^\top P_1 + P_1(A_i - L_i^- C_i)\right) e + 2e^\top P_1 G_i (f_a - \hat{f}_a) \\ &= e^\top \left(A_i^\top P_1 + P_1 A_i - Y_i C_i - C_i^\top Y_i^\top\right) e + 2e^\top P_1 G_i (f_a - \hat{f}_a). \end{aligned}$$

From (5a), we get

$$\dot{V} \leq -2\alpha(e^\top P_1 e) + 2e^\top P_1 G_i (f_a - \hat{f}_a). \tag{7}$$

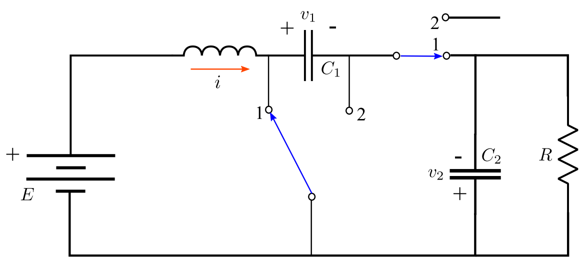

Fig. 2. Switched DC–DC converter.

For $F_iC_ie \neq 0$, this yields

$$\begin{aligned}
\dot{V} &= -2\alpha V + 2e^\top P_1 G_i f_a - 2e^\top P_1 G_i \hat{f}_a \\
&\le -2\alpha V + 2\|e^\top P_1 G_i\|\|f_a\| - 2\rho \left(e^\top P_1 G_i \frac{F_iC_ie}{\|F_iC_ie\|} \right) \\
&\le -2\alpha V + 2\rho_a \|F_iC_ie\| - 2\rho \frac{\|F_iC_ie\|^2}{\|F_iC_ie\|} \\
&\le -2\alpha V,
\end{aligned}$$

because $\rho \ge \rho_a$, where ρ_a is defined in Assumption 3.

During sliding motion, that is, when $F_iC_ie = 0$, since $G_i^\top P_1 e = F_iC_ie$ from (5b), we get $\dot{V} \le -2\alpha V$ by replacing $G_i^\top P_1 e = 0$ in (7).

Thus, $\dot{V}(t) \le -2\alpha V(t)$ for all $t \ge t_0$ excluding switching times. Thus V satisfies condition (ii) of Lemma 1. Hence, by Lemma 1 we can construct $\bar{V}(t) = \exp(-2\alpha(t - t_0))V(t_0)$ to be an upper bound of $V(t)$. Thus, V is a global Lyapunov-like function for the system (4), since $V(0) = 0$, $V > 0$, $V \le \bar{V}$ for all t and $\dot{\bar{V}} \le -2\alpha\bar{V}$. Since $\dot{V} \le -2\alpha V$ within all open intervals between switching, and V decreases at switching times, this implies that the observer error is exponentially convergent with a decay rate α. This concludes the proof. □

Remark 2. The LMI condition (5d) guarantees that the Lyapunov-like energy function $V(\cdot)$ is non-increasing at state jumps. This dissipation of energy during state jumps models the behavior of some switched physical systems in which the continuity of the state trajectory would require a hidden input, for example, in circuits where parallel capacitors are switched in and out.

To demonstrate a physical system where the phenomenon of dissipative state jumps is common, we provide the following example.

Example 1. Consider the DC–DC converter in Fig. 2 presented in [39]. The only two admissible switch positions are $(1, 1)$ and $(2, 2)$, which will be denoted as mode 1 and mode 2, respectively. The DC–DC converter can be modeled by

$$\frac{d}{dt}\begin{bmatrix} i(t) \\ v_1(t) \\ v_2(t) \end{bmatrix} = A_\sigma \begin{bmatrix} i(t) \\ v_1(t) \\ v_2(t) \end{bmatrix} + BE(t) \tag{8}$$

where

$$A_1 = \begin{bmatrix} 0 & \frac{1}{L} & 0 \\ \frac{-1}{C_1 + C_2} & \frac{-1}{R(C_1 + C_2)} & 0 \\ \frac{-1}{C_1 + C_2} & \frac{-1}{R(C_1 + C_2)} & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & \frac{-1}{L} & 0 \\ \frac{1}{C_1} & 0 & 0 \\ 0 & 0 & \frac{-1}{RC_2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L} \\ 0 \\ 0 \end{bmatrix}.$$

When the system switches from mode 1 to mode 2, there are no state jumps, hence: $\Theta_{1,2} = I$ and $\Gamma_{1,2} = 0$. However, when switching from mode 2 to mode 1 at time t_1, conservation of charge requires that

$$v_1(t_1^+) = v_2(t_1^+) = \frac{C_1 v_1(t_1^-) + C_2 v_2(t_1^-)}{C_1 + C_2}.$$

This implies

$$\Theta_{2,1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{C_1}{C_1 + C_2} & \dfrac{C_2}{C_1 + C_2} \\ 0 & \dfrac{C_1}{C_1 + C_2} & \dfrac{C_2}{C_1 + C_2} \end{bmatrix}, \tag{9}$$

and $\Gamma_{2,1} = 0$.

We consider the candidate energy function $V(x) = x^\top Px$, where $x^\top = [i, v_1, v_2]$ and

$$P = \begin{bmatrix} L & 0 & 0 \\ 0 & C_1 & 0 \\ 0 & 0 & C_2 \end{bmatrix} \succ 0.$$

Clearly,

$$\Theta_{1,2}^\top P \Theta_{1,2} - P = 0,$$

as $\Theta_{1,2} = I$. We also observe that

$$\Theta_{2,1}^\top P \Theta_{2,1} - P = \frac{C_1 C_2}{C_1 + C_2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \preceq 0$$

for any $C_1, C_2 > 0$. Hence, the energy function $V(x)$ is non-increasing at state jumps for any mode switch. Summarily, the example satisfies the condition (5d).

3.2. Actuator fault reconstruction with ideal sliding

We begin with the following technical lemma.

Lemma 2. *Let $\mathcal{S}_i = F_i C_i$. Suppose the conditions in Theorem 1 are satisfied. Then for each $i = 1, \ldots, N$, $\mathcal{S}_i G_i \in \mathbb{R}^{m_2 \times m_2}$ is positive definite.*

Proof. Recall that $F_i C_i = G_i^\top P_1$ by (5b) and G_1 is full column rank by Assumption 2. Since

$$\operatorname{rank}(\mathcal{S}_i G_i) = \operatorname{rank}(F_i C_i G_i) = \operatorname{rank}(G_i^\top P_1 G_i),$$

it is sufficient to show that $G_i^\top P_1 G_i$ is full column rank.

To this end, suppose we choose $x \in \mathbb{R}^n$ such that $G_1^\top P_1 G_1 x = 0$. This implies $x^\top G_1^\top P G_1 x = 0$. Choose $y = G_1 x$. As $P_1 \succ 0$, $y^\top P_1 y = 0$ if and only if $y = 0$, that is, if $G_1 x = 0$. But G_1 has full column rank, so $G_1 x = 0$ indicates $x = 0$. Hence, $G_1^\top P_1 G_1 x = 0$ only if $x = 0$, which implies that $\mathcal{S}_i G_i$ is full column rank. For $x \neq 0$, $x^\top G_1^\top P_1 G_1 x > 0$ as $P_1 \succ 0$. Hence, $\mathcal{S}_i G_i \succ 0$. This concludes the proof. □

We need the following assumption for the following technical result.

Assumption 5. The initial state of the plant belongs to a bounded set, and there exists a known positive scalar ρ_x such that $\|x(t_0)\| \leq \rho_x$, where t_0 denotes the initial time.

Note that this assumption is not required for the results obtained prior to this point. We now state a result which guarantees finite time convergence to the ith sliding manifold at each active mode $i \in \{1, \ldots, N\}$.

Theorem 2. *Suppose Assumption 5 holds. Let the sequence of switching times be denoted by $\mathcal{D}_s$, the minimum time between switches be T_{min}. For each $\tau_s \in (0, T_{min})$, there exists a $\rho \geq \rho_a$ such that the observer $\mathcal{O}^-$ error trajectory converges to the sliding manifold Σ_i within $[t_k, t_k + \tau_s] \subset [t_k, t_{k+1}]$ for any mode $i = 1, \ldots, N$.*

Proof. Fix the mode i and consider the quadratic function

$$V_{s_i}(e) = \frac{1}{2} e^\top \mathcal{S}_i^\top \mathcal{S}_i e.$$

Here $\mathcal{S}_i = F_i C_i$ for each $i = 1, \ldots, N$, and V_{s_i} is positive definite for every $i = 1, \ldots, N$, since $\mathcal{S}_i^\top S_i$ are positive definite matrices.

Our objective is to show that $\dot{V}_{s_i} \le -\eta\|S_i e\|$, for some $\eta > 0$. To this end, we compute the time-derivative of V_{s_i} prior to reaching the sliding manifold. This yields

$$\begin{aligned}
\dot{V}_{s_i} &= e^\top \mathcal{S}_i^\top \mathcal{S}_i \dot{e} \\
&= (\mathcal{S}_i e)^\top (\mathcal{S}_i \dot{e}) \\
&= (\mathcal{S}_i e)^\top \left(\mathcal{S}_i (A_i - L_i^- C_i) e + \mathcal{S}_i G_i (f_a - \hat{f}_a) \right) \\
&\le (\mathcal{S}_i e)^\top \left(\|\mathcal{S}_i (A_i - L_i^- C_i)\| \|e\| + \|\mathcal{S}_i G_i\| \|f_a\| - \mathcal{S}_i G_i \hat{f}_a \right) \\
&\le (\mathcal{S}_i e)^\top \left(k_0 \|e(t_i)\| + \|\mathcal{S}_i G_i\| \rho_a - \mathcal{S}_i G_i \hat{f}_a \right)
\end{aligned} \tag{10}$$

where $k_0 = \|\mathcal{S}_i (A_i - L_i^- C_i)\| c_0$, with c_0 defined in (6). Note that for $\mathcal{S}_i e \neq 0$,

$$(\mathcal{S}_i e)^\top \mathcal{S}_i G_i \hat{f}_a = \rho e^\top \mathcal{S}_i^\top \mathcal{S}_i G_i \frac{\mathcal{S}_i e}{\|\mathcal{S}_i e\|}.$$

From Lemma 2, we know that $\mathcal{S}_i G_i \succ 0$, hence,

$$\begin{aligned}
-\rho e^\top \mathcal{S}_i^\top \mathcal{S}_i G_i \frac{\mathcal{S}_i e}{\|\mathcal{S}_i e\|} &\le -\rho \lambda_{\min}(\mathcal{S}_i G_i) e^\top \mathcal{S}_i^\top \frac{\mathcal{S}_i e}{\|\mathcal{S}_i e\|} \\
&= -\rho \lambda_{\min}(\mathcal{S}_i G_i) \frac{\|\mathcal{S}_i e\|^2}{\|\mathcal{S}_i e\|} \\
&= -\rho \lambda_{\min}(\mathcal{S}_i G_i) \|\mathcal{S}_i e\|.
\end{aligned}$$

Replacing in (10), we get,

$$\begin{aligned}
\dot{V}_{s_i} &= \frac{1}{2} \frac{d}{dt} \left(e^\top \mathcal{S}_i^\top \mathcal{S}_i e \right) \\
&\le \left[k_0 \|e(t_i)\| + \|\mathcal{S}_i G_i\| \rho_a - \rho \lambda_{\min}(\mathcal{S}_i G_i) \right] \|\mathcal{S}_i e\|.
\end{aligned}$$

Since the initial condition of the plant x_0 is bounded by Assumption 5, the initial state estimation error $\|e(t_0)\| \le \rho_x + \|\hat{x}(t_0)\|$ is bounded and this bound is known because $\|\hat{x}(t_0)\|$ is known. Then from (6), we obtain that each $\|e(t_i)\|$ is bounded above by $\|e_0\|$. Thus, we can choose ρ large enough to guarantee $\dot{V}_{s_i} < -\eta \|\mathcal{S}_i e\|$ for any $\eta > 0$. Using arguments in [40], we conclude that the initial error converges in finite time to the sliding manifold Σ_i in the ith mode.

An estimate of the time required to converge to the sliding manifold Σ_i in the ith mode is given by $\tau_{s_i} \le \frac{\|\mathcal{S}_i e_0\|}{\eta}$, as discussed in [41]. Choosing ρ such that $\eta \le \frac{\|\mathcal{S}_i e\|}{\tau_s}$ for all modes $i = 1, \ldots, N$, we ensure that for every subsystem, the error dynamics converge to the sliding surface within the finite time τ_s. This concludes the proof. □

Theorem 2 implies that by selecting the observer gain ρ adequately large, following a switch into mode i at time t_k, the observer error trajectories reach the sliding manifold Σ_i within $t_k + \tau_s$. Let the switched sliding surface in the ith mode be given by

$$\Sigma_i(e) \triangleq \{ e \in \mathbb{R}^n : \mathcal{S}_i e = 0 \}$$

for each mode $i = 1, \ldots, N$, where $\mathcal{S}_i = F_i C_i$. Under ideal sliding motion, that is for $t \in \mathcal{T}_s(t_0)$, we know from [42] that

$$\mathcal{S}_i \dot{e} = 0. \tag{11}$$

Recall that $\hat{f}_a$ can take finite values only: namely, $\{-\rho, 0, \rho\}$. Since f_a is an arbitrary fault signal, a meaningful estimate of the actuator fault estimation error during sliding motion using $\hat{f}_a$ is estimated using an equivalent output injection $\hat{f}_a^{\text{eq}}$, in the sense of [43]. The following result demonstrates that, on the sliding manifold Σ_i, the $\mathcal{L}_2$ error between the actuator fault and the equivalent output injection term is bounded by a decaying exponential function. This implies that arbitrarily accurate reconstructions of f_a are possible given sufficiently large times between switching.

Theorem 3. *Suppose Assumption 5 holds. Let the sequence of switching times be denoted by $\{t_k\}$ and let*

$$\mathcal{T}_s = \{ t \ge t_0 \mid e(t) \in \Sigma_{\sigma(t)} \}$$

denote the time intervals for which the $\mathcal{O}^-$ observer error trajectories are on the sliding manifold for the active mode. If the conditions in Theorem 1 are satisfied, then

$$\|f_a(t) - \hat{f}_a^{\text{eq}}(t)\| \le \beta e^{-\alpha(t - t_0)} \tag{12}$$

for some $\beta > 0$ and all $t \in \mathcal{T}_s$.

Proof. In sliding,

$$0 = \mathcal{S}_i \dot{e} = \mathcal{S}_i (A_i - L_i^- C_i) e + \mathcal{S}_i G_i (f_a - \hat{f}_a^{\mathrm{eq}}).$$

From Lemma 2, we know that $\mathcal{S}_i G_i$ is invertible. Thus,

$$\begin{aligned}\|f_a(t) - \hat{f}_a^{\mathrm{eq}}(t)\| &= \|(\mathcal{S}_i G_i)^{-1} \mathcal{S}_i (A_i - L_i^- C_i) e(t)\| \\ &\le \|(\mathcal{S}_i G_i)^{-1} \mathcal{S}_i (A_i - L_i^- C_i)\| \|e(t)\| \\ &\le \|(\mathcal{S}_i G_i)^{-1} \mathcal{S}_i (A_i - L_i^- C_i)\| c_0 \exp(-\alpha(t - t_0)) \|e(t_0)\|\end{aligned}$$

from (6). By choosing $\hat{x}_0$ from a bounded set of initial conditions, and from Assumption 5, we have deduced in the proof of Theorem 1 that $\|e(t_0)\|$ is bounded. Thus, we have

$$\|f_a(t) - \hat{f}_a^{\mathrm{eq}}(t)\| \le \beta \exp(-\alpha(t - t_0)),$$

where

$$\beta = \max_{i=1,\dots,N} c_0 \|(\mathcal{S}_i G_i)^{-1} S_i (A_i - L_i^- C_i)\| \|e(t_0)\| < \infty.$$

This concludes the proof. □

Combining the results of Theorems 2 and 3, we conclude that by choosing ρ sufficiently large, we can increase the measure of $\mathcal{T}_s$. On $\mathcal{T}_s$ the actuator fault error satisfies (12) which guarantees that the actuator fault error will continue to decrease for all $t \in \mathcal{T}_s$. Specifically, as $t \to \infty$ this error converges to zero while in sliding.

Remark 3. As discussed in [43,44], if the conditions in Theorem 3 are satisfied, an arbitrarily accurate estimate of the actuator fault f_a can be obtained asymptotically, via low-pass filtering of the discontinuous term $\hat{f}_a$.

Remark 4. An issue that remains to be addressed is: if a sensor fault f_s occurs at Δ_s, is it possible for the observer $\mathcal{O}^-$ to have a residual $r = \int_{t-\delta}^{t} |e_y(\tau')|\, d\tau'$ that is identically zero? The problem associated with this scenario is that the transition from observer $\mathcal{O}^-$ to $\mathcal{O}^+$ would not occur. To analyze this possibility, consider the output error e_y with observer $\mathcal{O}^-$ when $f_s \neq 0$:

$$e_y = y - \hat{y} = C_\sigma e + D_\sigma f_s. \tag{13}$$

For the residual r to be zero after $f_s \neq 0$, the error $e(t)$ must satisfy $C_\sigma e = -D_\sigma f_s$ almost everywhere. From (4), before the first switching time the error $e(t)$ has the form

$$e(t) = e^{(A_\sigma - L_\sigma^- C_\sigma)(t-\Delta_s)} e(\Delta_s) + \int_{t-\Delta_s}^{t} e^{(A_\sigma - L_\sigma^- C_\sigma)(t-\tau)} (G_\sigma (f_a(\tau) - \hat{f}_a(\tau))) d\tau. \tag{14}$$

Note that the sensor fault f_s does appear in the output $y(t)$ and consequently the sliding term $\hat{f}_a(\tau)$ from (2). From (14), we see that $e(t)$ is continuous between switching times, which implies $e_y(\Delta_s^+) = C_\sigma e(\Delta_s^-) + D_\sigma f_s(\Delta s^+)$. Hence, for almost all errors $e(\Delta_s^-)$, an abrupt sensor fault f_s causes the output error e_y of the observer $\mathcal{O}^-$ to diverge from the sliding surface for a non-zero amount of time. This implies the existence of a sufficiently small positive scalar $\bar{r}$ such that $r(\Delta_s + \delta) \ge \bar{r}$, which in turn forces the integral in (2) to be non-zero.

Next we discuss the $\mathcal{O}^+$ observer design.

4. Design of post sensor-fault observer $\mathcal{O}^+$

The error dynamics of the observer $\mathcal{O}^+$ are given by,

$$\dot{e} = (A_\sigma - L_\sigma^+ C_\sigma) e + L_\sigma^+ D_\sigma f_s + G_\sigma f_a, \tag{15a}$$

$$e(t_k^+) = \Theta_{\sigma(t_k^-),\sigma(t_k^+)} e(t_k^-). \tag{15b}$$

Lumping the uncertain terms together as $w = \begin{bmatrix} f_s^\top & f_a^\top \end{bmatrix}^\top$, we rewrite (15a) as,

$$\dot{e} = (A_\sigma - L_\sigma^+ C_\sigma) e + \Gamma_\sigma w, \tag{16}$$

where $\Gamma_\sigma = \begin{bmatrix} L_\sigma^+ D_\sigma & G_\sigma \end{bmatrix}$. Note that

$$\|w(t)\| \le \rho_w \tag{17}$$

for all t, where $\rho_w = \max\{\rho_a, \rho_s\}$ and ρ_a, ρ_s are presented in Assumption 3.

The $\mathcal{O}^+$ observer is designed to be $\mathcal{L}_\infty$-stable with a specified performance level (p.l.), according to the following definition.

Definition 2 (*$\mathcal{L}_\infty$ stability with p.l. γ*)**.** The switched linear system (15) with performance output

$$z = He$$

is globally uniformly $\mathcal{L}_\infty$ stable with a specified performance level γ if the following conditions are satisfied for any arbitrary switching signal $\sigma(t)$.

($\mathcal{P}1$) **Global uniform asymptotic stability.** The nominal error system $\dot{e} = (A_\sigma - L_\sigma^+ C_\sigma)e$, is globally uniformly asymptotically stable with respect to the origin.

($\mathcal{P}2$) **Global uniform boundedness of the error state.** For every initial error e_0 and every disturbance input w, there is a bound $\beta(e_0, \|w(\cdot)\|_\infty)$ so that for every initial condition, $e(t_0) = e_0$, we have

$$\|e(t)\| \le \beta(e_0, \|w(\cdot)\|_\infty), \qquad \text{for all} \quad t \ge t_0$$

for all $t \ge t_0$.

($\mathcal{P}3$) **Output response for zero initial error.** For zero initial error, $e(t_0) = 0$, and every disturbance input w, we have

$$\|z(t)\| \le \gamma \|w(\cdot)\|_\infty,$$

for all $t \ge t_0$.

($\mathcal{P}4$) **Ultimate output response.** For every initial error $e(t_0) = e_0$, and every disturbance input w, we have

$$\limsup_{t\to\infty} \|z(t)\| \le \gamma \|w(\cdot)\|_\infty. \tag{18}$$

Moreover, convergence is uniform with respect to t_0.

For additional background, we refer the reader to [38,45].

Remark 5. The matrix H is used to prioritize the performance level γ of the observer $\mathcal{O}^+$ on linear combinations of the error states. For example, let $x \triangleq \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top \in \mathbb{R}^3$ and x_1 is the only state of interest after a sensor fault occurs. Then $H = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ can be selected to reflect this prioritization of x_1. As per Definition 2, if γ can be made small, $\limsup_{t\to\infty} \|He(t)\| = \limsup_{t\to\infty} \|\hat{x}_1(t) - x_1(t)\|$ will be made small. If all the states are equally important, $H = I$, the identity matrix.

We now present sufficient conditions for the design of the $\mathcal{O}^+$ observer gains.

Theorem 4. *Let $z = H_i e$ be the desired performance output. Suppose that Assumptions 1–4 hold and there exist matrices $P_2 = P_2^\top \succ 0$, $Y_1, \ldots, Y_N$ and positive scalars μ_0, μ_1, α such that the following conditions hold,*

$$\begin{bmatrix} Z_i & Y_i D_i & P_2 G_i \\ D_i^\top Y_i^\top & -2\mu_0 \alpha I & 0 \\ G_i^\top P_2 & 0 & -2\mu_0 \alpha I \end{bmatrix} \preceq 0 \tag{19a}$$

$$\Theta_{i,j}^\top P_2 \Theta_{i,j} - P_2 \preceq 0 \tag{19b}$$

$$P_2 - \mu_1 H_i^\top H_i \succeq 0 \tag{19c}$$

for $i, j \in \{1, \ldots, N\}$where

$$Z_i = A_i^\top P_2 + P_2 A_i - C_i^\top Y_i^\top - Y_i C_i + 2\alpha P_2.$$

Then the observer with gains $L_i^+ = Y_i P_2^{-1}$ has error dynamics which are $\mathcal{L}_\infty$ stable with performance level $\gamma = \sqrt{\mu_0/\mu_1}$.

Proof. Consider a quadratic function

$$V(t) \triangleq V(e(t)) = e^\top P_2 e.$$

Consider any open time-interval (t_k, t_{k+1}) between switches. Within any such open interval, the function V is differentiable with respect to t. Then, the time-derivative of V evaluated on the dynamics (15) for any such interval is given by

$$\begin{aligned} \dot{V} &= 2e^\top P_2 \dot{e}, \\ &= 2e^\top P_2 (A_i - L_i^+ C_i)e + 2e^\top P_2 L_i^+ D_i f_s + 2e^\top P_2 G_i f_a. \end{aligned} \tag{20}$$

Replacing $Y_i = P_2 L_i^+$ in (19a) and using the equality (20), we get,

$$
\begin{aligned}
0 &\geq \begin{bmatrix} e \\ f_s \\ f_a \end{bmatrix}^\top \begin{bmatrix} Z_i & Y_i D_i & P_2 G_i \\ D_i^\top Y_i^\top & -2\mu_0 \alpha I & 0 \\ G_i^\top P_2 & 0 & -2\mu_0 \alpha I \end{bmatrix} \begin{bmatrix} e \\ f_s \\ f_a \end{bmatrix} \\
&= 2e^\top P_2 (A_i - L_i^+ C_i) e + 2e^\top P_2 L_i^+ D_i f_s + 2e^\top P_2 G_i f_a - 2\mu_0 \alpha (f_s^2 + f_a^2) + 2\alpha e^\top P_2 e \\
&= 2e^\top P_2 (A_i - L_i^+ C_i) e + 2e^\top P_2 L_i^+ D_i f_s + 2e^\top P_2 G_i f_a - 2\mu_0 \alpha w^\top w + 2\alpha e^\top P_2 e \\
&= \dot{V} - 2\mu_0 \alpha w^\top w + 2\alpha e^\top P_2 e, \\
&= \dot{V} + 2\alpha V - 2\mu_0 \alpha \|w\|^2.
\end{aligned}
$$

This implies

$$
\dot{V} \leq -2\alpha V + 2\alpha \mu_0 \|w\|_\infty^2 \tag{21}
$$

for any open interval (t_k, t_{k+1}).

We want to show that the Lyapunov-like function V is non-increasing for all $t \geq t_0$, where t_0 is the initial time.

Clearly $V(t_0) \leq V(t_0)$. Now, consider the first switching time t_1 and the interval (t_0, t_1) without switching. Pre-multiplying both sides by $e^{-2\alpha t}$, we get

$$
e^{-2\alpha t} \dot{V} \leq -2\alpha e^{-2\alpha t} V + 2\alpha \mu_0 e^{-2\alpha t} \|w\|_\infty^2,
$$

which yields,

$$
\frac{d}{dt}(e^{-2\alpha t} V) \leq 2\alpha \mu_0 e^{-2\alpha t} \|w\|_\infty^2 .
$$

By the Grönwall inequality, we get

$$
V(t) \leq V(t_0) e^{-2\alpha(t - t_0)} + 2\alpha \mu_0 \int_{t_0}^{t} e^{-2\alpha(t-\tau)} \|w(\cdot)\|_\infty^2 \, d\tau
$$

for any $t \in (t_0, t_1)$.

At time t_1, a switch occurs from mode i to mode j, where $i, j \in \{1, \dots, N\}$. By the inequality (19b), we have

$$
\begin{aligned}
V(t_1^+) &= e(t_1^+)^\top P_2 e(t_1^+) \\
&= e(t_1^-)^\top \Theta_{i,j}^\top P_2 \Theta_{i,j} e(t_1^-), \\
&\leq e(t_1^-)^\top P_2 e(t_1^-) \\
&= V(t_1^-).
\end{aligned}
$$

Hence,

$$
V(t) \leq V(t_0) e^{-2\alpha(t - t_0)} + 2\alpha \mu_0 \int_{t_0}^{t} e^{-2\alpha(t-\tau)} \|w(\cdot)\|_\infty^2 \, d\tau \tag{22}
$$

for any $t \in [t_0, t_1]$.

Now, we consider the time interval (t_1, t_2), where t_2 is the next switching time. Then, from (21), we get

$$
\begin{aligned}
V(t) &\leq V(t_1^+) e^{-2\alpha(t - t_1)} + 2\alpha \mu_0 \int_{t_1}^{t} e^{-2\alpha(t-\tau)} \|w(\cdot)\|_\infty^2 \, d\tau \\
&\leq V(t_1^-) e^{-2\alpha(t - t_1)} + 2\alpha \mu_0 \int_{t_1}^{t} e^{-2\alpha(t-\tau)} \|w(\cdot)\|_\infty^2 \, d\tau \\
&\leq V(t_0) e^{-2\alpha(t - t_0)} + 2\alpha \mu_0 \left(\int_{t_1}^{t} e^{-2\alpha(t-\tau)} \|w(\cdot)\|_\infty^2 \, d\tau + \int_{t_0}^{t_1} e^{-2\alpha(t_1-\tau)} \|w(\cdot)\|_\infty^2 \, d\tau \right) \\
&= V(t_0) e^{-2\alpha(t - t_0)} + 2\alpha \mu_0 \int_{t_0}^{t} e^{-2\alpha(t-\tau)} \|w(\cdot)\|_\infty^2 \, d\tau,
\end{aligned}
$$

for all $t \in [t_0, t_2)$. By similar arguments as before, we deduce that this inequality holds for all $t \in [t_0, t_2]$.

Using inductive arguments, we get that (22) holds for any $t \geq t_0$. This yields

$$
\begin{aligned}
V(t) &\leq V(t_0) e^{-2\alpha(t - t_0)} + 2\alpha \mu_0 \int_{t_0}^{t} e^{-2\alpha(t-\tau)} \|w(\cdot)\|_\infty^2 \, d\tau \\
&\leq V(t_0) e^{-2\alpha(t - t_0)} + \mu_0 \|w\|_\infty^2
\end{aligned} \tag{23}
$$

for all $t \geq t_0$.

Hence,

$$\limsup_{t\to\infty} V(t) \le \mu_0 \|w(\cdot)\|_\infty^2 . \tag{24}$$

We will now demonstrate that this is sufficient for the error system (15) to be $\mathcal{L}_\infty$ stable with performance level $\sqrt{\mu_0/\mu_1}$.

Suppose $w = 0$. Then, from (21), we get $\dot V \le -2\alpha V$, which guarantees that the $\mathcal{O}^+$ error dynamics are globally uniformly exponentially stable. This demonstrates property $(\mathcal{P}1)$ in Definition 2.

Next, we consider

$$\Omega = \{e : e^\top P_2 e \le \mu_0 \|w(\cdot)\|_\infty^2\} \tag{25}$$

which is a sub-level set of the Lyapunov-like function V. Then we know that Ω is attractive from the inequality (24).

From (19c), we get

$$e^\top P_2 e - \mu_1 e^\top H_i^\top H_i e \ge 0$$

which implies

$$\mu_1 \|z\|^2 \le V.$$

Using (24) yields

$$\limsup_{t\to\infty} \mu_1 \|z(t)\|^2 \le \mu_0 \|w(\cdot)\|_\infty^2,$$

hence

$$\limsup_{t\to\infty} \|z(t)\| \le \gamma \|w(\cdot)\|_\infty,$$

with $\gamma = \sqrt{\mu_0/\mu_1}$, which demonstrates property $(\mathcal{P}4)$ of Definition 2.

Furthermore, we note that if $e(t_0) \notin \Omega$, then inequality (23) yields

$$\begin{aligned} \lambda_{\min}(P_2)\|e(t)\|^2 &\le V(t) \\ &\le V(t_0)e^{-2\alpha(t-t_0)} + \mu_0 \|w\|_\infty^2 \\ &\le V(t_0) + \mu_0 \|w\|_\infty^2 . \end{aligned}$$

Thus,

$$\|e(t)\| \le \beta(e(t_0), \|w\|_\infty),$$

where

$$\beta(e(t_0), \|w\|_\infty) = \sqrt{\frac{V(t_0) + \mu_0 \|w\|_\infty^2}{\lambda_{\min}(P_2)}}.$$

Now consider the case when $e(t_0) \in \Omega$. We want to show that Ω is disturbance invariant, as discussed in Definition 1.

Assume the contrary, that is, suppose there exists a time $t_3 > t_0$ such that $V(e(t_3)) \notin \Omega$. Since (19b) guarantees that V is non-increasing at switching times, V must cross the boundary of Ω in an interval $[t_1, t_3] \subset [t_0, \infty)$ where V is continuous, that is, no switching occurs in $[t_1, t_3]$.

Since V is continuous and crosses the boundary of Ω, there must exist a time $t_2 \in [t_1, t_3]$ where $V(t_2) = \mu_0 \|w\|_\infty^2$ and $\dot V(t_2) > 0$, which, by (21), is a contradiction. Hence, Ω is disturbance invariant and satisfies

$$e(t)^\top P_2 e(t) \le \mu_0 \|w(\cdot)\|_\infty^2$$

for all $t \ge t_0$.

Thus, for any $e(t_0) \in \Omega$, we obtain

$$\|e(t)\| \le \beta(e(t_0), \|w(\cdot)\|_\infty),$$

where

$$\beta(e(t_0), \|w(\cdot)\|_\infty) = \|w(\cdot)\|_\infty \sqrt{\frac{\mu_0}{\lambda_{\min}(P_2)}}.$$

This proves the property $(\mathcal{P}2)$ of Definition 2.

Furthermore, from (19b), we get

$$\begin{aligned} \|z(t)\|^2 &\le e(t)^\top P_2 e(t) \\ &\le \mu_0 \|w(\cdot)\|_\infty^2, \end{aligned}$$

for all $t \geq t_0$. It follows that

$$\limsup_{t\to\infty} \|z(t)\| \leq \gamma \|w(\cdot)\|,$$

where $\gamma = \sqrt{\mu_0/\mu_1}$. This demonstrates property $(\mathcal{P}3)$ of Definition 2 and concludes the proof. $\square$

Remark 6. Recall that we considered $\Delta_a < \Delta_s$, that is, the lifetime of the sensors is larger than the actuators. Although this is generally true in practice, it may be possible for $\Delta_s < \Delta_a$, in which case, the observer $\mathcal{O}^+$ is triggered, which provides ultimate boundedness of the estimation error until the sensor is fixed.

Remark 7. If α is considered to be a variable, then the observer design inequalities (5), (19) are not linear matrix inequalities. To convert this to LMIs, a line search with respect to α may be performed to optimize for the decay rate.

Remark 8. In practice, we minimize $\mu_0 - \mu_1$ subject to the conditions (19) to ensure that γ is minimized. A small value of γ implies that the ultimate bound on the performance output z is less conservative, that is, the estimation error $H_i e$ is smaller.

5. Example

In this section, we present a simulated example to illustrate the performance of the dual-mode observer with a numerical example. We consider a SLS of the form (1), based on proposed in [28] with the $N = 3$ subsystems. The SLS considered is the one described by (1) parametrized by

$$A_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & -2 & -1 & -1 \\ 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -2 & 1 \\ 1 & -1 & 0 & -1 & -1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1 & 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & -1 \\ -1 & -2 & -1 & 0 & -1 \\ 0 & -1 & 1 & -1 & 1 \\ 1 & -1 & 0 & -1 & 0 \end{bmatrix}, \; G_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \; G_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ -1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \; G_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & -1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^\top, \; C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^\top, \; C_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}^\top,$$

$$\Theta_{2,3} = \begin{bmatrix} 0.11 & 0 & 0 & 0 & 0 \\ 0 & 0.83 & 0 & 0 & 0 \\ 0 & 0 & 0.34 & 0 & 0 \\ 0 & 0 & 0 & 0.29 & 0 \\ 0 & 0 & 0 & 0 & 0.75 \end{bmatrix}, \quad \Theta_{3,1} = \begin{bmatrix} 0.01 & 0 & 0 & 0 & 0 \\ 0 & 0.04 & 0 & 0 & 0 \\ 0 & 0 & 0.67 & 0 & 0 \\ 0 & 0 & 0 & 0.60 & 0 \\ 0 & 0 & 0 & 0 & 0.53 \end{bmatrix},$$

$$\Gamma_{2,3} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^\top, \quad \Gamma_{3,1} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \end{bmatrix}^\top.$$

In this simulation, we consider an arbitrarily generated switching sequence $\sigma(t)$ where switching between subsystems occurs after every 1 s. The system in (1) is subject to actuator fault at time $\Delta_a = 50$ s which is to be detected by the proposed observer $\mathcal{O}^-$. Following the actuator fault, at time $\Delta_s = 120$ s, a sensor fault $f_s \equiv 0.5$ occurs. In Figs. 3 and 4, the green window is the time window at which only the actuator fault occurs. In the purple window, both actuator and sensor faults occur simultaneously. We consider $\rho_a = 0.5$ and $\rho_s = 0.5$ as the bounds on our actuator and sensor fault, respectively.

For time $t \in [0, \Delta_s)$, the observer $\mathcal{O}^-$ is designed using Theorem 1 which reconstructs the actuator fault in addition to the states. Solving the LMIs (5) using CVX [46] with $\alpha = 1$, we obtain the matrix

$$P_1 = \begin{bmatrix} 33.12 & 11.71 & -12.96 & -0.98 & 3.63 \\ 11.71 & 32.46 & -9.33 & -2.30 & -3.59 \\ -12.96 & -9.33 & 12.96 & 0.98 & -3.63 \\ -0.98 & -2.30 & 0.98 & 9.76 & 1.32 \\ 3.63 & -3.59 & -3.63 & 1.32 & 7.23 \end{bmatrix}$$

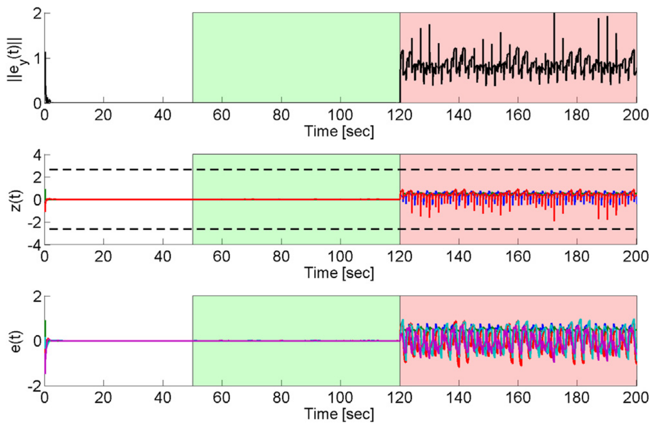

Fig. 3. Performance of observer $\mathcal{O}^+$ in the presence of both actuator and sensor faults and state jumps. (Top) Norm of the output error. The sensor fault occurs at $t = 120$ s and continues to $t = 200$ s (shown in light purple window). The sensor fault is detected by computing a residual r exceeding the predefined threshold $\bar{r} = 0.1$. (Middle) The performance output z satisfies the ultimate bounding condition (18). The black lines denote $\pm\gamma\rho_w = 2.635$. (Bottom) The state estimation error of all five states. Note that after the sensor fault occurs, the states are ultimately bounded. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

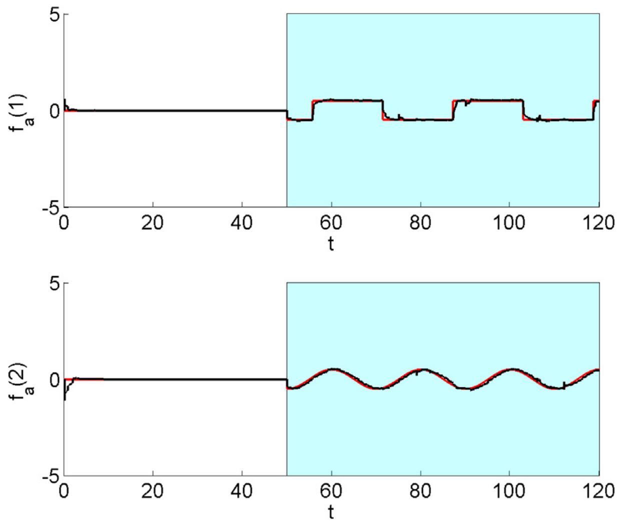

Fig. 4. Estimation of the actuator faults. The actuator fault occurs at $t = 50$ s and continues till $t = 120$ s, as shown by the green patch. The red line denotes the actual actuator fault and the black line denotes the estimated actuator fault signal after low-pass filtering the nonlinear injection term of $\mathcal{O}^-$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

and observer $\mathcal{O}^-$ gains given by

$$L_1^- = \begin{bmatrix} 9.47 & 3.63 & 5.05 \\ 4.47 & 3.13 & 3.20 \\ 20.24 & 9.15 & 13.72 \\ -0.13 & 4.63 & -0.94 \\ 23.85 & 12.36 & 18.17 \end{bmatrix}, \quad L_2^- = \begin{bmatrix} 2.33 & 3.10 & -0.71 \\ 1.40 & 1.87 & -0.56 \\ 5.21 & 7.62 & -1.77 \\ 3.35 & 3.77 & -1.57 \\ 7.54 & 8.26 & -1.84 \end{bmatrix}, \quad L_3^- = \begin{bmatrix} 2.75 & -1.69 & 1.83 \\ 2.42 & -0.22 & 1.02 \\ 5.99 & -6.68 & 5.11 \\ 2.06 & 1.57 & -1.94 \\ 10.61 & -7.02 & 8.28 \end{bmatrix}$$

$$F_1 = \begin{bmatrix} 12.96 & -1.34 & -9.66 \\ -1.01 & 18.21 & 0.32 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -12.96 & 1.34 & 9.66 \\ -1.01 & 18.21 & 0.32 \end{bmatrix}, \quad F_3 = \begin{bmatrix} 12.98 & -1.34 & -9.66 \\ 1.01 & -18.21 & -0.32 \end{bmatrix}.$$

We also compute the finite convergence time to the sliding surface $\tau_s = 0.06$ s.

The sliding mode gain is chosen to be $\rho = 400$. In Fig. 3, we show the performance of the observer $\mathcal{O}^-$ for $t \in [0, 120)$. Note that the state and output estimation errors are driven to zero by the observer. Furthermore, in Fig. 4, we see that the actuator fault signal is reconstructed from $t = \Delta_a = 50$ s, as expected. The actuator faults are of the form,

$$f_a = 0.5\begin{bmatrix} \mathrm{sq}(0.2t) & \cos\left(\frac{2\pi t}{7}\right)\end{bmatrix}^\top,$$

where $\mathrm{sq}(t)$ is a periodic square wave. We use a low-pass filter of the form, $H(s) = \frac{1}{s\tau+1}$, where $\tau = 1$. The low-pass filter extracts the actuator fault from the nonlinear injection term as demonstrated in [47], and show that the fault estimate $\hat{f}_a$ is highly accurate. Additionally, we note that at time $\Delta_s = 120$ s, the norm of output error signal $\|y - \hat{y}\|$ becomes non-zero consistently. This in turn makes the residual r, defined in (2), exceed the user-defined threshold of $\bar{r} = 1$ calculated over a window of $\delta = 1$ s, where δ is described in (2). This triggers the switch to the post sensor-fault observer $\mathcal{O}^+$.

For time $t > \Delta_s$, the observer $\mathcal{O}^+$ is designed as described in Theorem 4 which guarantees that the state estimates converge to a disturbance invariant set about the origin. Note that

$$\rho_w \triangleq \|w(\cdot)\|_\infty = \max\{\rho_a, \rho_s\} = 0.5.$$

The gains of the observer $\mathcal{O}^+$ are computed to be

$$L_1^+ = \begin{bmatrix} 2.58 & 0.53 & -0.72 \\ -0.07 & 0.99 & -0.13 \\ 1.13 & 0.15 & -0.35 \\ 2.13 & 1.22 & -0.51 \\ 0.68 & -0.06 & 0.14 \end{bmatrix}, \quad L_2^+ = \begin{bmatrix} 2.89 & 0.82 & -1.03 \\ -0.14 & 1.08 & -0.80 \\ 1.91 & 0.99 & -2.05 \\ 2.83 & 1.27 & -1.76 \\ 1.22 & 0.13 & -0.46 \end{bmatrix}, \quad L_3^+ = \begin{bmatrix} 1.78 & -0.27 & 0.73 \\ 0.21 & 0.52 & -0.62 \\ 0.98 & -1.10 & -0.38 \\ 1.73 & -0.73 & 0.24 \\ 0.57 & -0.54 & 0.41 \end{bmatrix}$$

using CVX by solving (19) with $\alpha = 1$. The performance level is computed to be $\gamma = 5.27$, and the disturbance invariant set is described by (25) is parametrized by $\mu_0\rho_w^2 = 1.41$ and

$$P_2 = \begin{bmatrix} 1.68 & -0.45 & -0.96 & -0.41 & -0.23 \\ -0.45 & 0.88 & 0.40 & -0.03 & 0.40 \\ -0.96 & 0.40 & 1.38 & 0.20 & -0.38 \\ -0.41 & -0.03 & 0.20 & 0.71 & -0.04 \\ -0.23 & 0.40 & -0.38 & -0.04 & 0.90 \end{bmatrix}.$$

The output error is guaranteed to satisfy the condition (18), as illustrated in Fig. 3. After $\mathcal{O}^-$ detects a NRZ sensor fault (Assumption 4), we switch the observer to $\mathcal{O}^+$ permanently. Fault tolerant controller strategies can then be implemented to ensure safety of operation.

6. Conclusions

In this paper, we propose a methodology of designing dual-mode observers for fault detection and estimation of switched linear systems. LMI-based sufficient conditions are provided for the design of the observer gain matrices. The performance of the observer strategy is illustrated through simulation. An open problem is to extend these results for the construction of fault-tolerant controllers.

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