On Optimal Collision Avoidance and Formation Switching on Riemannian Manifolds

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Research Project

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Abstract

The problems of optimal collision avoidance and optimal formation switching for multiple agents moving on a Riemannian manifold are studied. It is assumed that the underlying manifold admits a group of isometries, with respect to which the Lagrangian function is invariant. Various necessary conditions on optimal solutions are obtained through variational analysis, and the results are illustrated by examples. A generalization of the problems is also presented.

1 Introduction

We study two related problems for multiple agents moving on a Riemannian manifold: optimal collision avoidance (OCA) and optimal formation switching (OFS). In both cases, a number of agents move from a set of initial positions to a set of destination positions within a certain time interval. Their trajectories must satisfy the separation condition that at any time the Riemannian distance between any two of them is at least r for some positive r. In the OFS problem, there are further constraints on the distances between certain pairs of agents. The optimal trajectories are the ones that minimize the weighted sum of the energies of individual agents, with the weights representing the priorities of the agents.

The motivating application for this research is aircraft conflict resolution [10, 13], in which the underlying manifold is $\mathbb{R}^2$ or $\mathbb{R}^3$, and r is 5 nautical miles for en route aircraft. Related applications can be, for example, multiple mobile robots cooperating to carry a common object, or a multi-link reconfigurable robot performing configuration switchings. In this paper we only consider holonomic constraints, as opposed to the many papers dealing with nonholonomic constraints, such as [2, 4, 20]. Other relevant work can be found in [12, 21].

It is often the case that the underlying manifold admits a group of symmetries. Without the separation constraints, the classical Noether theorem [1, 15] can be used to reduce the degrees of freedom of the problems by establishing the conservation of certain quantities called *momentum maps*. The OCA and OFS problems we consider here have boundary constraints, and in the case of the OFS problem, even the state spaces themselves are not smooth. However, since the constraints are also invariant with respect to the symmetry group, the conservation laws still hold. Bounds on the conserved quantities can be derived through second variation and topological consideration. These bounds apply uniformly for optimal solutions of both the OCA and the OFS problems, and

can be improved by taking into consideration specific structures of the problems. Our results can be further generalized, for example, to OCA and OFS problems of bodies with arbitrary shapes.

This paper is organized as follows. In Section 2, we formulate the OCA and the OFS problems for k agents moving on a Riemannian manifold M, and introduce the symmetry assumption on M assumed throughout the rest of the paper. Then in Section 3, based on the symmetry assumption, we derive various necessary conditions that apply *uniformly* for optimal solutions to *all* OCA and OFS problems. In particular, using some preliminary results in Section 3.1, we show in Section 3.2 that a version of the classical Noether theorem, namely, the preservation of momentum maps, still apply in our problems that are nonsmooth in nature. The possible values of the conserved quantities are also restricted by optimality, as is shown in Section 3.3 and 3.4 by second variation and topological consideration, respectively. In Section 4, an interesting example will be presented to show that our necessary conditions, based solely on the symmetry assumption, are in general not sufficient, especially when k is large. Section 5 contains a natural generalization of our results to the OCA and OFS problems for bodies on M. The paper is concluded in Section 6 with some discussion on possible future directions. Appendix A and Appendix B provide the proofs of several lemmas useful in Section 3. Throughout the paper, the results are illustrated using several recurrent examples: the Euclidean space $\mathbb{R}^n$, the sphere $\mathbf{S}^n$, a group G with a bi-invariant metric, the Grassmann and the Stiefel manifolds.

One important aspect of the problems left largely untouched in this study is the existence (and uniqueness) of solutions to the OCA and OFS problems. In this paper we have restricted our consideration of solutions to continuous and piecewise smooth curves for simplicity. However, even when the underlying manifold is simple, it is already a nontrivial task to prove that solutions exist in this category for arbitrary starting and destination positions of the agents. Therefore, all the obtained results should be understood to hold under the provision that solutions exist.

2 Problem Formulation

Let M be a C^∞ Riemannian manifold. For each $q \in M$, denote by $\langle \cdot, \cdot \rangle_q$ and $\| \cdot \|_q$ (or simply $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$) the Riemannian metric and the corresponding norm on the tangent space T_qM respectively. Given $t_0, t_1 \in \mathbb{R}$ with $t_0 \leq t_1$, the arc length of a curve $\gamma : [t_0, t_1] \to M$ is defined as $\int_{t_0}^{t_1} \|\dot{\gamma}(t)\| \, dt$. Note that unless otherwise stated, in this paper we shall always assume implicitly

that curves in M are continuous and piecewise C^∞. The distance between two arbitrary points q_1 and q_2 in M, denoted by $d_M(q_1, q_2)$, is by definition the infimum of the arc lengths of all the curves connecting q_1 and q_2. A geodesic in M is a locally distance-minimizing curve. More precisely, $\gamma : [t_0, t_1] \to M$ is a geodesic if and only if for any $t \in (t_0, t_1)$, there exists an $\epsilon > 0$ small enough such that the arc length of γ restricted on $[t-\epsilon, t+\epsilon]$ is equal to $d_M(\gamma(t-\epsilon), \gamma(t+\epsilon))$. In this paper, it is assumed that M is complete and connected, and that all the geodesics in M are parameterized proportionally to arc length.

Let $L : TM \to \mathbb{R}$ be a smooth function (the *Lagrangian function*) defined on the tangent bundle of M such that it is nonnegative and convex on each fiber. As an example one can take $L = \frac{1}{2}\|\cdot\|^2$, i.e., $L(v) = \frac{1}{2}\|v\|_q^2$, $\forall v \in T_qM, q \in M$. For each curve $\gamma : [t_0, t_1] \to M$, define its *cost* as

$$J(\gamma) = \int_{t_0}^{t_1} L[\dot{\gamma}(t)]\, dt. \tag{1}$$

The curves joining two fixed points in M with minimal cost are extremals of the functional J, which in any canonical local coordinates of TM, say, $(x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n)$, $n = \dim(M)$, are characterized by the Euler-Lagrange equations [1]:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i}, \quad i = 1, \ldots, n.$$

For $L = \frac{1}{2}\|\cdot\|^2$, the above equations describe the geodesics in M.

Consider an (ordered) k-tuple of points of M, $\langle q_i \rangle_{i=1}^k = (q_1, \ldots, q_k)$, where k is a positive integer. We say that $\langle q_i \rangle_{i=1}^k$ satisfies the *r-separation condition* for some positive r if $d_M(q_i, q_j) \geq r$ for all $i \neq j$. Let $\langle a_i \rangle_{i=1}^k$ and $\langle b_i \rangle_{i=1}^k$ be two k-tuples of points of M, each of which satisfies the r-separation condition. $\langle a_i \rangle_{i=1}^k$ is called the *starting position* and $\langle b_i \rangle_{i=1}^k$ the *destination position.*

Let $\gamma = \langle \gamma_i \rangle_{i=1}^k$ be a k-tuple of curves in M defined on $[t_0, t_1]$ such that $\gamma_i(t_0) = a_i, \gamma_i(t_1) = b_i$, for $i = 1, \ldots, k$. One can think of γ as the joint trajectory of k agents moving on M that start from $\langle a_i \rangle_{i=1}^k$ at time t_0 and end at $\langle b_i \rangle_{i=1}^k$ at time t_1. γ is said to be *collision-free* if the k-tuple $\langle \gamma_i(t) \rangle_{i=1}^k$ satisfies the r-separation condition for each $t \in [t_0, t_1]$. Equivalently, if the agents are Riemannian disks of radius $\frac{r}{2}$ in M whose centers follow γ, then γ is collision-free if and only if no two agents overlap during $[t_0, t_1]$. Naturally, r is chosen to be small enough so that it is possible to pack k disks of radius $\frac{r}{2}$ in M.

The first problem we are going to study is

Problem 1 (Optimal Collision Avoidance (OCA)) *Among all collision-free $\gamma = \langle \gamma_i \rangle_{i=1}^k$ that*

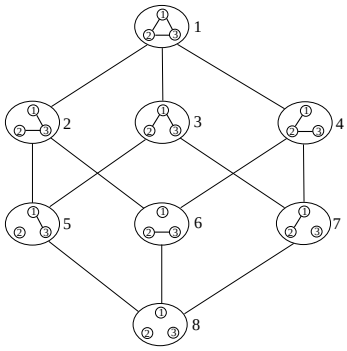

Figure 1: Hasse diagram of $\mathcal{F}$ when $k = 3$, $M = \mathbb{R}^2$.

start from $\langle a_i \rangle_{i=1}^k$ *at time* t_0 *and end at* $\langle b_i \rangle_{i=1}^k$ *at time* t_1*, find the one (or ones) minimizing the cost*

$$J(\gamma) = \sum_{i=1}^{k} \lambda_i J(\gamma_i). \tag{2}$$

Here $\langle \lambda_i \rangle_{i=1}^k$ is a k-tuple of positive numbers representing the relative priorities of the k agents, while $J(\gamma_i)$ is defined in (1) for each i.

There is an alternative way of formulating the OCA problem. By viewing each k-tuple of points of M as a single point in $M^{(k)} = M \times \cdots \times M$, γ under consideration becomes a curve in $M^{(k)}$ starting from $(a_1, \ldots, a_k)$ at time t_0 and ending at $(b_1, \ldots, b_k)$ at time t_1, while avoiding the obstacle

$$W = \cup_{i \neq j} \{(q_1, \ldots, q_k) \in M^{(k)} : d_M(q_i, q_j) < r\}. \tag{3}$$

As a result, solutions to the OCA problem are cost-minimizing curves in $M^{(k)} \setminus W$ connecting two fixed points. In particular, if $L = \frac{1}{2}\|\cdot\|^2$, solutions are geodesics in $M^{(k)} \setminus W$, a manifold with nonsmooth boundary.

To introduce the second problem we need some notions. For a k-tuple $\langle q_i \rangle_{i=1}^k$ of points of M satisfying the r-separation condition, a graph $(\mathcal{V}, \mathcal{E})$ can be constructed as following: the set of vertices is $\mathcal{V} = \{1, \ldots, k\}$; the set $\mathcal{E}$ of edges is such that an edge (i, j) exists between vertex i and vertex j if and only if $d_M(q_i, q_j) = r$. $(\mathcal{V}, \mathcal{E})$ is called the *formation pattern* of $\langle q_i \rangle_{i=1}^k$. Let $\gamma = \langle \gamma_i \rangle_{i=1}^k$ be a collision-free k-tuple of curves in M defined on $[t_0, t_1]$. Then, for each $t \in [t_0, t_1]$, the formation pattern of γ at time t is defined to be the formation pattern of $\langle \gamma_i(t) \rangle_{i=1}^k$.

Remark 1 *Depending on* M, r*, and* k*, not all graphs with* k *vertices can be realized as the formation pattern of some* k*-tuple* $\langle q_i \rangle_{i=1}^k$ *of points of* M *satisfying the* r*-separation condition. For example,*

the complete graph of four vertices is not the formation pattern of any such $\langle q_i \rangle_{i=1}^4$ *when* $M = \mathbb{R}^2$*. In fact, each feasible formation pattern* $(\mathcal{V}, \mathcal{E})$ *corresponds to a nonempty subset of* $M^{(k)} \setminus W$*, namely, those* $(q_1, \ldots, q_k) \in M^{(k)} \setminus W$ *satisfying that* $d_M(q_i, q_j) = r$ *if* $(i, j) \in \mathcal{E}$ *and* $d_M(q_i, q_j) > r$ *otherwise. In particular, if* $\mathcal{E}$ *contains no edges, then* $(\mathcal{V}, \mathcal{E})$ *corresponds to the interior of* $M^{(k)} \setminus W$*.*

Denote by $\mathcal{F}$ the set of all formation patterns. A partial order $\prec$ is defined on $\mathcal{F}$ such that two formation patterns $(\mathcal{V}_1, \mathcal{E}_1) \prec (\mathcal{V}_2, \mathcal{E}_2)$ if and only if $(\mathcal{V}_1, \mathcal{E}_1)$ is a subgraph of $(\mathcal{V}_2, \mathcal{E}_2)$. With this partial order, $\mathcal{F}$ can be rendered graphically as a *Hasse diagram*, in which each element of $\mathcal{F}$ is drawn as a node, and between these nodes line segments are drawn according to the following rules:

1. If $(\mathcal{V}_1, \mathcal{E}_1) \prec (\mathcal{V}_2, \mathcal{E}_2)$, then the node corresponding to $(\mathcal{V}_1, \mathcal{E}_1)$ is placed lower than the node corresponding to $(\mathcal{V}_2, \mathcal{E}_2)$;
2. There is a line segment upward from $(\mathcal{V}_1, \mathcal{E}_1)$ to $(\mathcal{V}_2, \mathcal{E}_2)$ if and only if $(\mathcal{V}_1, \mathcal{E}_1) \prec (\mathcal{V}_2, \mathcal{E}_2)$ and there exists no other $(\mathcal{V}, \mathcal{E}) \in \mathcal{F}$ such that $(\mathcal{V}_1, \mathcal{E}_1) \prec (\mathcal{V}, \mathcal{E})$ and $(\mathcal{V}, \mathcal{E}) \prec (\mathcal{V}_2, \mathcal{E}_2)$.

As an example, Figure 1 plots the Hasse diagram of $\mathcal{F}$ in the case $M = \mathbb{R}^2$ and $k = 3$.

Now we define the second problem.

Problem 2 (Optimal Formation Switching (OFS)) *Let* $\tilde{\mathcal{F}}$ *be a subset of* $\mathcal{F}$ *to which the formation patterns of both* $\langle a_i \rangle_{i=1}^k$ *and* $\langle b_i \rangle_{i=1}^k$ *belong. Among all collision-free* $\gamma = \langle \gamma_i \rangle_{i=1}^k$ *that start from* $\langle a_i \rangle_{i=1}^k$ *at time* t_0*, end in* $\langle b_i \rangle_{i=1}^k$ *at time* t_1*, and satisfy the constraint that the formation pattern of* γ *at any time* $t \in [t_0, t_1]$ *belongs to* $\tilde{\mathcal{F}}$*, find the one (or ones) minimizing the cost (2).*

The OFS problem is a natural generalization of the OCA problem in that γ as a curve in $M^{(k)} \setminus W$ is required to lie in a subset of $M^{(k)} \setminus W$ obtained by piecing together cells of various dimensions, each of which corresponds to a formation pattern in $\tilde{\mathcal{F}}$. In the example shown in Figure 1, one can choose $\tilde{\mathcal{F}}$ to consist of formation patterns 1, 2, 3, and 4, thus requiring that all three agents, each of which is of radius $\frac{r}{2}$, "contact" one another either directly or indirectly via the third agent at any time in the joint trajectory. As another example, $\tilde{\mathcal{F}}$ can be chosen to consist of formation patterns 1, 3, 4, and 7. So agent 1 and agent 2 are required to be bound together at all time; and the OFS problem becomes the optimal collision avoidance between agent 3 and this two-agent subsystem, i.e., between a disk and an object of two disks glued together.

Remark 2 *Solutions to the OFS problem may not exist for some choices of $\tilde{\mathcal{F}}$ and starting and destination positions. For example, suppose that in Figure 1, $\tilde{\mathcal{F}}$ consists of only formation pattern 8. So the corresponding subset of $M^{(k)} \setminus W = \mathbb{R}^6 \setminus W$ is its interior. If the starting and destination positions correspond to two points in int$(\mathbb{R}^6 \setminus W)$ that are 'invisible' to each other with respect to the obstacle W, then the OFS problem does not admit a solution. In general, to ensure that solutions exist to the OFS problem, it is sufficient (though not necessary) to require that the subset of $M^{(k)} \setminus W$ corresponding to $\tilde{\mathcal{F}}$ be closed and that the two points corresponding to the starting and destination positions be in the same connected component of this subset. To meet the first requirement, $\tilde{\mathcal{F}}$ needs to have the following property: for each $(\mathcal{V}, \mathcal{E}) \in \tilde{\mathcal{F}}$, any feasible formation pattern $(\mathcal{V}_1, \mathcal{E}_1)$ such that $(\mathcal{V}, \mathcal{E}) \prec (\mathcal{V}_1, \mathcal{E}_1)$ is also an element of $\tilde{\mathcal{F}}$. To meet the second requirement, it is enough to establish the existence of one collision-free $\langle \gamma_i \rangle_{i=1}^k$ from the starting to the destination position whose formation pattern is always in $\tilde{\mathcal{F}}$.*

Remark 3 *Solutions to the OFS problem are expected to be less regular than those to the OCA problem. For example, it is found in [9] that when M is a Euclidean space and $L = \frac{1}{2}\|\cdot\|^2$, solutions to the OCA problem are always C^1 (though not C^2 in general), while solutions to the OFS problem that are not C^1 can be easily constructed.*

Instead of studying the OCA and OFS problems on general Riemannian manifolds, in this paper we focus on a special case by making the following assumptions.

Assumption 1 (Symmetry) *There is a Lie group G such that*

1. *$\Phi : G \times M \to M$ is a C^∞ left action of G on M by isometry;*
2. *The Lagrangian function L is G-invariant.*

The meaning of the above assumptions is explained in the following. For brevity, we write $gq \triangleq \Phi(g, q)$ for $g \in G$ and $q \in M$. For each $g \in G$, define $\Phi_g : M \to M$ to be the map $\Phi_g : q \mapsto gq$, $\forall q \in M$. For each $q \in M$, define $\Phi^q : G \to M$ to be the map $\Phi^q : g \mapsto gq$, $\forall g \in G$. Both Φ_g and Φ^q are C^∞ maps. That Φ is a left action on M is equivalent to that $\Phi_{g_1 g_2} = \Phi_{g_1} \circ \Phi_{g_2}$ for $g_1, g_2 \in G$, and that $\Phi_e = \mathrm{id}_M$, where e is the identity element of G and id_M is the identity map on M. For each $g \in G$, the first assumption implies that Φ_g is an isometry of M, while the second assumption implies that $L \circ d\Phi_g = L$. Here $d\Phi_g : TM \to TM$ is the tangent map of Φ_g.

We give several simple examples of M and G satisfying the above assumptions.

Example 1 Let $M = \mathbb{R}^n$ be a Euclidean space with the usual metric. Let $G = \mathbf{SE}_n$, the group of orientation-preserving isometries of $\mathbb{R}^n$. Choose $L = \frac{1}{2}\|\cdot\|^2$.

Example 2 Let $M = \mathbf{S}^{n-1}$ be the unit $(n-1)$-sphere for some $n \geq 2$ with the standard metric. Let $G = \mathbf{SO}_n$, the group of orientation-preserving n-by-n orthogonal matrices. Let $L = \frac{1}{2}\|\cdot\|^2$.

Example 3 Let $M = G$ be a Lie group equipped with a left invariant Riemannian metric. Then the group multiplication $G \times G \to G$ is a left action of G on itself by isometry. Let $L : TG \to \mathbb{R}$ be any left invariant function. Such L correspond in a one-to-one way with nonnegative and convex functions $T_eG \to \mathbb{R}$.

More examples will be presented later.

3 Necessary Conditions

In this section, we will derive necessary conditions for optimal solutions to the OCA and OFS problems on a Riemannian manifold M satisfying Assumption 1. It should be pointed out that some of the results, especially those in Section 3.2, can be derived from the more elegant Hamiltonian or symplectic point of view. In this paper, however, we adopt the more direct Lagrangian viewpoint for two reasons: the nonsmooth nature of the problems can be more easily dealt with this way; and as a byproduct, further optimality conditions can be obtained, such as those in Section 3.3 and 3.4.

3.1 Variations of Curves in G

We first introduce some notions and results on smooth variations of curves in G that are useful in later sections. All the results in this section are well known in the literature.

Definition 1 *Let $h_0 : [t_0, t_1] \to G$ be a C^∞ curve in G. A (smooth)* variation *of h_0 is a C^∞ map $h : (-\epsilon, \epsilon) \times [t_0, t_1] \to G$ for some small positive number ϵ such that $h(0, \cdot) = h_0(\cdot)$. If in addition $h(\cdot, t_0) \equiv h_0(t_0)$ and $h(\cdot, t_1) \equiv h_0(t_1)$, then the variation h is called* proper.

Let h be a variation of h_0 as in Definition 1. For each $s \in (-\epsilon, \epsilon)$, denote by $h_s(\cdot) = h(s, \cdot)$, which is a curve in G defined on $[t_0, t_1]$. Notice that this notation is consistent at $s = 0$ by the above

definition. Thus the variation h can be alternatively specified by a smoothly varying family of curves $\{h_s\}_{s\in(-\epsilon,\epsilon)}$. The condition that h is a proper variation is equivalent to that all curves in this family have the same starting and ending points respectively.

For each $(s,t) \in (-\epsilon,\epsilon) \times [t_0,t_1]$, denote $\dot{h}(s,t) \triangleq \frac{\partial h}{\partial t}(s,t) = dh(\frac{\partial}{\partial t}|_{(s,t)})$, $h'(s,t) \triangleq \frac{\partial h}{\partial s}(s,t) = dh(\frac{\partial}{\partial s}|_{(s,t)})$, where dot and prime indicate differentiations with respect to t and s respectively. Both $\dot{h}(s,t)$ and $h'(s,t)$ belong to the tangent space of G at $h(s,t)$. We can pull them back via left multiplication to the tangent space of G at the identity element e, i.e., the Lie algebra $\mathfrak{g} = T_eG$ of G. Thus we define

$$\xi(s,t) \triangleq h(s,t)^{-1}\dot{h}(s,t) \in \mathfrak{g}, \qquad \eta(s,t) \triangleq h(s,t)^{-1}h'(s,t) \in \mathfrak{g}.$$

Here to simplify notation we use $h(s,t)^{-1}\dot{h}(s,t)$ to denote $dm_{h(s,t)^{-1}}[\dot{h}(s,t)]$ (for any $g \in G$, $m_g : G \to G$ stands for the left multiplication by g, while $dm_g : TG \to TG$ is its tangent map). Similarly for $h(s,t)^{-1}h'(s,t)$. This kind of notational simplification will be carried out in the following without further explanation.

Define $\dot{\xi}(s,t) = \frac{\partial \xi}{\partial t}(s,t)$ and $\xi'(s,t) = \frac{\partial \xi}{\partial s}(s,t)$, both of which belong to $T_{\xi(s,t)}\mathfrak{g} \simeq \mathfrak{g}$. Similarly we can define $\dot{\eta}(s,t)$, $\eta'(s,t) \in \mathfrak{g}$. Denote by $[\cdot,\cdot]$ the Lie bracket of $\mathfrak{g}$. Then

Lemma 1 *At any* $(s,t) \in (-\epsilon,\epsilon) \times [t_0,t_1]$,

$$\xi' = \dot{\eta} + [\xi,\eta]. \tag{4}$$

See Appendix A for a proof of Lemma 1. Proofs can also be found in [3], and in the case of matrix Lie groups, in [14].

In line with our previous notation that $h_s(\cdot) = h(s,\cdot)$ for each $s \in (-\epsilon,\epsilon)$, we denote $\dot{h}_s(\cdot) = \dot{h}(s,\cdot)$ and $h'_s(\cdot) = h'(s,\cdot)$. We shall also write $\xi_s(\cdot) = \xi(s,\cdot)$, $\dot{\xi}_s(\cdot) = \dot{\xi}(s,\cdot)$, and $\xi'_s(\cdot) = \xi'(s,\cdot)$. Similarly for $\eta_s(\cdot)$, $\dot{\eta}_s(\cdot)$ and $\eta'_s(\cdot)$. So equation (4) can be written as $\xi'_s = \dot{\eta}_s + [\xi_s,\eta_s]$ for all s.

We now apply Lemma 1 to a very special case. Denote by c_e the constant map that maps every $t \in [t_0,t_1]$ to the identity e in G, i.e., $c_e(\cdot) \equiv e$. Suppose that h is a *proper* variation of $h_0 = c_e$. Then since $h_0 \equiv e$, we have $\dot{h}_0 \equiv 0$, hence $\xi_0 \equiv 0$. Since h is a proper variation, we have $h'(\cdot,t_0) = h'(\cdot,t_1) \equiv 0$, hence $\eta(\cdot,t_0) = \eta(\cdot,t_1) \equiv 0$. Define

$$\chi = \xi'_0, \tag{5}$$

which is a C^∞ map from $[t_0, t_1]$ to $\mathfrak{g}$. By Lemma 1, $\chi = \dot{\eta}_0 + [\xi_0, \eta_0] = \dot{\eta}_0$, where the second equality follows since $\xi_0 \equiv 0$. Therefore, $\int_{t_0}^{t_1} \chi = \int_{t_0}^{t_1} \dot{\eta}_0 = \eta_0(t_1) - \eta_0(t_0) = 0$. Conversely, given any C^∞ map $\chi : [t_0, t_1] \to \mathfrak{g}$ with $\int_{t_0}^{t_1} \chi = 0$, define $h(s,t) = \exp[s \int_{t_0}^{t} \chi]$, $\forall (s,t) \in (-\epsilon, \epsilon) \times [t_0, t_1]$ for some $\epsilon > 0$ small enough, where exp is the exponential map of G. One can verify that h is indeed a proper variation of c_e for which ξ_0' coincides with χ. Therefore,

Lemma 2 *The necessary and sufficient condition for a C^∞ map $\chi : [t_0, t_1] \to \mathfrak{g}$ to be realized as $\chi = \xi_0'$ where $\xi = h^{-1}\dot{h}$ for some C^∞ proper variation h of c_e is that*

$$\int_{t_0}^{t_1} \chi = 0.$$

Remark 4 *The result in Lemma 2 is also a direct consequence of Proposition 1.14.1 in [6], since χ is an element of the Lie algebra $P(\mathfrak{g})_0^{alg}$ of the Banach Lie group $P(\mathfrak{g})_0$ defined there.*

3.2 First Variation

Suppose that $\gamma = \langle \gamma_i \rangle_{i=1}^k$ is an optimal solution to the OCA (or OFS) problem that starts from $\langle a_i \rangle_{i=1}^k$ at time t_0 and ends in $\langle b_i \rangle_{i=1}^k$ at time t_1. Necessary conditions on γ can be derived in the following way. Let $h : (-\epsilon, \epsilon) \times [t_0, t_1] \to G$ be a C^∞ proper variation of the constant map c_e for some small $\epsilon > 0$. In the same notations as in Section 3.1, for each $s \in (-\epsilon, \epsilon)$, h_s is a C^∞ curve in G both starting and ending at e, hence can be used to define a k-tuple of curves $\gamma_s = \langle \gamma_{s,i} \rangle_{i=1}^k$ in M by

$$\gamma_{s,i}(\cdot) = h_s(\cdot)\gamma_i(\cdot), \quad i = 1, \ldots, k,$$

which also starts from $\langle a_i \rangle_{i=1}^k$ at time t_0 and ends in $\langle b_i \rangle_{i=1}^k$ at time t_1. Moreover, γ_s is collision-free, and has the same formation pattern as γ at any time $t \in [t_0, t_1]$, since $\Phi_{h_s(t)}$ is an isometry of M by Assumption 1. Note that $h_0 \equiv e$ implies that $\gamma_0 = \gamma$. Define

$$J(s) \triangleq J(\gamma_s), \quad \forall s \in (-\epsilon, \epsilon). \tag{6}$$

$J(s)$ is a C^∞ function since h is a C^∞ variation. A necessary condition for γ to be optimal is that $J(s)$ assumes its minimum at $s = 0$. In particular, this implies that $J'(0) = 0$ and $J''(0) \geq 0$. The implications of these two conditions will be studied in this and the next sections respectively.

We now compute $J'(0)$. For each $(s,t) \in (-\epsilon,\epsilon) \times [t_0,t_1]$, and each $i = 1,\ldots,k$, we have[1]

$$L[\dot{\gamma}_{s,i}] = L[\dot{h}_s\gamma_i + h_s\dot{\gamma}_i] = L[h_s\xi_s\gamma_i + h_s\dot{\gamma}_i] = L[h_s(\xi_s\gamma_i + \dot{\gamma}_i)] = L[\xi_s\gamma_i + \dot{\gamma}_i].$$

Here $\dot{h}_s\gamma_i$ denotes $d\Phi^{\gamma_i}(\dot{h}_s)$, and $h_s\dot{\gamma}_i$ denotes $d\Phi_{h_s}(\dot{\gamma}_i)$, which both belong to $T_{h_s\gamma_i}M$. We also use the notation that $\dot{h}_s\gamma_i = h_s\xi_s\gamma_i$, which makes sense since $(g_1g_2)q = g_1(g_2q)$, $\forall g_1,g_2 \in G$, $q \in M$. Note that the last equality follows by the G-invariance of L. The cost of γ_s is then

$$J(s) = \sum_{i=1}^{k} \lambda_i \int_{t_0}^{t_1} L[\xi_s\gamma_i + \dot{\gamma}_i]\, dt. \tag{7}$$

For any vector space E, denote by $(\cdot,\cdot) : E^* \times E \to \mathbb{R}$ the natural pairing between E and its dual E^*, i.e., $(\alpha, v) = \alpha(v)$, $\forall \alpha \in E^*$, $v \in E$. Differentiating (7) with respect to s, we have

$$J'(s) = \sum_{i=1}^{k} \lambda_i \int_{t_0}^{t_1} (\mathbb{D}L_{\xi_s\gamma_i+\dot{\gamma}_i}, \xi_s'\gamma_i)\, dt. \tag{8}$$

Here we identify the tangent space at $\xi_s\gamma_i + \dot{\gamma}_i$ of $T_{\gamma_i}M$ with $T_{\gamma_i}M$ itself, so $\xi_s'\gamma_i \in T_{\gamma_i}M$; $\mathbb{D}L_{\xi_s\gamma_i+\dot{\gamma}_i}$ is the fiberwise differential of L, or more precisely, the differential of $L|_{T_{\gamma_i}M}$, evaluated at $\xi_s\gamma_i + \dot{\gamma}_i \in T_{\gamma_i}M$, and thought of as an element of $T^*_{\gamma_i}M$. At $s = 0$, we have $\xi_0 = 0$ and $\xi_0' = \chi$. Therefore,

$$J'(0) = \sum_{i=1}^{k} \lambda_i \int_{t_0}^{t_1} (\mathbb{D}L_{\dot{\gamma}_i}, d\Phi^{\gamma_i}(\chi))\, dt = \int_{t_0}^{t_1} (\sum_{i=1}^{k} \lambda_i (d\Phi^{\gamma_i})^* \mathbb{D}L_{\dot{\gamma}_i}, \chi)\, dt, \tag{9}$$

where $(d\Phi^{\gamma_i})^* : T^*_{\gamma_i}M \to \mathfrak{g}^*$ is the dual of $d\Phi^{\gamma_i} : \mathfrak{g} \to T_{\gamma_i}M$ defined by

$$((d\Phi^{\gamma_i})^*\alpha, \zeta) = (\alpha, d\Phi^{\gamma_i}(\zeta)), \quad \forall \alpha \in T^*_{\gamma_i}M,\ \zeta \in \mathfrak{g}. \tag{10}$$

From (9) and Lemma 2, the condition that $J'(0) = 0$ for all eligible variations h is equivalent to

$$\int_{t_0}^{t_1} (\sum_{i=1}^{k} \lambda_i (d\Phi^{\gamma_i})^* \mathbb{D}L_{\dot{\gamma}_i}, \chi)\, dt = 0 \tag{11}$$

for all C^∞ map $\chi : [t_0,t_1] \to \mathfrak{g}$ such that $\int_{t_0}^{t_1} \chi = 0$. Since $\sum_{i=1}^{k} \lambda_i (d\Phi^{\gamma_i})^* \mathbb{D}L_{\dot{\gamma}_i}$ is piecewise C^∞ (though not necessarily continuous) in $\mathfrak{g}^*$, condition (11) implies that $\sum_{i=1}^{k} \lambda_i (d\Phi^{\gamma_i})^* \mathbb{D}L_{\dot{\gamma}_i}$ is constant for all $t \in [t_0,t_1]$ whenever $\dot{\gamma}_i$'s are well-defined, for otherwise one can always choose a χ with $\int_{t_0}^{t_1} \chi = 0$ such that (11) fails to hold. Therefore,

[1] Since γ_i is only piecewise C^∞, this and all equations that follow should be understood to hold only at those t where $\dot{\gamma}_i$'s are well defined. In addition, the parameter t is implicit in these equations for brevity.

Theorem 1 (Noether) *Suppose $\gamma = \langle \gamma_i \rangle_{i=1}^k$ is an optimal solution to the OCA (or OFS) problem. Then there exists a constant $\nu_0 \in \mathfrak{g}^*$ such that*

$$\nu \triangleq \sum_{i=1}^{k} \lambda_i (d\Phi^{\gamma_i})^* \mathbb{D}L_{\dot{\gamma}_i} \equiv \nu_0 \tag{12}$$

for all $t \in [t_0, t_1]$ where $\dot{\gamma}_i$'s are well defined.

Remark 5 *The action of G on M induces an action of G on $M^{(k)}$ naturally, which can be cotangent lifted to an action of G on $T^*(M^{(k)})$. The ν defined in (12) is in fact the momentum map for this last action evaluated along the curve $(\gamma_1, \ldots, \gamma_k, \lambda_1 \mathbb{D}L_{\dot{\gamma}_1}, \ldots, \lambda_k \mathbb{D}L_{\dot{\gamma}_k})$ in $T^*(M^{(k)})$. Note also that although for Theorem 1 to hold it is not necessary that the action of G on M be transitive, usually the most descriptive results are obtained when this is the case (as in all our examples).*

If $L = \frac{1}{2}\|\cdot\|^2$, then the conclusion of Theorem 1 can be simplified by canonically identifying each $v \in T_{\gamma_i}M$ with the element in $T^*_{\gamma_i}M$ defined by $u \mapsto \langle v, u \rangle$, $\forall u \in T_{\gamma_i}M$. Thus $\mathbb{D}L_{\dot{\gamma}_i}$ is identified with $\dot{\gamma}_i$, and (12) becomes

$$\sum_{i=1}^{k} \lambda_i (d\Phi^{\gamma_i})^* \dot{\gamma}_i \equiv \nu_0 \in \mathfrak{g}^*, \tag{13}$$

where $(d\Phi^{\gamma_i})^* : T_{\gamma_i}M \to \mathfrak{g}^*$ is now defined by

$$((d\Phi^{\gamma_i})^* v, \zeta) = \langle v, d\Phi^{\gamma_i}(\zeta) \rangle, \quad \forall v \in T_{\gamma_i}M,\ \zeta \in \mathfrak{g}. \tag{14}$$

Furthermore, there is occasionally a natural choice for a metric on $\mathfrak{g}$, which can be used to identify $\mathfrak{g}$ and $\mathfrak{g}^*$. In this case, the conserved quantity ν can be thought of taking values in $\mathfrak{g}$.

In Example 1, it can be shown that the conserved quantities are the total linear momentum and the total (generalized) angular momentum of a k-particle system moving on $\mathbb{R}^n$. In Example 2, (12) becomes the conservation of total angular momentum of a k-particle system moving on $\mathbf{S}^{n-1}$. In either case, the k particles have masses $\lambda_1, \ldots, \lambda_k$, respectively. We show this only for Example 2 in the following.

Example 4 ($G = \mathbf{SO}_n$, $M = \mathbf{S}^{n-1}$) Let $M = \mathbf{S}^{n-1} = \{(x_1, \ldots, x_n)^t \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\}$, $G = \mathbf{SO}_n = \{A \in \mathbb{R}^{n\times n} : A^t A = I_n, \det A = 1\}$, and the action Φ be left matrix multiplication. For each $q \in \mathbf{S}^{n-1}$, the tangent space $T_q\mathbf{S}^{n-1} = \{v \in \mathbb{R}^n : v^t q = 0\}$ is equipped with the standard metric. Then $\mathbf{SO}_n$ acts on $\mathbf{S}^{(n-1)}$ by isometry. The Lie algebra of $\mathbf{SO}_n$ is $\mathfrak{so}_n = \{X \in \mathbb{R}^{n\times n} : X + X^t = 0\}$. Choose $L = \frac{1}{2}\|\cdot\|^2$. Suppose that a k-tuple of curves in $\mathbf{S}^{n-1}$, $\gamma = \langle \gamma_i \rangle_{i=1}^k$, is an optimal solution

to the OCA (or OFS) problem defined on $[t_0, t_1]$. At each time $t \in [t_0, t_1]$, let $v \in T_{\gamma_i}\mathbf{S}^{n-1}$ and $X \in \mathfrak{so}_n$ be arbitrary. Then

$$\langle v, d\Phi^{\gamma_i}(X)\rangle = \langle v, X\gamma_i\rangle = v^t X\gamma_i = \mathrm{tr}(\gamma_i v^t X) = \langle v\gamma_i^t, X\rangle_F = \frac{1}{2}\langle v\gamma_i^t - \gamma_i v^t, X\rangle_F,$$

where $\langle\cdot,\cdot\rangle_F$ is the Frobenius inner product on $\mathbb{R}^{n\times n}$ defined by $\langle Y, Z\rangle_F = \mathrm{tr}(Y^t Z)$ for $Y, Z \in \mathbb{R}^{n\times n}$. The last equality follows since X is skew-symmetric. So by (14),

$$((d\Phi^{\gamma_i})^* v, X) = \frac{1}{2}\langle v\gamma_i^t - \gamma_i v^t, X\rangle_F, \quad \forall X \in \mathfrak{so}_n.$$

Note that $v\gamma_i^t - \gamma_i v^t \in \mathfrak{so}_n$. So if $\mathfrak{so}_n$ is identified with $\mathfrak{so}_n^*$ using the metric $\frac{1}{2}\langle\cdot,\cdot\rangle_F$, then the above equation implies that $(d\Phi^{\gamma_i})^* v = v\gamma_i^t - \gamma_i v^t$. Hence (13) becomes

$$\sum_{i=1}^{k} \lambda_i(\dot\gamma_i\gamma_i^t - \gamma_i\dot\gamma_i^t) \equiv \nu_0 \in \mathfrak{so}_n. \tag{15}$$

Or equivalently, $\sum_{i=1}^k \lambda_i(\dot\gamma_i \wedge \gamma_i)$ is constant. In particular, if $n = 2$ ($G = \mathbf{SO}_3$, $M = \mathbf{S}^2$), then equation (15) can be written as $\sum_{i=1}^k \lambda_i(\gamma_i \times \dot\gamma_i) \equiv \Omega_0$ for some $\Omega_0 \in \mathbb{R}^3$, where $\times$ is the vector product. This is exactly the conservation of total angular momentum.

Next we study a special case of Example 3 for which the conclusion of Theorem 1 takes an especially simple form.

Example 5 (Lie Group with a Bi-Invariant Metric) Let G be a Lie group with a bi-invariant Riemannian metric. Such G include all compact Lie groups (see [5]). Let $M = G$, and let the action Φ be the left group multiplication. Choose $L = \frac{1}{2}\|\cdot\|^2$. Suppose that $\gamma = \langle\gamma_i\rangle_{i=1}^k$ is a solution to the OCA (or OFS) problem. Then at each time t, $\forall v \in T_{\gamma_i}G$, $\zeta \in \mathfrak{g}$,

$$\langle v, d\Phi^{\gamma_i}(\zeta)\rangle = \langle v, \zeta\gamma_i\rangle = \langle v\gamma_i^{-1}\gamma_i, \zeta\gamma_i\rangle = \langle v\gamma_i^{-1}, \zeta\rangle \quad \Rightarrow \quad ((d\Phi^{\gamma_i})^* v, \zeta) = \langle v\gamma_i^{-1}, \zeta\rangle. \tag{16}$$

Under the canonical identification of $\mathfrak{g}$ with $\mathfrak{g}^*$ via $\langle\cdot,\cdot\rangle$, the right hand side is equivalent to $(d\Phi^{\gamma_i})^* v = v\gamma_i^{-1} \in \mathfrak{g}$. Therefore the conservation law (13) is

$$\sum_{i=1}^{k} \lambda_i\dot\gamma_i\gamma_i^{-1} \equiv \nu_0 \in \mathfrak{g}. \tag{17}$$

In the simplest case, if G is a Euclidean space $\mathbb{R}^n$ (a flat n-torus $\mathbf{T}^n$) with the canonical metric and with addition (modulo $\mathbb{Z}^n$) as the group operation, then (17) implies the conservation of total linear momentum since $\dot\gamma_i\gamma_i^{-1} = \dot\gamma_i$. For another example, consider $G = \mathbf{SO}_n$. A left invariant Riemannian metric on $\mathbf{SO}_n$ can be established by first specifying its restriction on $\mathfrak{so}_n$ to be $\frac{1}{2}\langle\cdot,\cdot\rangle_F$, and then

extending it to all other fibers by left translation. It is easy to see that the metric thus defined is also right invariant, hence bi-invariant. Let $L = \frac{1}{2}\|\cdot\|^2$. Then (17) holds for solutions $\gamma = \langle\gamma_i\rangle_{i=1}^k$ to the OCA and OFS problems on $\mathbf{SO}_n$.

A large class of examples can be constructed from the last one by considering the quotient spaces of G under certain subgroups H, i.e., the symmetric spaces G/H.

Example 6 (Grassmann Manifold) Let $\mathbf{SO}_n$ be equipped with the bi-invariant Riemannian metric described in Example 5. Let p be an integer, $1 \le p \le n$. Denote by H_p the subgroup $\begin{bmatrix} \mathbf{SO}_p & 0 \\ 0 & \mathbf{SO}_{n-p} \end{bmatrix} \simeq \mathbf{SO}_p \times \mathbf{SO}_{n-p}$ of $\mathbf{SO}_n$. Define $G_{n,p} \triangleq \mathbf{SO}_n/H_p$ to be the set of left cosets of H_p in $\mathbf{SO}_n$, and let $\pi : \mathbf{SO}_n \to G_{n,p}$ be the natural projection. Elements of $G_{n,p}$ are $\pi(A) = AH_p$, $\forall A \in \mathbf{SO}_n$. For each $\pi(A) \in G_{n,p}$, the subspace of $\mathbb{R}^n$ spanned by the first p column vectors of A is the same for all $A \in \pi(A)$, hence there is a one-to-one correspondence between $G_{n,p}$ and set of p-dimensional subspaces of $\mathbb{R}^n$. Since H_p is a closed subgroup of $\mathbf{SO}_n$, $G_{n,p}$ admits a natural differential structure, and is called a *Grassmann manifold.* At each $A \in \mathbf{SO}_n$, the tangent space of $\mathbf{SO}_n$ has the orthogonal decomposition [7]:

$$T_A\mathbf{SO}_n = \mathrm{vert}_A\mathbf{SO}_n \oplus \mathrm{hor}_A\mathbf{SO}_n.$$

The *vertical space* $\mathrm{vert}_A\mathbf{SO}_n = A \begin{bmatrix} \mathfrak{so}_p & 0 \\ 0 & \mathfrak{so}_{n-p} \end{bmatrix}$ is the tangent space of AH_p at A; the *horizontal space* $\mathrm{hor}_A\mathbf{SO}_n$ consists of all those matrices of the form $A \begin{bmatrix} 0 & -X^t \\ X & 0 \end{bmatrix}$ for some $X \in \mathbb{R}^{(n-p)\times p}$. Note that $d\pi : \mathrm{hor}_A\mathbf{SO}_n \to T_{\pi(A)}G_{n,p}$ is a vector space isomorphism. The restriction of the metric on $T_A\mathbf{SO}_n$ defines a metric on $\mathrm{hor}_A\mathbf{SO}_n$ as

$$\langle A \begin{bmatrix} 0 & -X_1^t \\ X_1 & 0 \end{bmatrix}, A \begin{bmatrix} 0 & -X_2^t \\ X_2 & 0 \end{bmatrix} \rangle = \langle X_1, X_2 \rangle_F, \quad \forall X_1, X_2 \in \mathbb{R}^{(n-p)\times p}. \tag{18}$$

An important observation is that there is a unique Riemannian metric on $G_{n,p}$ that makes $d\pi : \mathrm{hor}_A\mathbf{SO}_n \to T_{\pi(A)}G_{n,p}$ an isometry for each $\pi(A) \in G_{n,p}$, regardless of the choice of $A \in \pi(A)$. Such a metric exists because the metric on $\mathbf{SO}_n$ is right invariant. Moreover, this metric on $G_{n,p}$ is invariant under the induced left action of $\mathbf{SO}_n$ on $G_{n,p}$ since the metric on $\mathbf{SO}_n$ is left invariant. In the terminology of [17], $\pi : \mathbf{SO}_n \to G_{n,p}$ is a *Riemannian submersion*, and when viewed as a principal H_p-bundle over $G_{n,p}$, $\mathbf{SO}_n$ has a metric of *constant bi-invariant type.* See [17] for more details on (sub-riemannian) metrics of principal bundles.

Suppose $L = \frac{1}{2}\|\cdot\|^2$. Let $\gamma = \langle\gamma_i\rangle_{i=1}^k$ be a k-tuple of curves in $G_{n,p}$ that is a solution to the OCA (or OFS) problem. For each $i = 1, \ldots, k$, let A_i be a lifting of γ_i in $\mathbf{SO}_n$ in the sense that $\pi(A_i) = \gamma_i$. In other words, the first p column vectors of $A_i \in \mathbf{SO}_n$ span the subspace $\gamma_i \in G_{n,p}$. We can choose A_i to be continuous and piecewise C^∞. At each time t, choose arbitrary $X \in \mathfrak{so}_n$ and $v \in T_{\gamma_i}G_{n,p}$, and let V be the unique element of $\mathrm{hor}_{A_i}\mathbf{SO}_n$ such that $d\pi(V) = v$. Then, using the fact that $d\pi : \mathrm{hor}_{A_i}\mathbf{SO}_n \to T_{\gamma_i}G_{n,p}$ is an isometry, we have

$$\langle v, d\Phi^{\gamma_i}(X)\rangle_{T_{\gamma_i}G_{n,p}} = \langle V, P_{A_i}(XA_i)\rangle_{\mathrm{hor}_{A_i}\mathbf{SO}_n} = \langle V, XA_i\rangle_{T_{A_i}\mathbf{SO}_n} = \langle VA_i^t, X\rangle_{\mathfrak{so}_n},$$

where P_{A_i} is the orthogonal projection $T_{A_i}\mathbf{SO}_n \to \mathrm{hor}_{A_i}\mathbf{SO}_n$. Here for clarity we indicate in subscript the associated tangent space of each inner product. Therefore, $(d\Phi^{\gamma_i})^* v = VA_i^t \in \mathfrak{so}_n \simeq \mathfrak{so}_n^*$. Finally, notice that $\dot\gamma_i = d\pi[P_{A_i}(\dot A_i)]$. So (13) becomes

$$\sum_{i=1}^k \lambda_i (d\Phi^{\gamma_i})^* \dot\gamma_i = \sum_{i=1}^k \lambda_i P_{A_i}(\dot A_i) A_i^t \equiv \nu_0 \in \mathfrak{so}_n. \tag{19}$$

Example 7 (Stiefel Manifold) Denote by K_p the subgroup $\begin{bmatrix} I_p & 0 \\ 0 & \mathbf{SO}_{n-p} \end{bmatrix} \simeq \mathbf{SO}_{n-p}$ of $\mathbf{SO}_n$. Then the quotient space $V_{n,p} \triangleq \mathbf{SO}_n/K_p$ is called a *Stiefel manifold*. Elements in $V_{n,p}$ correspond in a one-to-one way to the orthonormal p-frames of $\mathbb{R}^n$. At each $A \in \mathbf{SO}_n$, the horizontal space is now $A\begin{bmatrix} I_p & 0 \\ 0 & \mathfrak{so}_{n-p} \end{bmatrix}$, while the vertical space consists of matrices of the form $A\begin{bmatrix} Y & -X^t \\ X & 0 \end{bmatrix}$ for $X \in \mathbb{R}^{(n-p)\times p}$, $Y \in \mathbb{R}^{p\times p}$, $Y + Y^t = 0$. The metric on $\mathbf{SO}_n$ restricts to a metric on the horizontal space as

$$\langle A\begin{bmatrix} Y_1 & -X_1^t \\ X_1 & 0 \end{bmatrix}, A\begin{bmatrix} Y_2 & -X_2^t \\ X_2 & 0 \end{bmatrix}\rangle = \langle X_1, X_2\rangle_F + \frac{1}{2}\langle Y_1, Y_2\rangle_F,$$

which can be used to define a Riemannian metric on $V_{n,p}$ such that $d\pi$ is an isometry from the horizontal space at each $A \in \mathbf{SO}_n$ to $T_{\pi(A)}V_{n,p}$ (π is now the natural projection from $\mathbf{SO}_n$ to $V_{n,p}$). The metric thus defined is invariant under the induced left action of $\mathbf{SO}_n$ on $V_{n,p}$. Suppose $L = \frac{1}{2}\|\cdot\|^2$. By similar arguments as in Example 6, we can show that if $\gamma = \langle\gamma_i\rangle_{i=1}^k$ is a solution to the OCA (or OFS) problem on $V_{n,p}$, and if A_i is a lifting of γ_i in $\mathbf{SO}_n$, $i = 1, \ldots, k$, then

$$\sum_{i=1}^k \lambda_i \hat P_{A_i}(\dot A_i) A_i^t \equiv \nu_0 \in \mathfrak{so}_n. \tag{20}$$

Here $\hat P_{A_i}$ is the orthogonal projection from $T_{A_i}\mathbf{SO}_n$ to the horizontal space of $\mathbf{SO}_n$ at A_i.

3.3 Second Variation

Further optimality conditions can be obtained through second variation. Suppose that $\gamma = \langle \gamma_i \rangle_{i=1}^k$ is an optimal solution to the OCA (or OFS) problem defined on $[t_0, t_1]$. Let $J(s)$, $s \in (-\epsilon, \epsilon)$, be defined as in (7). Differentiating equation (8) with respect to s at $s = 0$, we have

$$\begin{aligned} J''(0) &= \sum_{i=1}^k \lambda_i \int_{t_0}^{t_1} [(\mathbb{D}L_{\dot\gamma_i}, \xi_0''\gamma_i) + \mathbb{D}^2 L_{\dot\gamma_i}(\xi_0'\gamma_i, \xi_0'\gamma_i)]\, dt \\ &= \int_{t_0}^{t_1} (\sum_{i=1}^k \lambda_i (d\Phi^{\gamma_i})^* \mathbb{D}L_{\dot\gamma_i}, \xi_0'')\, dt + \int_{t_0}^{t_1} \sum_{i=1}^k \lambda_i \mathbb{D}^2 L_{\dot\gamma_i}(\xi_0'\gamma_i, \xi_0'\gamma_i)\, dt. \end{aligned} \tag{21}$$

Here $\xi_0'' \in \mathfrak{g}$, and $\mathbb{D}^2 L_{\dot\gamma_i} : T_{\gamma_i}M \times T_{\gamma_i}M \to \mathbb{R}$ is the fiberwise second order derivative (Hessian) of L on $T_{\gamma_i}M$ evaluated at $\dot\gamma_i$. By Theorem 1, the first term in (21) can be written as

$$\begin{aligned} \int_{t_0}^{t_1} (\nu_0, \xi_0'')\, dt &= \frac{d}{ds}\Big|_{s=0} (\nu_0, \int_{t_0}^{t_1} \xi_s'\, dt) \qquad = \frac{d}{ds}\Big|_{s=0} (\nu_0, \int_{t_0}^{t_1} (\dot\eta_s + [\xi_s, \eta_s])\, dt) \\ &= \frac{d}{ds}\Big|_{s=0} (\nu_0, \int_{t_0}^{t_1} [\xi_s, \eta_s]\, dt) = (\nu_0, \int_{t_0}^{t_1} ([\xi_0', \eta_0] + [\xi_0, \eta_0'])\, dt) \\ &= (\nu_0, \int_{t_0}^{t_1} [\dot\eta_0, \eta_0]\, dt), \end{aligned} \tag{22}$$

where we have used Lemma 1 and the following facts: $\eta_s(t_0) = \eta_s(t_1) = 0$; $\xi_0 \equiv 0$; and $\xi_0' = \chi = \dot\eta_0$ by the remark immediately following equation (5). For the second term in (21), define

$$\mathbb{I}_t(\zeta_1, \zeta_2) \triangleq \sum_{i=1}^k \lambda_i \mathbb{D}^2 L_{\dot\gamma_i}(\zeta_1\gamma_i, \zeta_2\gamma_i), \quad \forall \zeta_1, \zeta_2 \in \mathfrak{g}, \tag{23}$$

for each $t \in [t_0, t_1]$. Then $\mathbb{I}_t(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is a nonnegative definite quadratic form on $\mathfrak{g}$, since $\lambda_i > 0$ and L is convex on each fiber of TM by assumption. If in particular $L = \frac{1}{2}\|\cdot\|^2$, then $\mathbb{D}^2 L_v(\cdot,\cdot) = \langle \cdot, \cdot \rangle_q$ for any $v \in T_qM$, $q \in M$. Hence $\mathbb{I}_t$ in this case is defined by

$$\mathbb{I}_t(\zeta_1, \zeta_2) \triangleq \sum_{i=1}^k \lambda_i \langle \zeta_1\gamma_i, \zeta_2\gamma_i \rangle, \quad \forall \zeta_1, \zeta_2 \in \mathfrak{g}. \tag{24}$$

In mechanics, $\mathbb{I}_t$ defined in (24) is called the *moment of inertia tensor* ([15, 17]) for the action of G on $M^{(k)}$ (with the metric $\prod_{i=1}^k \lambda_i \langle \cdot, \cdot \rangle$) evaluated at $(\gamma_1, \ldots, \gamma_k)$. By substituting (22) and (23) into (21), we obtain

$$J''(0) = (\nu_0, \int_{t_0}^{t_1} [\dot\eta_0, \eta_0]\, dt) + \int_{t_0}^{t_1} \mathbb{I}_t(\dot\eta_0, \dot\eta_0)\, dt.$$

In order for γ to be optimal, we must have $J''(0) \geq 0$ for all feasible η_0. Therefore,

Theorem 2 *Suppose that $\langle\gamma_i\rangle_{i=1}^k$ is an optimal solution to the OCA (or OFS) problem. Let $\nu_0 \in \mathfrak{g}^*$ be defined as in Theorem 1. Then for any C^∞ curve $\eta_0 : [t_0, t_1] \to \mathfrak{g}$ such that $\eta_0(t_0) = \eta_0(t_1) = 0$,*

$$(\nu_0, \int_{t_0}^{t_1} [\dot\eta_0, \eta_0]\, dt) + \int_{t_0}^{t_1} \mathbb{I}_t(\dot\eta_0, \dot\eta_0)\, dt \geq 0. \tag{25}$$

Remark 6 *Consider the OCA problem with $L = \frac{1}{2}\|\cdot\|^2$. If $k = 1$, then solutions $\gamma = \gamma_1$ are geodesics of M. If the action Φ is transitive, then any local proper variation of γ in M can be generated as $h\gamma$ by some proper variation h of c_e in G. So in this case Theorem 2 characterizes the first* conjugate point *along γ. See Example 8 below. If $k > 1$, then solutions γ are geodesics in $M^{(k)} \setminus W$, a manifold with boundary whose dimension is usually much larger than that of G. The variations of γ in the form of $h\gamma$ can only perturb the k components of γ uniformly by multiplying from the left the* same *elements of G. Hence the condition in Theorem 2 is in general only necessary for the local optimality of γ. See Section 4 for an example.*

It is often difficult to apply Theorem 2 directly. In the following we shall derive some of its implications that are easier to check. Note that if $\dim(\mathfrak{g}) = 1$ (or if $\mathfrak{g}$ is abelian), condition (25) holds trivially. So we shall assume that $\dim(\mathfrak{g}) > 1$.

Choose an arbitrary inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$, whose corresponding norm is denoted by $\|\cdot\|_{\mathfrak{g}}$. In many cases there is a natural choice for $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$. At each time t, define the *spectral radius* of $\mathbb{I}_t$ as

$$\rho(\mathbb{I}_t) = \inf\{\lambda \in \mathbb{R} : \lambda\langle\cdot,\cdot\rangle_{\mathfrak{g}} - \mathbb{I}_t(\cdot,\cdot) \text{ is nonnegative definite on } \mathfrak{g}\}.$$

Then $\rho(\mathbb{I}_t) \geq 0$ is the largest eigenvalue of the symmetric matrix representing $\mathbb{I}_t$ in any orthonormal basis of $\mathfrak{g}$. For any subspace $\mathfrak{h}$ of $\mathfrak{g}$, the restriction $\mathbb{I}_t|_{\mathfrak{h}}$ is still nonnegative definite. Define

$$\rho(\mathbb{I}_t; \mathfrak{h}) = \rho(\mathbb{I}_t|_{\mathfrak{h}}) = \inf\{\lambda \in \mathbb{R} : \lambda\langle\cdot,\cdot\rangle_{\mathfrak{g}} - \mathbb{I}_t(\cdot,\cdot) \text{ is nonnegative definite on } \mathfrak{h}\}. \tag{26}$$

An immediate result of definition (26) is

$$\mathbb{I}_t(\zeta_1, \zeta_2) \leq \rho(\mathbb{I}_t; \mathfrak{h})\ \langle\zeta_1, \zeta_2\rangle_{\mathfrak{g}}, \quad \forall \zeta_1, \zeta_2 \in \mathfrak{h}.$$

Pick a two dimensional subspace $\mathfrak{h}$ of $\mathfrak{g}$, and let $\{\zeta_1, \zeta_2\}$ be an orthonormal basis of $\mathfrak{h}$. Denote

$$\zeta_0 = [\zeta_1, \zeta_2]. \tag{27}$$

Now consider condition (25) in the special case when η_0 as a curve in $\mathfrak{g}$ is contained entirely in $\mathfrak{h}$. So there exist C^∞ functions $x_1, x_2 : [t_0, t_1] \to \mathbb{R}$ such that $\eta_0 = x_1\zeta_1 + x_2\zeta_2$. The constraints that

$\eta_0(t_0) = \eta_0(t_1) = 0$ imply that $x_1(t_0) = x_1(t_1) = 0$ and $x_2(t_0) = x_2(t_1) = 0$. Moreover,

$$[\dot{\eta}_0, \eta_0] = [\dot{x}_1\zeta_1 + \dot{x}_2\zeta_2, x_1\zeta_1 + x_2\zeta_2] = (\dot{x}_1 x_2 - x_1\dot{x}_2)\zeta_0.$$

Therefore, on the left hand side of inequality (25), the first term becomes

$$(\nu_0, \int_{t_0}^{t_1} [\dot{\eta}_0, \eta_0]\, dt) = \nu_0(\zeta_0) \int_{t_0}^{t_1} (\dot{x}_1 x_2 - x_1\dot{x}_2)\, dt = -2\nu_0(\zeta_0) S_{\eta_0},$$

where S_{η_0} is the (oriented) planar area encircled by η_0 in $\mathfrak{h}$. The second term is dominated by

$$\int_{t_0}^{t_1} \mathbb{I}_t(\dot{\eta}_0, \dot{\eta}_0)\, dt \leq \int_{t_0}^{t_1} \rho(\mathbb{I}_t; \mathfrak{h}) \|\dot{\eta}_0\|_{\mathfrak{g}}^2\, dt \leq \sup_{t_0 \leq t \leq t_1} \rho(\mathbb{I}_t; \mathfrak{h}) \int_{t_0}^{t_1} \|\dot{\eta}_0\|_{\mathfrak{g}}^2\, dt = 2E_{\eta_0} \sup_{t_0 \leq t \leq t_1} \rho(\mathbb{I}_t; \mathfrak{h}),$$

where $E_{\eta_0} = \frac{1}{2}\int_{t_0}^{t_1} \|\dot{\eta}_0\|_{\mathfrak{g}}^2\, dt$ is the energy of the curve η_0. As a result, (25) implies

$$\nu_0(\zeta_0) S_{\eta_0} \leq E_{\eta_0} \sup_{t_0 \leq t \leq t_1} \rho(\mathbb{I}_t; \mathfrak{h}). \tag{28}$$

By reversing the parameterization of η_0 in (28), the sign of the left hand side is flipped, while the right hand side remains unchanged. So

$$|\nu_0(\zeta_0)|\, |S_{\eta_0}| \leq E_{\eta_0} \sup_{t_0 \leq t \leq t_1} \rho(\mathbb{I}_t; \mathfrak{h}). \tag{29}$$

Since (29) holds for all η_0, and $\sup_{t_0 \leq t \leq t_1} \rho(\mathbb{I}_t; \mathfrak{h})$ is independent of the choice of η_0, we have

$$|\nu_0(\zeta_0)| \leq \sup_{t_0 \leq t \leq t_1} \rho(\mathbb{I}_t; \mathfrak{h}) \inf_{\eta_0} \frac{E_{\eta_0}}{|S_{\eta_0}|}, \tag{30}$$

where the infimum is taken over all closed curves η_0 in $\mathfrak{h}$ with $\eta_0(t_0) = \eta_0(t_1) = 0$ and $S_{\eta_0} \neq 0$.

Denote by $L_{\eta_0} = \int_{t_0}^{t_1} \|\dot{\eta}_0\|_{\mathfrak{g}}\, dt$ the arc length of η_0. It is well known [16] that

$$E_{\eta_0} \geq \frac{L_{\eta_0}^2}{2(t_1 - t_0)},$$

with equality if and only if η_0 has constant speed. Since $|S_{\eta_0}|$ is independent of the parameterizations of η_0, we can always choose η_0 with constant speed, so (30) is equivalent to

$$|\nu_0(\zeta_0)| \leq \frac{1}{2(t_1 - t_0)} \sup_{t_0 \leq t \leq t_1} \rho(\mathbb{I}_t; \mathfrak{h}) \inf_{\eta_0} \frac{L_{\eta_0}^2}{|S_{\eta_0}|}. \tag{31}$$

The following theorem is a famous result whose proof can be found in, for example, [8].

Theorem 3 (Isoperimetric Problem) *Using a string of fixed length, one can encircle the maximal area by arranging the string into a circle. Or equivalently, among all the closed curves that enclose a fixed area, the one with the shortest length is a circle.*

From this theorem, it is easy to see that $L^2_{\eta_0}/|S_{\eta_0}|$ achieves its infimum when η_0 draws a circle in $\mathfrak{h}$ of arbitrary radius through the origin, and the infimum is 4π. Hence (31) can be written as

$$|\nu_0(\zeta_0)| \leq \frac{2\pi}{t_1 - t_0} \sup_{t_0 \leq t \leq t_1} \rho(\mathbb{I}_t; \mathfrak{h}).$$

Summing up, we have

Corollary 1 *Suppose $\gamma = \langle \gamma_i \rangle_{i=1}^k$ is an optimal solution to the OCA (or OFS) problem, and ν_0 is defined as in Theorem 1. Let $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ be an arbitrary inner product on $\mathfrak{g}$. Then*

$$|\nu_0([\zeta_1, \zeta_2])| \leq \frac{2\pi}{t_1 - t_0} \sup_{t_0 \leq t \leq t_1} \rho(\mathbb{I}_t; \mathfrak{h}), \tag{32}$$

for any orthonormal pair $\zeta_1, \zeta_2 \in \mathfrak{g}$. Here $\mathfrak{h} = span\{\zeta_1, \zeta_2\}$, and $\rho(\mathbb{I}_t; \mathfrak{h})$ is defined in (26).

Remark 7 *The choice of the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ affects both the choices of ζ_1, ζ_2 and the values of $\rho(\mathbb{I}_t, \mathfrak{h})$, so in this sense the conclusion of Corollary 1 is not intrinsic.*

In certain cases, Corollary 1 takes an especially simple form. Suppose $L = \frac{1}{2}\|\cdot\|^2$ and that the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ is chosen such that

$$\langle \zeta_1 q, \zeta_2 q \rangle = \langle \zeta_1, \zeta_2 \rangle_{\mathfrak{g}}, \quad \forall \zeta_1, \zeta_2 \in \mathfrak{g}, q \in M. \tag{33}$$

Then at each t, $\mathbb{I}_t(\zeta_1, \zeta_2) = \sum_{i=1}^k \lambda_i \langle \zeta_1 \gamma_i, \zeta_2 \gamma_i \rangle = (\sum_{i=1}^k \lambda_i) \langle \zeta_1, \zeta_2 \rangle_{\mathfrak{g}}$, $\forall \zeta_1, \zeta_2 \in \mathfrak{g}$, which implies that

$$\rho(\mathbb{I}_t; \mathfrak{h}) = \rho(\mathbb{I}_t) = \sum_{i=1}^k \lambda_i$$

for any t and any two dimensional subspace $\mathfrak{h}$ of $\mathfrak{g}$. Therefore,

Corollary 2 *If in addition to the hypotheses of Corollary 1, we have $L = \frac{1}{2}\|\cdot\|^2$, and $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ satisfying condition (33), then for any orthonormal pair $\zeta_1, \zeta_2 \in \mathfrak{g}$,*

$$|\nu_0([\zeta_1, \zeta_2])| \leq \frac{2\pi \sum_{i=1}^k \lambda_i}{t_1 - t_0}. \tag{34}$$

Example 8 ($G = \mathbf{SO}_3$, $M = \mathbf{S}^2$) Consider Example 4 with $n = 3$. So $M = \mathbf{S}^2$ with the standard metric, $G = \mathbf{SO}_3$, and the action Φ is the matrix left multiplication. As in Example 4, an inner product on $\mathfrak{so}_3$ is given by $\langle \cdot, \cdot \rangle_{\mathfrak{so}_3} = \frac{1}{2}\langle \cdot, \cdot \rangle_F$. Then an orthonormal basis of $\mathfrak{so}_3$ can be chosen as:

$$W_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad W_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that $[W_1, W_2] = W_3$, $[W_2, W_3] = W_1$, $[W_3, W_1] = W_2$. So the linear map defined by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mapsto x_1 W_1 + x_2 W_2 + x_3 W_3 = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \in \mathfrak{so}_3$$

is both an isometry and a Lie algebra isomorphism between $\mathfrak{so}_3$ and $(\mathbb{R}^3, \times)$.

Suppose $L = \frac{1}{2}\|\cdot\|^2$, $k = 1$, and $\lambda_1 = 1$. Then solutions to the OCA problem are geodesics (i.e., great circles) on $\mathbf{S}^2$. Consider as an example $\gamma(t) = (\cos t, \sin t, 0)^t \in \mathbf{S}^2$, $\forall t \in [t_0, t_1]$, where $t_0 = -\frac{T}{2}$, $t_1 = \frac{T}{2}$ for some $T \in [0, 2\pi)$. One can check that in (15), $\dot{\gamma}\gamma^t - \gamma\dot{\gamma}^t \equiv \nu_0 = W_3$. Define a curve η_0 in $\mathfrak{so}_3$ by $\eta_0 = x_1 W_1 + x_2 W_2$, where $x_1(t) = \sin t(\cos t - \cos\frac{T}{2})$, $x_2(t) = -\cos t(\cos t - \cos\frac{T}{2})$, for $t \in [-\frac{T}{2}, \frac{T}{2}]$. η_0 satisfies $\eta_0(-\frac{T}{2}) = \eta_0(\frac{T}{2}) = 0$. So by Theorem 2, we must have

$$\langle \nu_0, \int_{-\frac{T}{2}}^{\frac{T}{2}} [\dot{\eta}_0, \eta_0]\, dt \rangle_{\mathfrak{so}_3} + \int_{-\frac{T}{2}}^{\frac{T}{2}} \mathbb{I}_t(\dot{\eta}_0, \dot{\eta}_0)\, dt = T(1 + \cos T)/2 - \sin T \geq 0$$

in order for γ to be optimal (in this case, distance-minimizing). Simple calculus shows that this is equivalent to $T \leq \pi$. In other words, γ fails to be distance-minimizing once extended beyond two antipodal points. Note that the mysterious looking η_0 defined above is in fact conceived for a proper variation h of c_e in $\mathbf{SO}_3$ such that for small s, $h_s\gamma$ is the intersection of $\mathbf{S}^2$ with a plane that passes through $\gamma(-\frac{T}{2})$ and $\gamma(\frac{T}{2})$ and makes an angle s with the xy plane.

On the other hand, consider the 2-plane $\mathfrak{h}$ in $\mathfrak{so}_3$ spanned by W_1 and W_2. Since at each t,

$$\mathbb{I}_t(x_1 W_1 + x_2 W_2, x_1 W_1 + x_2 W_2) = (x_1 \sin t - x_2 \cos t)^2 \leq x_1^2 + x_2^2, \quad \forall x_1, x_2 \in \mathbb{R},$$

with equality if and only if $x_1 = \lambda \sin t$, $x_2 = -\lambda \cos t$ for some $\lambda \in \mathbb{R}$, we have $\rho(\mathbb{I}_t; \mathfrak{h}) = 1$. Note that $\nu_0([W_1, W_2]) = \nu_0(W_3) = \langle W_3, W_3 \rangle_{\mathfrak{so}_3} = 1$. So Corollary 1 implies that $T \leq 2\pi$, which is of course only a necessary condition for the optimality of γ.

Example 9 (Lie Group with a Bi-Invariant Metric) Let G be a Lie group with a bi-invariant metric $\langle \cdot, \cdot \rangle$, and $L = \frac{1}{2}\|\cdot\|^2$. Choose the metric $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ to be the restriction of $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$, which obviously satisfies condition (33). Hence Corollary 2 holds, where $\nu_0 \in \mathfrak{g}$ is given by (17). In particular, let $G = \mathbf{SO}_3$ be equipped with the bi-invariant metric described in Example 5. Then since the set of $[X_1, X_2]$ for orthonormal pairs $X_1, X_2 \in \mathfrak{so}_3$ is the unit sphere in $\mathfrak{so}_3$, (34) implies

$$\|\nu_0\|_{\mathfrak{so}_3} \leq \frac{2\pi \sum_{i=1}^k \lambda_i}{t_1 - t_0}. \tag{35}$$

If $k = 1$, $\lambda_1 = 1$, then solutions γ to the OCA problem are geodesics in $\mathbf{SO}_3$. Consider the example $\gamma(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\forall t \in [t_0, t_1]$. Then $\nu_0 = \dot{\gamma}\gamma^{-1} = W_3$. So (35) becomes $t_1 - t_0 \leq 2\pi$, which indeed characterizes the first conjugate point along the geodesic γ in $\mathbf{SO}_3$. In fact, since $d\Phi^q : \mathfrak{so}_3 \to T_q\mathbf{SO}_3$ is surjective for any $q \in \mathbf{SO}_3$, Theorem 2 is necessary *and* sufficient for the second order optimality of γ. A closer look at condition (25) reveals that the "best" attempt to break it is to choose η_0 to be a curve contained in the two dimensional plane $\mathfrak{h}$ orthogonal to ν_0, for otherwise, one can always replace it with its projection onto $\mathfrak{h}$ such that the first term in (25) remains the same, while the second term is decreased.

Example 10 (Grassmann Manifold) We continue the discussion in Example 6, where $G = \mathbf{SO}_n$, $M = G_{n,p}$, and $L = \frac{1}{2}\|\cdot\|^2$. Suppose that $\gamma = \langle\gamma_i\rangle_{i=1}^k$ is an optimal solution to the OCA (or OFS) problem defined on $[t_0, t_1]$, and that $\langle A_i\rangle_{i=1}^k$ is a lifting of γ in $\mathbf{SO}_n$. At each time t, we have

$$\begin{aligned} \mathbb{I}_t(X, X) &= \sum_{i=1}^k \lambda_i \langle d\Phi^{\gamma_i}(X), d\Phi^{\gamma_i}(X)\rangle_{T_{\gamma_i}G_{n,p}} = \sum_{i=1}^k \lambda_i \langle P_{A_i}(XA_i), P_{A_i}(XA_i)\rangle_{T_{A_i}\mathbf{SO}_n} \\ &\leq \sum_{i=1}^k \lambda_i \langle XA_i, XA_i\rangle_{T_{A_i}\mathbf{SO}_n} = \sum_{i=1}^k \lambda_i \|X\|^2_{\mathfrak{so}_n}, \quad \forall X \in \mathfrak{so}_n. \end{aligned} \tag{36}$$

where equality holds if and only if $XA_i \in \mathrm{hor}_{A_i}\mathbf{SO}_n$ for each i. Hence $\rho(\mathbb{I}_t) \leq \sum_{i=1}^k \lambda_i$, and Corollary 1 implies that

$$|\nu_0([X_1, X_2])| \leq \frac{2\pi \sum_{i=1}^k \lambda_i}{t_1 - t_0} \tag{37}$$

for all orthonormal pairs $X_1, X_2 \in \mathfrak{so}_n$, where ν_0 is defined in (19). In particular, if $n = 3$, then the set of possible $[X_1, X_2]$ is the unit sphere in $\mathfrak{so}_3$. Hence (37) reduces to

$$\|\nu_0\|_{\mathfrak{so}_3} \leq \frac{2\pi \sum_{i=1}^k \lambda_i}{t_1 - t_0}.$$

Example 11 (Stiefel Manifold) For the Stiefel manifold $V_{n,p}$ studied in Example 7, a similar argument shows that (37) still holds, with ν_0 now defined in (20).

3.4 A Topological Optimality Condition

Suppose that M is a Riemannian manifold satisfying Assumption 1 with the symmetry group G. Throughout this section, it is assumed that $L = \frac{1}{2}\|\cdot\|^2$. Let $\gamma = \langle\gamma_i\rangle_{i=1}^k$ be an optimal solution to the OCA (or OFS) problem defined on $[t_0, t_1]$. We have proved in Theorem 1 that the quantity

ν_0 is conserved along γ. In this section, additional optimality conditions will be derived based on topological properties of M. Roughly speaking, Theorem 4 below states that for every possible way of embedding a circle in G, ν_0 is bounded when evaluated along the corresponding direction in $\mathfrak{g}$, for otherwise one can always "go the other way" around the circle.

Denote by $\mathbf{T}^1 = \mathbb{R}/2\pi\mathbb{Z} = \{\theta \mod 2\pi : \theta \in \mathbb{R}\}$ with the quotient metric of $\mathbb{R}$. $\mathbf{T}^1$ is a Lie group under addition modulo 2π, and its Lie algebra is isomorphic to $\mathbb{R}$ under the correspondence $\lambda\frac{\partial}{\partial\theta} \in T_0\mathbf{T}^1 \mapsto \lambda \in \mathbb{R}$. Hence we shall denote it by $\mathbb{R}$. Suppose that there exists a Lie group homomorphism $\varphi : \mathbf{T}^1 \to G$. Then $d\varphi : \mathbb{R} \to \mathfrak{g}$ is a Lie algebra homomorphism.

Let $h_0 : [t_0, t_1] \to \mathbf{T}^1$ be a curve in $\mathbf{T}^1$ starting from and ending at the identity 0. Define

$$\xi_0(t) = h_0(t)^{-1}\dot{h}_0(t), \quad \forall t \in [t_0, t_1]. \tag{38}$$

Then ξ_0 is a curve[2] in the Lie algebra $\mathbb{R}$ such that $\int_{t_0}^{t_1} \xi_0 = 2m\pi$ for some $m \in \mathbb{Z}$. Conversely, every curve ξ_0 in $\mathbb{R}$ satisfying $\frac{1}{2\pi}\int_{t_0}^{t_1} \xi_0 \in \mathbb{Z}$ can be realized as $h_0^{-1}\dot{h}_0$ for some curve h_0 in $\mathbf{T}^1$ that starts from and ends at 0. Fix one such h_0. Then $\varphi(h_0)$ is a curve in G that starts from and ends at e. Define a new k-tuple of curves in M by $\varphi(h_0)\gamma = \langle\varphi(h_0)\gamma_i\rangle_{i=1}^k$, which has the same starting and destination positions as γ. At each $t \in [t_0, t_1]$, since φ is a homomorphism, the following diagram commutes:

$$\begin{array}{ccc} T_0\mathbf{T}^1 & \xrightarrow{d\varphi} & T_eG \\ {\scriptstyle dm_{h_0^{-1}}}\uparrow & & \uparrow{\scriptstyle dm_{\varphi(h_0^{-1})}} \\ T_{h_0}\mathbf{T}^1 & \xrightarrow{d\varphi} & T_{\varphi(h_0)}G \end{array}$$

where m stands for left group multiplication. We then have

$$\frac{d}{dt}[\varphi(h_0)\gamma_i] = d\varphi(\dot{h}_0)\gamma_i + \varphi(h_0)\dot{\gamma}_i = \varphi(h_0)[\varphi(h_0^{-1})d\varphi(\dot{h}_0)\gamma_i + \dot{\gamma}_i] = \varphi(h_0)[d\varphi(h_0^{-1}\dot{h}_0)\gamma_i + \dot{\gamma}_i].$$

Therefore, the cost of $\varphi(h_0)\gamma$ is

$$\begin{aligned} J[\varphi(h_0)\gamma] &= \sum_{i=1}^k \lambda_i \int_{t_0}^{t_1} \frac{1}{2}\|\varphi(h_0)[d\varphi(\xi_0)\gamma_i + \dot{\gamma}_i]\|^2\, dt = \sum_{i=1}^k \lambda_i \int_{t_0}^{t_1} \frac{1}{2}\|d\varphi(\xi_0)\gamma_i + \dot{\gamma}_i\|^2\, dt \\ &= \int_{t_0}^{t_1} \left[\frac{1}{2}\sum_{i=1}^k \lambda_i\|\dot{\gamma}_i\|^2 + \frac{1}{2}\sum_{i=1}^k \lambda_i\|d\varphi(\xi_0)\gamma_i\|^2 + \sum_{i=1}^k \lambda_i((d\Phi^{\gamma_i})^*\dot{\gamma}_i, d\varphi(\xi_0))\right] dt \\ &= J(\gamma) + \int_{t_0}^{t_1} \frac{1}{2}\{\mathbb{I}_t[d\varphi(\xi_0), d\varphi(\xi_0)] + (\nu_0, d\varphi(\xi_0))\}\, dt, \end{aligned}$$

where in the last step we have used Theorem 1, and $\mathbb{I}_t$ is defined in (24). Denote by $\varphi^*\mathbb{I}_t$ the pull back of $\mathbb{I}_t$ via φ defined by $\varphi^*\mathbb{I}_t(x_1, x_2) \triangleq \mathbb{I}_t[d\varphi(x_1), d\varphi(x_2)]$, $\forall x_1, x_2 \in \mathbb{R}$. Then $\varphi^*\mathbb{I}_t : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

[2] In this section, h_0 is assumed to be continuous and piecewise C^∞, so ξ_0 is piecewise C^∞.

is a quadratic function, and is obviously of the form

$$\varphi^*\mathbb{I}_t(x_1, x_2) = \|\varphi^*\mathbb{I}_t\| x_1 x_2, \quad \forall x_1, x_2 \in \mathbb{R},$$

where $\|\varphi^*\mathbb{I}_t\| \geq 0$ is the spectral radius of $\varphi^*\mathbb{I}_t$ given by

$$\|\varphi^*\mathbb{I}_t\| \triangleq \varphi^*\mathbb{I}_t(1,1) = \mathbb{I}_t[d\varphi(1), d\varphi(1)]. \tag{39}$$

Similarly, denote by $\varphi^*\nu_0 \in \mathbb{R}^* \simeq \mathbb{R}$ the pull back of ν_0 via φ such that $(\nu_0, d\varphi(x)) = (\varphi^*\nu_0)x$, $\forall x \in \mathbb{R}$. From the above equations, the difference between the cost of $\varphi(h_0)\gamma$ and γ is given by

$$\Delta J(\xi_0) \triangleq J[\varphi(h_0)\gamma] - J(\gamma) = \int_{t_0}^{t_1} \left[\frac{1}{2}\|\varphi^*\mathbb{I}_t\|\xi_0^2 + (\varphi^*\nu_0)\xi_0\right] dt. \tag{40}$$

In the following, we shall assume that $\|\varphi^*\mathbb{I}_t\| > 0$ for almost all $t \in [t_0, t_1]$. A necessary condition for γ to be optimal is that $\Delta J(\xi_0) \geq 0$ for all possible ξ_0. By (40), this is equivalent to

$$\int_{t_0}^{t_1} \left[\frac{1}{2}\|\varphi^*\mathbb{I}_t\|\xi_0^2 + (\varphi^*\nu_0)\xi_0\right] dt \geq 0, \tag{41}$$

for all curves ξ_0 in $\mathbb{R}$ such that $\frac{1}{2\pi}\int_{t_0}^{t_1} \xi_0 \in \mathbb{Z}$.

Fix an $m \in \mathbb{Z}$. To find the ξ_0 that minimizes $\Delta J(\xi_0)$ subject to the constraint that $\int_{t_0}^{t_1} \xi_0 = 2m\pi$, we use the Lagrangian multiplier approach, and define

$$\mathcal{L}(\xi_0, \lambda) \triangleq \Delta J(\xi_0) + \lambda(\int_{t_0}^{t_1} \xi_0 - 2m\pi) = \int_{t_0}^{t_1} \left[\frac{1}{2}\|\varphi^*\mathbb{I}_t\|\xi_0^2 + (\lambda + \varphi^*\nu_0)\xi_0\right] dt - 2\lambda m\pi$$

for $\lambda \in \mathbb{R}$. Note that $\mathcal{L}(\xi_0, \lambda)$ is minimized when $\xi_0 = -(\lambda + \varphi^*\nu_0)/\|\varphi^*\mathbb{I}_t\|$. The constraint that $\int_{t_0}^{t_1} \xi_0 = 2m\pi$ implies that $\lambda + \varphi^*\nu_0 = -2m\pi/[\int_{t_0}^{t_1} \frac{1}{\|\varphi^*\mathbb{I}_t\|} dt]^{-1}$. Hence

$$\xi_0 = \frac{2m\pi}{\|\varphi^*\mathbb{I}_t\| \int_{t_0}^{t_1} \frac{1}{\|\varphi^*\mathbb{I}_t\|} dt} \tag{42}$$

minimizes $\Delta J(\xi_0)$ among ξ_0 such that $\int_{t_0}^{t_1} \xi_0 = 2m\pi$. Substituting (42) into (41), we have

$$m^2\pi + m(\varphi^*\nu_0) \int_{t_0}^{t_1} \frac{1}{\|\varphi^*\mathbb{I}_t\|} dt \geq 0.$$

Note that the above inequality must hold for all $m \in \mathbb{Z}$, which implies

$$|\varphi^*\nu_0| \leq \pi \left[\int_{t_0}^{t_1} \frac{1}{\|\varphi^*\mathbb{I}_t\|} dt\right]^{-1}.$$

Therefore, we have

Theorem 4 *Suppose that $\gamma = \langle\gamma_i\rangle_{i=1}^k$ is a solution to the OCA (or OFS) problem with $L = \frac{1}{2}\|\cdot\|^2$. Let ν_0 and $\mathbb{I}_t$ be defined in (13) and (24) respectively. Then, for any Lie group homomorphism $\varphi : \mathbf{T}^1 \to G$ such that $\|\varphi^*\mathbb{I}_t\| > 0$ for almost all $t \in [t_0, t_1]$,*

$$|\varphi^*\nu_0| \leq \pi\left[\int_{t_0}^{t_1} \frac{1}{\|\varphi^*\mathbb{I}_t\|}\, dt\right]^{-1}. \tag{43}$$

Example 12 ($G = M = \mathbf{T}^n$) Let $G = \mathbf{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the flat n-torus with the metric inherited from $\mathbb{R}^n$. $\mathbf{T}^n$ is a Lie group under componentwise modulo $\mathbb{Z}$ addition, and its metric is bi-invariant. Its Lie algebra is $\mathbb{R}^n$ with trivial Lie bracket, and is equipped with the standard metric. Let $\gamma = \langle\gamma_i\rangle_{i=1}^k$ be a solution to the OCA (or OFS) problem. Then in Example 5 we show that

$$\sum_{i=1}^k \lambda_i\dot{\gamma}_i\gamma_i^{-1} = \sum_{i=1}^k \lambda_i\dot{\gamma}_i \equiv \nu_0 \in \mathbb{R}^n.$$

Pick any $z \in \mathbb{Z}^n$, $z \neq 0$. The map $\varphi(\theta \mod 2\pi) = \frac{\theta}{2\pi}z \mod \mathbb{Z}^n$, $\forall\theta \in \mathbb{R}$, is a homomorphism from $\mathbf{T}^1$ to $\mathbf{T}^n$ with $d\varphi(1) = z/2\pi$. So $\varphi^*\nu_0 = \langle\nu_0, d\varphi(1)\rangle = \langle\nu_0, z\rangle/2\pi$, and $\|\varphi^*\mathbb{I}_t\| = \mathbb{I}_t[d\varphi(1), d\varphi(1)] = \sum_{i=1}^k \lambda_i\|z/2\pi\|^2$. As a result, Theorem 4 implies that

$$|\langle\nu_0, \frac{z}{\|z\|^2}\rangle| \leq \frac{\sum_{i=1}^k \lambda_i}{2(t_1 - t_0)}, \quad \text{for all } z \in \mathbb{Z}^n,\ z \neq 0. \tag{44}$$

In particular, if $\nu_0 = (\nu_{0,1}, \ldots, \nu_{0,n})^t$ in coordinates, and $z = e_j$ is the element in $\mathbb{R}^n$ with the j-th coordinate 1 and the rest 0, then a necessary condition of (44) is

$$|\nu_{0,j}| \leq \frac{\sum_{i=1}^k \lambda_i}{2(t_1 - t_0)}, \quad j = 1, \ldots, n. \tag{45}$$

It can be verified that (45) is also sufficient for (44).

Example 13 ($G = \mathbf{SO}^2$, $M = \mathbb{R}^2$) Suppose $G = \mathbf{SO}_2$, $M = \mathbb{R}^2$ with the standard metric, and G acts on M by matrix multiplication. As before, choose the metric on $\mathfrak{so}_2$ to be $\frac{1}{2}\langle\cdot,\cdot\rangle_F$. By following the same arguments as in Example 4 of Section 3.2, we conclude that for any solution $\gamma = \langle\gamma_i\rangle_{i=1}^k$ to the OCA (or OFS) problem,

$$\sum_{i=1}^k \lambda_i(\dot{\gamma}_i\gamma_i^t - \gamma_i\dot{\gamma}_i^t) \equiv \nu_0 \in \mathfrak{so}_2.$$

Note that $\mathbf{SO}_2 \simeq \mathbf{T}^1$ under the isomorphism $\varphi(\theta \mod 2\pi) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $\forall\theta \in \mathbb{R}$, and $d\varphi(1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Hence $\varphi^*\nu_0 = \langle\nu_0, d\varphi(1)\rangle_{\mathfrak{so}_2} = \sum_{i=1}^k \lambda_i(\gamma_{i,1}\dot{\gamma}_{i,2} - \gamma_{i,2}\dot{\gamma}_{i,1})$ if we write each γ_i in

coordinates, and $\|\varphi^*\mathbb{I}_t\| = \sum_{i=1}^k \lambda_i \|d\varphi(1)\gamma_i\|^2 = \sum_{i=1}^k \lambda_i \|\gamma_i\|^2$. Therefore, by Theorem 4,

$$\left| \sum_{i=1}^k \lambda_i (\gamma_{i,1}\dot{\gamma}_{i,2} - \gamma_{i,2}\dot{\gamma}_{i,1}) \right| \leq \pi \left[\int_{t_0}^{t_1} \frac{dt}{\sum_{i=1}^k \lambda_i \|\gamma_i\|^2} \right]^{-1}. \tag{46}$$

In [11], equation (46) is derived by an elementary approach. For a concrete example, consider the case $k = 2$, $r = 2$, $\lambda_1 = \lambda_2 = 1$. Let $t_0 = 0$, $t_1 = T$, $\gamma_1(t) = -\gamma_2(t) = (\cos t, \sin t)^t$, $\forall t \in [0, T]$. Then (46) implies that $\gamma = \langle \gamma_i \rangle_{i=1}^2$ is not an optimal solution for the OCA problem if $T > \pi$, which is obvious since the two agents could rotate at constant speed in reversed direction around the origin with a smaller cost.

Example 14 ($G = M = \mathbf{SO}_n$) Let $G = \mathbf{SO}_n$ be equipped with the bi-invariant metric defined in Example 5. So the conserved quantity is $\nu_0 = \sum_{i=1}^k \lambda_i \dot{\gamma}_i \gamma_i^{-1} \in \mathfrak{so}_n$ along a solution $\gamma = \langle \gamma_i \rangle_{i=1}^k$ to the OCA (or OFS) problem. Let $X \in \mathfrak{so}_n$ be such that $e^{2\pi X} = I_n$, $X \neq 0$. Define a Lie group homomorphism $\varphi_X : \mathbf{T}^1 \to \mathbf{SO}_n$ by $\varphi_X(\theta \mod 2\pi) = e^{\theta X}$, $\forall \theta \in \mathbb{R}$. Then $d\varphi_X(1) = X$, $\varphi_X^* \nu_0 = \langle \nu_0, X \rangle_{\mathfrak{so}_n}$, and $\|\varphi_X^* \mathbb{I}_t\| = \sum_{i=1}^k \lambda_i \|X\|_{\mathfrak{so}_n}^2$. Theorem 4 thus implies that

$$|\langle \nu_0, X \rangle_{\mathfrak{so}_n}| \leq \pi \sum_{i=1}^k \lambda_i \|X\|_{\mathfrak{so}_n}^2 / (t_1 - t_0)$$

for all $X \in \mathfrak{so}_n$ such that $e^{2\pi X} = I_n$. By Lemma 9 in Appendix B, this is equivalent to

$$\|\nu_0\|_2 \leq \frac{\pi}{t_1 - t_0} \sum_{i=1}^k \lambda_i, \tag{47}$$

where $\|\nu_0\|_2$ is the L^2 norm of the matrix ν_0. Or equivalently,

$$\text{the maximum of the singular values of } \nu_0 \leq \frac{\pi}{t_1 - t_0} \sum_{i=1}^k \lambda_i.$$

Example 15 (Grassmann Manifold) Let $G = \mathbf{SO}_n$, $M = G_{n,p}$ be as in Example 6. So for any solution $\gamma = \langle \gamma_i \rangle_{i=1}^k$ to the OCA (or OFS) problem in $G_{n,p}$ and its lifting $\langle A_i \rangle_{i=1}^k$ in $\mathbf{SO}_n$, $\sum_{i=1}^k \lambda_i P_{A_i}(\dot{A}_i) A_i^t \equiv \nu_0 \in \mathfrak{so}_n$. Let $X \in \mathfrak{so}_n$ be such that $e^{2\pi X} = I_n$, $X \neq 0$. Define the homomorphism $\varphi_X : \mathbf{T}^1 \to \mathbf{SO}_n$ as in Example 14. Then $\varphi_X^* \nu_0 = \langle \nu_0, X \rangle_{\mathfrak{so}_n}$, and by (36), $\|\varphi_X^* \mathbb{I}_t\| = \mathbb{I}_t(X, X) \leq \sum_{i=1}^k \lambda_i \|X\|_{\mathfrak{so}_n}^2$. Theorem 4 then implies that

$$|\langle \nu_0, X \rangle_{\mathfrak{so}_n}| \leq \pi \sum_{i=1}^k \lambda_i \|X\|_{\mathfrak{so}_n}^2 / (t_1 - t_0), \tag{48}$$

for all $X \in \mathfrak{so}_n$ such that $e^{2\pi X} = I_n$. Therefore, by Lemma 9, the bound (47) still holds.

However, it is possible to improve this bound by considering an additional symmetry of $G_{n,p}$. Suppose $X \in \mathfrak{so}_n$ is chosen such that $e^{\pi X} = -I_n$ (such X exists only if n is even). Consider $\{\pm I_n\}$,

a discrete subgroup of $\mathbf{SO}_n$. The action of each of $\{\pm I_n\}$ on $G_{n,p}$ is the identity map, so Φ induces naturally an action of the quotient group $\mathbf{SO}_n/\{\pm I_n\}$ on $G_{n,p}$, which also satisfies Assumption 1 in Section 2. Since $\{\pm I_n\}$ is discrete, the Lie algebra of $\mathbf{SO}_n/\{\pm I_n\}$ is $\mathfrak{so}_n$, and the conserved quantity ν_0 and the map $\mathbb{I}_t$ remain the same for this induced action. Now the map $\varphi(\theta \mod 2\pi) = e^{\theta X/2}$, $\forall \theta \in \mathbb{R}$, is a homomorphism from T^1 to $\mathbf{SO}_n/\{\pm I_n\}$ with $d\varphi(1) = X/2$. So $\varphi^*\nu_0 = \langle \nu_0, X/2\rangle_{\mathfrak{so}_n}$, and $\|\varphi^*\mathbb{I}_t\| \leq \sum_{i=1}^k \lambda_i \|X/2\|^2_{\mathfrak{so}_n}$. Applying Theorem 4, we have

$$|\langle \nu_0, X\rangle_{\mathfrak{so}_n}| \leq \frac{\pi}{2}\sum_{i=1}^k \lambda_i \|X\|^2_{\mathfrak{so}_n}/(t_1 - t_0)$$

for all $X \in \mathfrak{so}_n$ such that $e^{\pi X} = -I_n$. By Lemma 10 in the Appendix, this implies that

$$\text{the average of the singular values of } \nu_0 \leq \frac{\pi}{2(t_1 - t_0)}\sum_{i=1}^k \lambda_i.$$

Example 16 (Stiefel Manifold) Let $G = \mathbf{SO}_n$, $M = V_{n,p}$ be defined in Example 7. For a solution $\gamma = \langle \gamma_i\rangle_{i=1}^k$ to the OCA (or OFS) problem on $V_{n,p}$, it is calculated in Example 7 that $\sum_{i=1}^k \lambda_i \hat{P}_{A_i}(\dot{A}_i)A_i^t \equiv \nu_0 \in \mathfrak{so}_n$, where $\langle A_i\rangle_{i=1}^k$ is a lifting of γ_i in $\mathbf{SO}_n$. By following the same steps as in the previous example, we conclude that (48), hence (47), still holds.

4 An Example

The necessary conditions derived in Section 3 apply for solutions to both the OCA problem and the OFS problem under any choice of $\tilde{\mathcal{F}}$. The price for this general applicability, however, is that in general they are far from sufficient, particularly when the number of agents is large. We show this in this section by a simple example, which is also of its own interest.

Consider the OCA problem with $M = \mathbb{R}^2$, $\lambda_1 = \ldots = \lambda_k = 1$, $r = 1$, $t_0 = 0$, $t_1 = T > 0$. Suppose that the starting position $\langle a_i\rangle_{i=1}^k$ is given by $\langle(\frac{2i-k-1}{2}, 0)^t\rangle_{i=1}^k$. In other words, at time $t = 0$ the k agents are aligned on the x-axis with a minimal allowed separation between consecutive ones and with their centroid at the origin. For each $t \geq 0$, denote by $R_t : \mathbb{R}^2 \to \mathbb{R}^2$ the rotation of $\mathbb{R}^2$ by an angle t counterclockwise. Suppose that the destination position is $\langle b_i\rangle_{i=1}^k = \langle R_T(a_i)\rangle_{i=1}^k$. By Theorem 1, any optimal solution $\gamma = \langle \gamma_i\rangle_{i=1}^k$ to the OCA problem must satisfy that $\sum_{i=1}^k \dot{\gamma}_i$ is constant, which after integration implies that $\sum_{i=1}^k \gamma_i \equiv 0$ since $\sum_{i=1}^k a_i = \sum_{i=1}^k b_i = 0$. Therefore, γ as a curve in $M^k = \mathbb{R}^{2k}$ lies in the subspace $V = \{(q_1, \ldots, q_k) \in \mathbb{R}^{2k} : \sum_{i=1}^k q_i = 0\}$. Taking into consideration the separation constraints, γ is contained in $(\mathbb{R}^{2k} \setminus W) \cap V$, where W is defined in

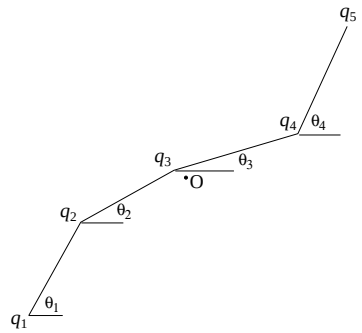

Figure 2: Coordinates of N when $k = 5$.

(3). Furthermore, by the remarks in Section 2, γ is a distance-minimizing geodesic in $(\mathbb{R}^{2k} \setminus W) \cap V$ connecting $a = (a_1, \ldots, a_k)$ to $b = (b_1, \ldots, b_k)$.

Now as a candidate for the optimal solution, consider γ given by $\gamma_i(t) = R_t(a_i)$, $\forall t \in [0, T]$, $i = 1, \ldots, k$, i.e., each agent rotates at constant angular velocity 1 around the origin during $[0, T]$. It can be checked that Theorem 4 implies that γ is not optimal if $T > \pi$. However, we will improve this bound by showing in the following that γ fails to be optimal once $T > \tau_k$, where $\tau_k \to 0$ as $k \to \infty$. To see this, note that γ is contained in a smooth component N of $(\mathbb{R}^{2k} \setminus W) \cap V$ given by

$$\{(q_1, \ldots, q_k) \in \mathbb{R}^{2k} : \|q_i - q_{i+1}\| = 1,\ i = 1, \ldots, k-1, \text{ and } \|q_i - q_j\| > 1 \text{ for all other } i \neq j\} \cap V.$$

N is a $k-1$ dimensional smooth submanifold of $\mathbb{R}^{2k}$, and admits (global) coordinates $(\theta_1, \ldots, \theta_{k-1})$, where θ_i is the angle $q_{i+1} - q_i \in \mathbb{R}^2$ makes with respect to the positive x-axis (see Figure 2). The coordinate map $f : (\theta_1, \ldots, \theta_{k-1}) \to (q_1, \ldots, q_k) \in N$ is defined by

$$q_i = q_1 + \sum_{j=1}^{i-1} \begin{bmatrix} \cos\theta_j \\ \sin\theta_j \end{bmatrix}, \quad i = 2, \ldots, k,$$

where q_1 is the unique element of $\mathbb{R}^2$ such that $\sum_{i=1}^{k} q_i = 0$ for q_i thus defined, namely,

$$q_1 = -\frac{1}{k} \sum_{j=1}^{k-1} (k-j) \begin{bmatrix} \cos\theta_j \\ \sin\theta_j \end{bmatrix}.$$

In these coordinates, γ corresponds to $\theta_i(t) = t$, $t \in [0, T]$, $i = 1, \ldots, k-1$.

At any $q \in N$, $\frac{\partial}{\partial\theta_1}, \ldots, \frac{\partial}{\partial\theta_{k-1}}$ form a basis of $T_q N$. In this basis, the Riemannian metric $\langle \cdot, \cdot \rangle$ of N that N inherits from $\mathbb{R}^{2k}$ as a submanifold can be computed as

$$g_{ij} \triangleq \langle \frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j} \rangle = \langle \frac{\partial f}{\partial\theta_i}, \frac{\partial f}{\partial\theta_j} \rangle_{\mathbb{R}^{2k}} = \Delta_{ij} \cos(\theta_i - \theta_j), \quad 1 \leq i, j \leq k-1,$$

where Δ_{ij} are the components of a matrix $\Delta = (\Delta_{ij})_{1\le i,j\le k-1} \in \mathbb{R}^{(k-1)\times(k-1)}$ defined by

$$\Delta_{ij} = \begin{cases} \frac{i(k-j)}{k} & \text{if } i \le j, \\ \frac{(k-i)j}{k} & \text{if } i > j. \end{cases} \tag{49}$$

Denote by $(g^{ij})_{1\le i,j\le k-1}$ the inverse matrix of $(g_{ij})_{1\le i,j\le k-1}$. The covariant derivative with respect to the Levi-Civita connection on N is given by $\nabla_{\frac{\partial}{\partial\theta_i}}\frac{\partial}{\partial\theta_j} = \sum_{m=1}^{k-1}\Gamma_{ij}^m\frac{\partial}{\partial\theta_m}$, where

$$\Gamma_{ij}^m = \frac{1}{2}\sum_{l=1}^{k-1}\{\frac{\partial g_{jl}}{\partial\theta_i} + \frac{\partial g_{li}}{\partial\theta_j} - \frac{\partial g_{ij}}{\partial\theta_k}\}g^{lm}, \quad 1 \le i,j,m \le k-1,$$

are the Christoffel symbols. From the above definition, one can compute that for $1 \le i,j,m \le k-1$,

$$\Gamma_{ij}^m = \begin{cases} 0 & \text{if } i \ne j, \\ \sum_{l=1}^{k-1}\Delta_{il}\sin(\theta_l - \theta_i)g^{lm} & \text{if } i = j. \end{cases} \tag{50}$$

Notice that $\theta_i = \theta_j$ for all i,j along γ. Therefore,

Lemma 3 *Along γ we have $\Gamma_{ij}^m = 0$, hence $\nabla_{\frac{\partial}{\partial\theta_i}}\frac{\partial}{\partial\theta_j} = 0$, for all $1 \le i,j,m \le k-1$.*

Since $\dot\gamma = \frac{\partial}{\partial\theta_1} + \cdots + \frac{\partial}{\partial\theta_{k-1}}$, by Lemma 3, $\nabla_{\dot\gamma}\frac{\partial}{\partial\theta_i} = 0$, i.e., $\frac{\partial}{\partial\theta_i}$ is parallel along γ, for each $i = 1,\ldots,k-1$. As a result, one can see that γ is a geodesic in N, since $\nabla_{\dot\gamma}\dot\gamma = 0$.

The curvature tensor of N is given by $R(\frac{\partial}{\partial\theta_i},\frac{\partial}{\partial\theta_j})\frac{\partial}{\partial\theta_l} = \sum_{m=1}^{k-1}R_{ijl}^m\frac{\partial}{\partial\theta_m}$, where ([5])

$$R_{ijl}^m = \sum_{\beta=1}^{k-1}\Gamma_{il}^\beta\Gamma_{j\beta}^m - \sum_{\beta=1}^{k-1}\Gamma_{jl}^\beta\Gamma_{i\beta}^m + \frac{\partial\Gamma_{il}^m}{\partial\theta_j} - \frac{\partial\Gamma_{jl}^m}{\partial\theta_i}, \quad 1 \le i,j,l,m \le k-1.$$

A Jacobi field X along γ satisfies the equation $\nabla_{\dot\gamma}\nabla_{\dot\gamma}X + R(\dot\gamma,X)\dot\gamma = 0$. Write $X = \sum_{i=1}^{k-1}x_i\frac{\partial}{\partial\theta_i}\big|_\gamma$ for some smooth $x_i : [0,T] \to \mathbb{R}$. Then $\nabla_{\dot\gamma}\nabla_{\dot\gamma}X = \sum_{i=1}^{k-1}\ddot x_i\frac{\partial}{\partial\theta_i}\big|_\gamma$, and $R(\dot\gamma,X)\dot\gamma = \sum_{i,j,l,m=1}^{k-1}R_{ijl}^m x_j\frac{\partial}{\partial\theta_m}\big|_\gamma$. So by denoting $x = (x_1,\ldots,x_{k-1})^t \in \mathbb{R}^{k-1}$, the Jacobi equation along γ is reduced to

$$\ddot x + B_k x = 0, \tag{51}$$

where $B_k = (b_{mj})_{1\le m,j\le k-1} \in \mathbb{R}^{(k-1)\times(k-1)}$ is defined by $b_{mj} = \sum_{i,l=1}^{k-1}R_{ijl}^m$. Note that here and in the rest of this section, all terms (such as R_{ijl}^m, the Christoffel symbols and their derivatives) in the equations are evaluated along γ. So by (50) and Lemma 3,

$$b_{mj} = \sum_{i,l}R_{ijl}^m = \sum_{i,l}\frac{\partial\Gamma_{il}^m}{\partial\theta_j} - \sum_{i,l}\frac{\partial\Gamma_{jl}^m}{\partial\theta_i} = \sum_i\frac{\partial\Gamma_{ii}^m}{\partial\theta_j} - \sum_i\frac{\partial\Gamma_{jj}^m}{\partial\theta_i}, \tag{52}$$

where the summations are all from 1 to $k-1$. The first term can be simplified to

$$\begin{aligned}\sum_i \frac{\partial \Gamma_{ii}^m}{\partial \theta_j} &= \sum_{i\neq j} \frac{\partial \Gamma_{ii}^m}{\partial \theta_j} + \frac{\partial \Gamma_{jj}^m}{\partial \theta_j} \\ &= \sum_{i\neq j} \frac{\partial}{\partial \theta_j}\Big[\sum_l \Delta_{il}\sin(\theta_l-\theta_i)g^{lm}\Big] + \frac{\partial}{\partial \theta_j}\Big[\sum_{l\neq j} \Delta_{jl}\sin(\theta_l-\theta_j)g^{lm}\Big] \\ &= \sum_{i\neq j}\Delta_{ij}g^{jm} - \sum_{l\neq j}\Delta_{jl}g^{lm},\end{aligned}$$

where we have used the fact that $\theta_1 = \ldots = \theta_{k-1}$ on γ. Similarly,

$$\begin{aligned}\sum_i \frac{\partial \Gamma_{jj}^m}{\partial \theta_i} &= \sum_{i\neq j} \frac{\partial}{\partial \theta_i}\Big[\sum_l \Delta_{jl}\sin(\theta_l-\theta_j)g^{lm}\Big] + \frac{\partial}{\partial \theta_j}\Big[\sum_{l\neq j} \Delta_{jl}\sin(\theta_l-\theta_j)g^{lm}\Big] \\ &= \sum_{i\neq j}\Delta_{ji}g^{im} - \sum_{l\neq j}\Delta_{jl}g^{lm}.\end{aligned}$$

Hence (52) implies that

$$b_{mj} = \sum_{i\neq j}\Delta_{ij}g^{jm} - \sum_{i\neq j}\Delta_{ji}g^{im} = (\sum_i \Delta_{ij})g^{jm} - \sum_i \Delta_{ji}g^{im}.$$

Since $g_{ji} = \Delta_{ji}$ on γ, $\sum_i \Delta_{ji}g^{im} = \sum_i g_{ji}g^{im} = \delta_{mj}$ by the definition of g^{im}. Moreover, $\sum_i \Delta_{ij} = \sum_{i\leq j}\frac{i(k-j)}{2} + \sum_{i>j}\frac{(k-i)j}{2} = \frac{j(k-j)}{2}$. Therefore,

$$b_{mj} = \frac{j(k-j)}{2}g^{jm} - \delta_{mj} = \frac{j(k-j)}{2}g^{mj} - \delta_{mj}, \tag{53}$$

by the symmetry of g^{mj}. Note that $(g^{mj})_{1\leq m,j\leq k-1} = [(g_{mj})_{1\leq m,j\leq k-1}]^{-1} = \Delta^{-1}$ on γ. So by (53),

Lemma 4 *$B_k = \Delta^{-1}\Lambda - I_k$, where $\Lambda = diag(\frac{k-1}{2},\ldots,\frac{i(k-i)}{2},\ldots,\frac{k-1}{2})_{1\leq i\leq k-1} \in \mathbb{R}^{(k-1)\times(k-1)}$.*

Remark 8 *B_k is a constant matrix independent of t since the metric of N is homogeneous along γ, or more precisely, for each $\tau > 0$, the map $(q_1,\ldots,q_k) \in N \mapsto (R_\tau(q_1),\ldots,R_\tau(q_k)) \in N$ is an isometry of N mapping $\gamma(t)$ to $\gamma(t+\tau)$ whose differential map takes $\frac{\partial}{\partial\theta_i}\big|_{\gamma(t)}$ to $\frac{\partial}{\partial\theta_i}\big|_{\gamma(t+\tau)}$ for each $i = 1,\ldots,k-1$, provided that both t and $t+\tau$ belong to $[0,T]$.*

To compute the eigenvalues of B_k, some preliminary results are needed. For each $l = 1,\ldots,k-1$, define $u_l = (1,\ldots,i^{l-1},\ldots,(k-1)^{l-1})^t \in \mathbb{R}^{k-1}$. Then $U \triangleq [u_1|\ldots|u_{k-1}] \in \mathbb{R}^{(k-1)\times(k-1)}$ is a Vandermonde matrix, hence nonsingular. The following two lemmas can be verified directly.

Lemma 5 *Let $\Delta \in \mathbb{R}^{(k-1)\times(k-1)}$ be defined in (49). Then*

$$\Delta^{-1} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

Lemma 6 *For each $l = 1, \ldots, k-1$,*

$$\Delta^{-1}\Lambda u_l = \frac{l(l+1)}{2} u_l + \textit{linear combination of } u_1, \ldots, u_{l-1}. \tag{54}$$

Written in matrices, (54) is equivalent to $\Delta^{-1}\Lambda U = U\Sigma$, i.e., $U^{-1}\Delta^{-1}\Lambda U = \Sigma$, where Σ is an upper triangular matrix whose elements on the main diagonal are $\frac{l(l+1)}{2}$, $l = 1, \ldots, k-1$. This implies that $\Delta^{-1}\Lambda$ and Σ have the same set of eigenvalues, namely, $\frac{l(l+1)}{2}$, $1 \le l \le k-1$, and that an eigenvector v_l of $\Delta^{-1}\Lambda$ corresponding to eigenvalue $\frac{l(l+1)}{2}$ is of the form u_l+ linear combination of $u_1, \ldots, u_{l-1}$. Hence, from Lemma 4, 5, and 6, we conclude that

Proposition 1 *B_k has $k-1$ distinctive eigenvalues $\mu_l = \frac{l(l+1)}{2} - 1$, for $l = 1, \ldots, k-1$.*

The exact expressions of the eigenvectors v_l are complicated, except for l at the two extremes.

Lemma 7 *B_k has the following eigenvectors:*

- $v_1 = (1, \ldots, 1)^t$ *for* $\mu_1 = 0$;
- $v_2 = (2-k, \ldots, 2i-k, \ldots, k-2)^t$ *for* $\mu_2 = 2$;
- $v_3 = [4k^2 - 10k + 9, \ldots, 5(2i-k)^2 - k^2 + 4, \ldots, 4k^2 - 10k + 9]^t$ *for* $\mu_3 = 5$;
- $v_{k-2} = [(2-k)\binom{k}{1}, \ldots, (-1)^{i+1}(2i-k)\binom{k}{i}, \ldots, (-1)^k(k-2)\binom{k}{k-1}]^t$ *for* $\mu_{k-2} = \frac{(k-1)(k-2)}{2} - 1$;
- $v_{k-1} = [\binom{k}{1}, \ldots, (-1)^{i+1}\binom{k}{i}, \ldots, (-1)^k\binom{k}{k-1}]^t$ *for* $\mu_{k-1} = \frac{k(k-1)}{2} - 1$.

Lemma 7 can be verified by direct computation. It is not hard to show that the i-th component of v_l is a polynomial function of $2i-k$ of degree $l-1$ consisting of even order terms only when l is odd and odd order terms only when l is even.

Remark 9 *B_k always has an eigenvalue 0 with the corresponding eigenvector $(1,\dots,1)^t$, a consequence of the fact that $\dot\gamma = \frac{\partial}{\partial\theta_1} + \cdots + \frac{\partial}{\partial\theta_{k-1}}$ is parallel along γ since γ is a geodesic of N.*

Note that $v_1,\dots,v_{k-1}$ form a basis of $\mathbb{R}^{k-1}$. If we write $x = \sum_{l=1}^{k-1} y_l v_l$ in this basis, then the Jacobi equation (51) is equivalent to $\ddot y_l + \mu_l y_l = 0$, $1 \le l \le k-1$. Assume that X, hence x, vanishes at $t=0$. Then $y_l(0)=0$, and solutions to the above equations are of the form $y_1(t) = c_1 t$, $y_l(t) = c_l \sin(\sqrt{\mu_l}t)$, $l = 2,\dots,k-1$, for some constants $c_1,\dots,c_{k-1}$. Therefore, the smallest $\tau_k > 0$ for which there is a nontrivial solution x such that $x(0) = x(\tau_k) = 0$ is

$$\tau_k = \frac{\pi}{\sqrt{\max_{1\le l\le k-1}\mu_l}} = \frac{\pi}{\sqrt{\mu_{k-1}}} = \frac{\pi\sqrt{2}}{\sqrt{(k-2)(k+1)}}.$$

In other words, the first conjugate point of $\gamma(0)$ along γ in N is $\gamma(\tau_k)$. Since a geodesic is no longer distance-minimizing after passing its first conjugate point, we have

Proposition 2 *γ is not an optimal solution to the OCA problem if $T > \tau_k$.*

Note that $\tau_k \sim \frac{1}{k}$ as $k \to \infty$. The result for the case $k=3$ is first proved in [11].

We will next explain what the better solutions than γ look like, at least infinitesimally, once $T > \tau_k$, or more generally, once $T > \pi/\sqrt{\mu_l}$, for $l = 2,\dots,k-1$. Let $\{\gamma_s\}_{-\epsilon<s<\epsilon}$ be a C^∞ proper variation of γ with variation field $X \triangleq \frac{\partial\gamma_s}{\partial s}\big|_{s=0}$. By the variation of energy formulas (see [5]), if we denote by $E(s)$ the energy of γ_s, then $E'(0)=0$, and

$$\frac{1}{2}E''(0) = -\int_0^T \langle X, \nabla_{\dot\gamma}\nabla_{\dot\gamma}X + R(\dot\gamma, X)\dot\gamma\rangle\, dt.$$

Write $X = \sum_{i=1}^{k-1} x_i \frac{\partial}{\partial\theta_i}\big|_\gamma$ in coordinates. Then $x = (x_1,\dots,x_{k-1})^t$ vanishes at 0 and T since $\{\gamma_s\}_{-\epsilon<s<\epsilon}$ is a proper variation. The above equation is thus reduced to

$$\frac{1}{2}E''(0) = -\int_0^T x^t\Delta(\ddot x + B_k x)\, dt.$$

Suppose now that $T > \pi/\sqrt{\mu_l}$ for some $2 \le l \le k-1$. Then, by choosing $\{\gamma_s\}_{-\epsilon<s<\epsilon}$ such that $x(t) = v_l \sin(\pi t/T)$ where v_l is an eigenvector of B_k for eigenvalue μ_l, we have

$$\frac{1}{2}E''(0) = -(\mu_l - \frac{\pi^2}{T^2})\int_0^T (v_l^t\Delta v_l)\sin^2(\pi t/T)\, dt < 0, \tag{55}$$

since $v_l^t\Delta v_l > 0$ and $\mu_l - \pi^2/T^2 > 0$. Therefore, the arc length of γ_s for s sufficiently close to 0 is smaller than that of γ. To sum up, the above analysis shows that for $T > \pi/\sqrt{\mu_l}$, a solution better than γ can be obtained by infinitesimally perturbing γ in such a way that, at each $t \in [0,T]$,

$(\theta_1, \ldots, \theta_{k-1})^t$ is incremented by an amount of $v_l \sin(\pi t/T) ds$. In particular, the signs of the components of v_l determine the shape of the $(k-1)$-rod link during such perturbations. For example, the alternating signs of the components of v_{k-1} indicate a perturbation where the $k-1$ rods are first folded into a saw-like shape during the first half of the time interval $[0, T]$, with the degree of folding of each rod depending on its position (in fact, proportional to $\binom{k}{l}$ for the l-th rod from the edge, $l = 1, \ldots, k-1$), and then straightened up during the later half of the time interval. In contrast, v_2 indicates the $k-1$ rods to bend into a bow-like shape, whereas the shape specified by v_{k-2} is a mixing (product) of the bending specified by v_2 and the folding specified by v_{k-1}. The efficiency of the perturbations specified by different v_l (provided $T > \pi/\sqrt{\mu_l}$) can be studied by comparing their respective $E''(0)$ under the requirement that $\int_0^T \|X\|^2 \, dt = \int_0^T (v_l^t \Delta v_l) \sin^2(\pi t/T) \, dt$ be constant, such that a smaller $E''(0)$ corresponds to a more efficient perturbation. In this respect, by (55), the larger the eigenvalue μ_l, the more efficient the perturbation specified by its corresponding eigenvector v_l. The most efficient perturbation is thus the one given by v_{k-1}.

5 Collision Avoidance of Bodies

The OCA and OFS problems studied in Section 3 can be thought of as optimal motion planning problems for k agents moving on a Riemannian manifold, with each agent a disk of radius $r/2$. The arguments in Section 3 can be generalized to the situation where agents have shape other than disks. To be precise, let M be a Riemannian manifold. Later in this section we will see in Remark 11 that M is in fact the frame bundle of the manifold the agents of shape are moving on.

Definition 2 (Body) *The shape of a* body *on M is specified by a map $\mathcal{S} : M \to 2^M$ that assigns to each $q \in M$ a subset $\mathcal{S}(q) \subset M$ corresponding to the subset of M the body occupies if it is at q. $\mathcal{S}$ is called the* shape (map) *of the body.*

Consider k bodies on M whose shapes are given by $\mathcal{S}_i$, $i = 1, \ldots, k$, respectively. Suppose that during the time interval $[t_0, t_1]$ their trajectories are given by a k-tuple of curves $\gamma = \langle \gamma_i \rangle_{i=1}^k$ in M. γ is called *collision-free* if $\mathcal{S}_i(\gamma_i)$, $i = 1, \ldots, k$, are disjoint at any time t. Fix the starting position $\langle a_i \rangle_{i=1}^k$ and the destination position $\langle b_i \rangle_{i=1}^k$ of the k bodies. Let $L : TM \to \mathbb{R}$ be the Lagrangian function that is nonnegative and convex on each fiber, and let the energy of γ, $J(\gamma)$, be defined by (2). Then the OCA problem for bodies is

Problem 3 (OCA of Bodies) *Among all collision-free γ that start from $\langle a_i \rangle_{i=1}^k$ at time t_0 and*

end in $\langle b_i \rangle_{i=1}^k$ *at time* t_1*, find the one (or ones) minimizing* $J(\gamma)$*.*

The OFS problem of bodies can be similarly formulated, which is omitted here for brevity.

In analogy to Assumption 1 in Section 3, we consider the following special case.

Assumption 2 *There is a* C^∞ *action* $\Phi : G \times M \to M$ *of a Lie group* G *on* M *such that*

1. *The shapes of the bodies are* G*-invariant. Namely,* $\Phi_g \circ \mathcal{S}_i = \mathcal{S}_i \circ \Phi_g$*,* $\forall g \in G$*,* $i = 1, \ldots, k$*;*
2. *The Lagrangian function* L *is* G*-invariant.*

Under Assumption 2, if G acts on M transitively, then each $\mathcal{S}_i$ is completely determined by $\mathcal{S}_i(q)$ at an arbitrary point $q \in M$. In general, one needs to specify $\mathcal{S}_i(q)$ for one q in each G-orbit of M to fully determine $\mathcal{S}_i$.

A key implication of Assumption 2 is that, if $\gamma = \langle \gamma_i \rangle_{i=1}^k$ is collision-free, so is $h_0\gamma = \langle h_0\gamma_i \rangle_{i=1}^k$ for any continuous and piecewise C^∞ curve $h_0 : [t_0, t_1] \to G$, since at any time t, $1 \le i < j \le k$,

$$\mathcal{S}_i(h_0\gamma_i) \cap \mathcal{S}_j(h_0\gamma_j) = \Phi_{h_0}[\mathcal{S}_i(\gamma_i)] \cap \Phi_{h_0}[\mathcal{S}_j(\gamma_j)] = \Phi_{h_0}[\mathcal{S}_i(\gamma_i) \cap \mathcal{S}_j(\gamma_j)],$$

hence $\mathcal{S}_i(h_0\gamma_i) \cap \mathcal{S}_j(h_0\gamma_j) = \emptyset$ if and only if $\mathcal{S}_i(\gamma_i) \cap \mathcal{S}_j(\gamma_j) = \emptyset$. This property enables one to apply the variational approach in Section 3 without modification. Therefore,

Theorem 5 *For the OCA and the OFS problems of bodies, all the necessary conditions derived in Section 3 remain true, including Theorem 1, 2, 4, and all their corollaries.*

As an example, consider $\mathbf{SE}_2$, which is a subgroup of $\mathbf{GL}(3, \mathbb{R})$ with elements of the form

$$A(x, y, \theta) \triangleq \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}, \quad \forall x, y, \theta \in \mathbb{R}.$$

$\mathbf{SE}_2$ acts on $\mathbb{R}^2 \simeq \mathbb{R}^2 \times \{1\} \subset \mathbb{R}^3$ by left matrix multiplication. Then $A(x, y, \theta)$ corresponds to the rigid body motion in $\mathbb{R}^2$ of a rotation by θ counterclockwise followed by a translation by $(x, y)^t$. The Lie algebra of $\mathbf{SE}_2$, $\mathfrak{se}_2$, is the set of all matrices of the form

$$\zeta(u, v, w) \triangleq \begin{bmatrix} 0 & -w & u \\ w & 0 & v \\ 0 & 0 & 0 \end{bmatrix}, \quad \forall u, v, w \in \mathbb{R}.$$

Define an inner product on $\mathfrak{se}_2$ by

$$\langle \zeta(u_1, v_1, w_1), \zeta(u_2, v_2, w_2) \rangle \triangleq u_1 u_2 + v_1 v_2 + \kappa w_1 w_2,$$

where $\kappa > 0$ is a constant, and extend it to a left invariant Riemannian metric $\langle \cdot, \cdot \rangle$ on $\mathbf{SE}_2$ through left translation. Consider Problem 3 with $M = \mathbf{SE}_2$ and the Lagrangian function $L = \frac{1}{2}\|\cdot\|^2$. Let $G = \mathbf{SE}_2$ and the action Φ be the group multiplication. Suppose that the shapes of the bodies are given by

$$\mathcal{S}_i[A(x, y, \theta)] = \{A(\hat{x}, \hat{y}, \hat{\theta}) \in \mathbf{SE}_2 : (\hat{x}, \hat{y})^t \in A(x, y, \theta) D_i\}, \quad \forall A(x, y, \theta) \in \mathbf{SE}_2, \tag{56}$$

where D_i is a subset of $\mathbb{R}^2$ containing the origin, for $i = 1, \ldots, k$. It is easy to verify that all $\mathcal{S}_i$ are G-invariant, hence Assumption 2 is satisfied.

Remark 10 *To justify the choice of $\mathcal{S}_i$ in (56), note that $\mathbf{SE}_2$ is the configuration space of a rigid body moving on $\mathbb{R}^2$, in the sense that each element $A(x, y, \theta) \in \mathbf{SE}_2$ can be thought of as a configuration of the rigid body whose pivot point is at $(x, y)^t \in \mathbb{R}^2$ and whose orientation is in the direction that makes an angle θ with the positive x-axis. The shape of the rigid body can be specified by the region $D \subset \mathbb{R}^2$ it occupies when it is in configuration $A(0, 0, 0)$, i.e., when it has its pivotal point at the origin and points at the positive x-axis. The region it occupies in any other configuration $A(x, y, \theta)$ is obtained by applying on D the rigid body motion that transforms configuration $A(0, 0, 0)$ to $A(x, y, \theta)$, hence the definition in (56). In this perspective, the problem can be alternatively formulated as the optimal motion planning problem for k rigid bodies in $\mathbb{R}^2$, such that no two of them can overlap at any time, and that the cost $\sum_{i=1}^{2} \frac{1}{2}\lambda_i \int_{t_0}^{t_1} (\dot{x}_i^2 + \dot{y}_i^2 + \kappa \dot{\theta}_i^2)\, dt$ is minimized.*

Let $\gamma = \langle \gamma_i \rangle_{i=1}^k = \langle A(x_i, y_i, \theta_i) \rangle_{i=1}^k$ be an optimal solution to Problem 3, where x_i, y_i, θ_i are continuous and piecewise C^∞ curves in $\mathbb{R}$ defined on $[t_0, t_1]$. Then it is easy to compute that $\sum_{i=1}^k \lambda_i (d\Phi^{\gamma_i})^* \dot{\gamma}_i = \zeta(\sum_{i=1}^k \lambda_i \dot{x}_i, \sum_{i=1}^k \lambda_i \dot{y}_i, \sum_{i=1}^k \lambda_i [\dot{\theta} - (x_i \dot{y}_i - \dot{x}_i y_i)/\kappa]) \in \mathfrak{se}_2$, which, by Theorem 1, should be constant for all t. In other words, the following quantities are conserved:

$$\sum_{i=1}^k \lambda_i \dot{x}_i, \quad \sum_{i=1}^k \lambda_i \dot{y}_i, \quad \sum_{i=1}^k \lambda_i [\dot{\theta}_i + (x_i \dot{y}_i - \dot{x}_i y_i)/\kappa].$$

In some simple cases, it is possible to construct the optimal solutions from these necessary conditions. Consider $k = 2$, and denote by $\langle A(x_i^0, y_i^0, \theta_i^0) \rangle_{i=1}^2$ and $\langle A(x_i^1, y_i^1, \theta_i^1) \rangle_{i=1}^2$ the starting and the destination positions respectively. Integrating the first two conserved quantities, we get

$$\sum_{i=1}^2 \lambda_i \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} = \frac{t_1 - t}{t_1 - t_0} \sum_{i=1}^2 \lambda_i \begin{bmatrix} x_i^0 \\ y_i^0 \end{bmatrix} + \frac{t - t_0}{t_1 - t_0} \sum_{i=1}^2 \lambda_i \begin{bmatrix} x_i^1 \\ y_i^1 \end{bmatrix}, \quad \forall t \in [t_0, t_1]. \tag{57}$$

Hence the weighted center of the two-body system moves at constant speed from the weighted center of their starting position to the weighted center of their destination position. Another fact we need is summarized in the following lemma.

Lemma 8 *Suppose that* $\gamma = \langle A(x_i, y_i, \theta_i)\rangle_{i=1}^k$ *is an optimal solution to the OCA problem of* k *bodies on* $\mathbf{SE}_2$ *with starting position* $\langle A(x_i^0, y_i^0, \theta_i^0)\rangle_{i=1}^k$ *and destination position* $\langle A(x_i^1, y_i^1, \theta_i^1)\rangle_{i=1}^k$. *Then for any* $(x, y)^t \in \mathbb{R}^2$, $\tilde{\gamma} = \langle A(x_i + \frac{t-t_0}{t_1-t_0}x, y_i + \frac{t-t_0}{t_1-t_0}y, \theta_i)\rangle_{i=1}^k$ *is an optimal solution of the OCA problem of the same* k *bodies on* $\mathbf{SE}_2$ *with starting position* $\langle A(x_i^0, y_i^0, \theta_i^0)\rangle_{i=1}^k$ *and destination position* $\langle A(x_i^1 + x, y_i^1 + y, \theta_i^1)\rangle_{i=1}^k$.

Proof: Note that $\tilde{\gamma}$ is collision-free if and only if γ is, and that the costs of γ and $\tilde{\gamma}$ are related by $J(\tilde{\gamma}) = J(\gamma)+$ some constant independent of γ. Hence the conclusion. ■

By Lemma 8, we may assume without loss of generality that

$$\sum_{i=1}^{2} \lambda_i \begin{bmatrix} x_i^0 \\ y_i^0 \end{bmatrix} = \sum_{i=1}^{2} \lambda_i \begin{bmatrix} x_i^1 \\ y_i^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{58}$$

So by (57), for all $t \in [t_0, t_1]$,

$$\sum_{i=1}^{2} \lambda_i \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{i.e.,} \quad \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} = -\frac{\lambda_1}{\lambda_2} \begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix}. \tag{59}$$

Assume in addition that D_2 is an open disk of radius r_2 centered at the origin. Then $\dot{\theta}_2$ must be constant since γ is collision-free under any reparameterization of θ_2. Hence θ_2 moves at constant speed from θ_2^0 to the nearest point in $\theta_2^1 + 2\pi\mathbb{Z}$ in $\mathbb{R}$. Since x_2, y_2 are related to x_1, y_1 as in (59), it remains only to specify x_1, y_1, θ_1. The set of feasible (x_1, y_1, θ_1) is

$$\begin{aligned}
F =& \{(x, y, \theta) : A(x, y, \theta)D_1 \cap (-\frac{\lambda_1}{\lambda_2}\begin{bmatrix} x \\ y \end{bmatrix} + D_2) = \emptyset\} \\
=& \{(x, y, \theta) : (A(0, 0, \theta)D_1 + \begin{bmatrix} x \\ y \end{bmatrix}) \cap (-\frac{\lambda_1}{\lambda_2}\begin{bmatrix} x \\ y \end{bmatrix} + D_2) = \emptyset\} \\
=& \{(x, y, \theta) : A(0, 0, \theta)D_1 \cap (-\frac{\lambda_1 + \lambda_2}{\lambda_2}\begin{bmatrix} x \\ y \end{bmatrix} + D_2) = \emptyset\} \\
=& \{(x, y, \theta) : \text{distance of } \begin{bmatrix} x \\ y \end{bmatrix} \text{ to } -\frac{\lambda_2}{\lambda_1 + \lambda_2}A(0, 0, \theta)D_1 \text{ is at least } \frac{\lambda_2 r_2}{\lambda_1 + \lambda_2}\},
\end{aligned}$$

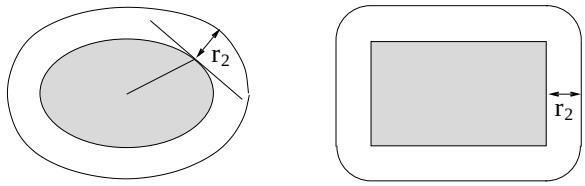

Figure 3: Two examples of outgrowing a set (shown by shaded areas) by r_2. Left: an ellipse; Right: a rectangle

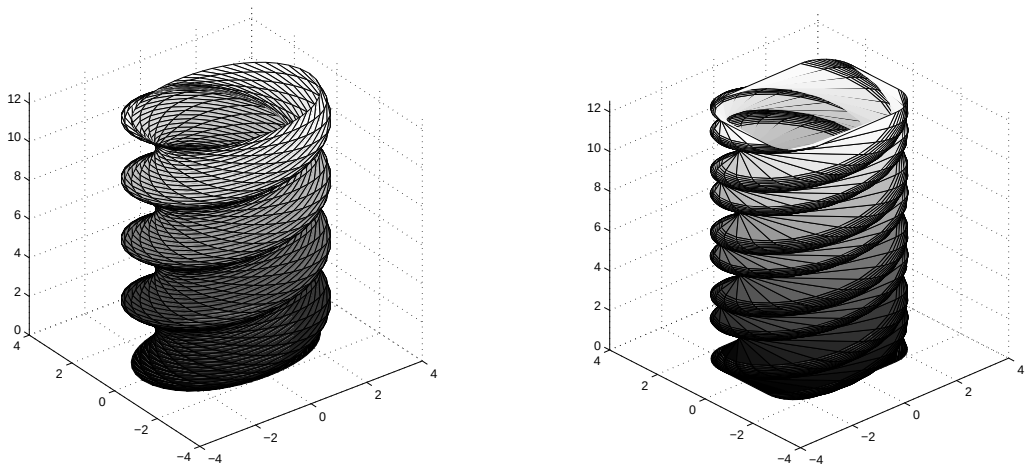

Figure 4: Plots of F when D_1 is an ellipse (left) and a rectangle (right).

which defines a *static* obstacle in $\mathbb{R}^3$. Denote by $F_\theta = \{(x, y) : (x, y, \theta) \in F\}$ a section of F. Then it is checked that $F_\theta = A(0, 0, \theta)F_0$, while F_0 can be obtained by first "outgrowing" $-D_1$ by r_2, and then scaling the resultant set by a factor of $\frac{\lambda_2}{\lambda_1+\lambda_2}$. See Figure 3 for two examples of how to outgrow a set D_1, and see Figure 4 for two examples of F.

The optimal solution corresponds to a curve (x_1, y_1, θ_1) in $\mathbb{R}^3 \setminus F$ that starts from $(x_1^0, y_1^0, \theta_1^0)$ at time t_0 and ends in $(x_1^1, y_1^1, \theta_1^1 + 2m\pi)$ for some integer m at time t_1, while minimizing the cost

$$J(\gamma) = \frac{1}{2}\sum_{i=1}^{2} \lambda_i \int_{t_0}^{t_1} (\dot{x}_i^2 + \dot{y}_i^2 + \kappa\dot{\theta}_i^2)\, dt = \frac{\lambda_1(\lambda_1 + \lambda_2)}{2\lambda_2} \int_{t_0}^{t_1} \left[\dot{x}_1^2 + \dot{y}_1^2 + \kappa_1\dot{\theta}_1^2\right] dt + C,$$

where $\kappa_1 = \frac{\lambda_2}{\lambda_1+\lambda_2}\kappa$, and C is a constant. By scaling the θ_1-axis by a factor of $\sqrt{\kappa_1}$, the integral above coincides with the usual definition of curve energy, and the problem is then reduced to finding the shortest curve between two points in the scaled feasible set. Except for very simple cases (for example, if D_1 is a disk of radius r_1 centered at the origin, then the problem is reduced to Problem 1 on $\mathbb{R}^2$ with $r = r_1 + r_2$, and solutions can be constructed geometrically), analytic solutions are not available. However, given the geometrical interpretation, there are various numerical algorithms to

solve it approximately, for example, the fast marching algorithm proposed in [18].

Remark 11 *A more natural formulation*[3] *of bodies of shape on a Riemannian manifold M is as following. Denote by FM the frame bundle over M. The fiber of FM over $q \in M$ consists all orthonormal frames in T_qM, with each frame thought of as an isomorphism from R^n to T_qM, $n = dim(M)$. Let D be a subset of $\mathbb{R}^n$ with nice properties such as being compact and convex, having smooth boundary, and containing the origin. Define a map $\Xi : FM \times D \to M$ that maps each orthonormal frame u in T_qM and each point $x \in D$ to $\exp_M[u(x)]$, where $\exp_M$ is the exponential map of M. Then a body of shape D on M can be defined by specifying $\Xi(u, D)$ as the subset of M it occupies when its position is q and its orientation is u. To ensure that $\Xi(u, D)$ resembles D, D should be within a ball of radius no larger than the injectivity radius of q, to avoid the folding caused by the singularities of the exponential map. After choosing a Lagrangian function $L : T(FM) \to \mathbb{R}$, one can study the OCA and OFS problems of bodies on the manifold M. In comparison, in the original formulation in this section, the manifold M in the definition of bodies is actually the frame bundle of a base manifold $\tilde{M}$ on which the bodies are moving, and useful shape maps $\mathcal{S}$ usually satisfy $\mathcal{S}(q) = \pi^{-1}(\tilde{D})$, $\forall q \in M$, where $\pi : M \to \tilde{M}$ is the natural projection and $\tilde{D}$ is a subset of $\tilde{M}$ containing $\pi(q)$. In this sense, the original formulation may be more general than necessary for the study of OCA and OFS problems of bodies.*

6 Conclusions

The problems of optimal collision avoidance and optimal formation control are studied for multiple agents moving on a Riemannian manifold with a group of symmetries. Some necessary conditions are given for the optimal solutions. In certain simple cases, these necessary conditions can be used to characterize the optimal solutions, whereas in general, they are not sufficient.

As a future direction, it will be interesting to see how the derived necessary conditions can aid in the numerical solution of the proposed problems.

[3] This formulation was suggested by Allen Knutson.

A Proof of Lemma 1

Define a $\mathfrak{g}$-valued left invariant 1-form ω on G by $\omega(v) = g^{-1}v$, $\forall v \in T_gG$, $g \in G$. By the Maurer-Cartan structure equation [19], $d\omega = -[\omega, \omega]$. Pulling this back via the map $h : (-\epsilon, \epsilon) \times [t_0, t_1] \to G$ yields $h^*(d\omega) = -[h^*\omega, h^*\omega]$. Evaluating both sides at the vector fields $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$, and noting that $\omega(\dot{h}) = \xi$ and $\omega(h') = \eta$ by definition, we obtain

$$-[h^*\omega(\frac{\partial}{\partial s}), h^*\omega(\frac{\partial}{\partial t})] = -[\omega(dh\frac{\partial}{\partial s}), \omega(dh\frac{\partial}{\partial t})] = -[\omega(h'), \omega(\dot{h})] = -[\eta, \xi] = [\xi, \eta],$$

$$h^*(d\omega)(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}) = d(h^*\omega)(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}) = \frac{\partial}{\partial s}[h^*\omega(\frac{\partial}{\partial t})] - \frac{\partial}{\partial t}[h^*\omega(\frac{\partial}{\partial s})] - h^*\omega([\frac{\partial}{\partial s}, \frac{\partial}{\partial t}]) = \xi' - \dot{\eta}.$$

The desired conclusion follows by equating the above two equations.[4]

B Two Lemmas Used in Section 3.4

Assume that $\mathfrak{so}_n$ $(n \geq 2)$ is equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{so}_n} = \frac{1}{2}\langle \cdot, \cdot \rangle_F$.

Lemma 9 *Suppose that $Y \in \mathfrak{so}_n$ and $\lambda > 0$ are constants. Then the following are equivalent:*

1. $|\langle Y, X \rangle_{\mathfrak{so}_n}| \leq \lambda \|X\|^2_{\mathfrak{so}_n}$ *for all* $X \in \mathfrak{so}_n$ *such that* $e^{2\pi X} = I_n$*;*
2. *The L^2-norm of Y, $\|Y\|_2$, is bounded by λ.*

Proof: $1 \to 2$: For any unit vector $v_1 \in \mathbb{R}^n$ such that $Yv_1 \neq 0$, define $v_2 = Yv_1/\|Yv_1\|$, which is a unit vector orthogonal to v_1 by the skew symmetry of Y. Let $v_1, v_2, \ldots, v_n$ be an orthonormal basis of $\mathbb{R}^n$, hence $A = [v_1|v_2|\ldots|v_n] \in \mathbf{O}_n$. Define $X = AZA^t$, where $Z = (z_{ij}) \in \mathfrak{so}_n$ is such that $z_{21} = -z_{12} = 1$, and $z_{ij} = 0$ otherwise. Then $X \in \mathfrak{so}_n$, and $e^{2\pi X} = I_n$. Hence $|\langle Y, X \rangle_{\mathfrak{so}_n}| \leq \lambda \|X\|^2_{\mathfrak{so}_n} = \lambda$. But $\langle Y, X \rangle_{\mathfrak{so}_n} = \langle Y, AZA^t \rangle_{\mathfrak{so}_n} = \langle A^tYA, Z \rangle_{\mathfrak{so}_n} = v_2^tYv_1 = \|Yv_1\|$. Therefore, $\|Yv_1\| \leq \lambda$. That this holds for every unit vector $v_1 \in \mathbb{R}^n$ implies that $\|Y\|_2 \leq \lambda$.

$2 \to 1$: For each $X \in \mathfrak{so}_n$ with $e^{2\pi X} = I_n$, there exist $A \in \mathbf{O}_n$ and $Z \in \mathfrak{so}_n$ such that $X = AZA^t$, where $Z = \mathrm{diag}(\begin{bmatrix} 0 & -m_1 \\ m_1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & -m_l \\ m_l & 0 \end{bmatrix}, 0, \ldots, 0)$ for some $m_1, \ldots, m_l \in \mathbb{Z}$ $(2l \leq n)$. Write

[4] The use of the Maurer-Cartan equation in this proof was suggested by Alan Weinstein.

$A = [u_1|v_1|\dots|u_l|v_l|w_1|\dots|w_{n-2l}]$ in column vectors. Then

$$|\langle Y, X\rangle_{\mathfrak{so}_n}| = |\langle A^t Y A, Z\rangle_{\mathfrak{so}_n}| = |\sum_{j=1}^{l} m_j v_j^t Y u_j| \le \sum_{j=1}^{l} |m_j| \cdot |v_j^t Y u_j| \le \|Y\|_2 \sum_{j=1}^{l} m_j^2 \le \lambda \|X\|_{\mathfrak{so}_n}^2,$$

since $|v_j^t Y u_j| \le \|v_j\| \|Y\|_2 \|u_j\| = \|Y\|_2$, and $\|X\|_{\mathfrak{so}_n}^2 = \|Z\|_{\mathfrak{so}_n}^2 = \sum_{j=1}^{l} m_j^2$. ■

Lemma 10 *Suppose that $n = 2l$ is even, and that $Y \in \mathfrak{so}_n$ and $\lambda > 0$ are constants such that $|\langle Y, X\rangle_{\mathfrak{so}_n}| \le \lambda \|X\|_{\mathfrak{so}_n}^2$ for all $X \in \mathfrak{so}_n$ satisfying $e^{\pi X} = -I_n$. Then*

$$\frac{1}{n} \sum_{j=1}^{n} \lambda_j \le \lambda,$$

where $\lambda_1, \dots, \lambda_n$ are the singular values of Y.

Proof: Since $Y \in \mathfrak{so}_n$, there exist $A \in \mathbf{O}_n$ and $\omega_1, \dots, \omega_l \ge 0$ such that $Y = AZA^t$, where $Z = \mathrm{diag}(\begin{bmatrix} 0 & -\omega_1 \\ \omega_1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & -\omega_l \\ \omega_l & 0 \end{bmatrix}) \in \mathfrak{so}_n$. Hence the singular values $\lambda_1, \dots, \lambda_n$ of Y are simply $\omega_1, \omega_1, \dots, \omega_l, \omega_l$. Define $X = A \cdot \mathrm{diag}(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}) \cdot A^t$. Then $X \in \mathfrak{so}_n$ and $e^{\pi X} = -I_n$. So by hypothesis, $|\langle Y, X\rangle_{\mathfrak{so}_n}| = \sum_{j=1}^{l} \omega_j \le \lambda \|X\|_{\mathfrak{so}_n}^2 = \lambda l$, which is the desired conclusion. ■

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