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Stability Analysis of Discrete-Time Piecewise-Linear Systems: A Generating Function Approach

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Abstract: This paper studies the exponential stability of a class of discrete-time piecewise-linear systems (DPLS). Some basic properties of the proposed DPLS are established, which enables the generating function approach to be used for the system stability analysis. By introducing the generating functions of DPLS and showing their properties, a sufficient and necessary condition for the exponential stability of DPLS is derived. Furthermore, the maximum exponential growth rate of system trajectories can be obtained exactly by computing the radii of convergence of the generating functions. The algorithm for computing the generating functions is developed and two examples are given to demonstrate the proposed approach.

Keywords: Exponential stability, generating function, growth rate, piecewise-linear systems.

1. INTRODUCTION

Piecewise-linear systems (PLS) have been receiving increasing attention by control community, since many practical control systems can be represented by PLS, especially when piecewise-linear components including deadzone, saturation, relays, and hysteresis are encountered, and many highly nonlinear systems can also be approximated by PLS [1]. In addition, they provide an equivalent framework to the well-known linear complementary systems [2] and mixed logical dynamical systems [3] from the model point of view.

Remarkable progress has been achieved on the stability problem of PLS. Representative approaches to the study of stability for PLS include the construction of common, multiple and surface Lyapunov functions [4-6], which have also been extended to the stabilization problem [7,8] and robust control problem [9,10].

For numerical computation, these approaches restrict the available classes of Lyapunov functions to the class of piecewise quadratic or higher-order functions, which makes the resulting stability conditions conservative. Therefore, a new research trend in the recent work [11-13] is to derive a computable sufficient and necessary condition for the stability of PLS. For a class of planar PLS (PPLS), Iwatani [11] derived an explicit and exact stability test, which was given in terms of the coefficients of the transfer functions of subsystems and was computationally tractable. Arapostathis [12] studied the behavior of PPLS directly, and it was shown that the asymptotic stability of PPLS can be fully modes. On the other hand, by introducing the definition of integral function, Liu [13] provided a necessary and sufficient stability test for PPLS. However, the above results only focus on the PPLS, there is still a few steps away from being effective in the stability analysis of general PLS.

In our previous work [14], a method based on the notion of generating function is proposed to study the stability of discrete-time switched linear systems under three types of switching rules: arbitrary switching, optimal switching and random switching. In this paper, we will extend this method to the discrete-time piecewise linear systems (D-PLS), i.e., switched linear systems with state-dependent switching.

This paper is organized as follows: The system model is described and some basic properties of DPLS are established in Section 2. In Section 3, the generating functions are defined, analyzed and used to characterize the exponential stability of DPLS. Their numerical computation algorithm is also developed. Section 4 shows two examples to demonstrate the generating functions approach and conclusions are drawn in Section 5.

Notation:

- $\text{Int}(S)$: The interior of a set S;
- $\overline{\text{Co}}S$: The convex hull of a set S;
- $\partial(L)$: For a state-space partition $\{R_i\}_{i=1}^{n}$, $\partial(L)$ denotes the index set of the regions R_i containing a set L,i.e. $\partial(L) = \{i \mid \partial(L) \subset R_i\}$
- $d(x,S)$: The distance from a point x to a set S
  $[1,m] := \{1, 2, \cdots, m\text{-}1, m\}$

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2. PROBLEM FORMULATION

2.1. Problem statement

We consider a class of discrete-time piecewise-linear systems represented by

$$x(t+1)=f(x(t)):=\begin{cases}A_1x & if \quad x(t)\in R_1\\ A_2x & if \quad x(t)\in R_2\\ \cdots & \\ A_Mx & if \quad x(t)\in R_M,\end{cases} \tag{1}$$

where A_i are nonsingular matrices and R_i are polyhedron cones with nonempty interior, of the forms

$$R_i=\{x \mid E_ix\le 0\} \tag{2}$$

such that, $R_i\cap R_j=L_{ij}$, $\bigcup_{i=1}^{M}R_i=\mathrm{R}^n$, where L_{ij} denotes the common boundary of R_i and R_j.

To express the system trajectories uniquely, we introduce the following definition of the trajectories of DPLS (1) from the view of autonomous switched systems.

Definition 1: Let $x(t;z,\sigma)$ denotes the solution of DPLS (1) with the initial state z and switch sequence $\sigma\in\Sigma(z)$, where $\Sigma(z)$ denotes the set of all possible switch sequences for DPLS with the initial state z.

Definition 2: At the equilibrium origin, the DPLS (1) is called:

- stable in the sense of Lyapunov if, for each $\varepsilon>0$, there exists $\delta_\varepsilon>0$, such that $\|z\|<\delta_\varepsilon\Rightarrow\|x(t;z,\sigma)\|\le\varepsilon$ for all $t\in Z^+$.
- asymptotically stable if it is stable and all the solutions $x(t;z,\sigma)$ converge to origin.
- exponentially stable if there exist $\delta,\kappa>0$ and $r\in(0,1)$ such that $\|z\|<\delta\Rightarrow\|x(t;z,\sigma)\|\le\kappa r^t\|z\|$ for all $t\in Z^+$ and $\sigma\in\Sigma(z)$.

Our objective are: (i) present a computable sufficient and necessary criterion of exponential stability for continuous-time DPLS (1); (ii) compute the exponential growth rate with conservatism, i.e., the exact r.

2.2. Refinement for state-space partition

This subsection proposes a procedure for recursively refining the state-space partition. We assume the procedure terminates after a finite number of steps, which ensures that some "nice" properties (see Lemma 1 and 2) of DPLS can be concluded under the obtained state-space partition. The key idea of the procedure is to separate each region R_i into several subregions $\{D_{ij}\}_{j=1}^m$ satisfying

$$D_{ij}=\{x\in R_i \ \&\ A_ix\in R_j\}. \tag{3}$$

Obiviously, the new state-space partition can be computed by solving

$$D_{ij}=\{x \mid E_ix\le 0 \ \&\ A_iE_jx\le 0\}. \tag{4}$$

Assumption 1: It is assumed that after finite steps of refinement algorithm, the state-space partition cannot be further refined, i.e., the new state-space partition is identical to the old one.

The discussion about Assumption 1 can be found in [15], a sufficient condition for Assumption 1 to hold have been provided. From Assumption 1 we can learn that the Algorithm 1 will convergence in the finite steps, and we call the obtained state-space partition as the final state-space partition. All the properties in the following are presented for the DPLS with the final state-space partition.

Remark 1: It is observed from the refining procedure that the final state-space partition satisfies that, for each region R_i, there exists a region R_j such that,

$$A_ix\in R_j,\quad \forall x\in R_i. \tag{5}$$

We call R_j the objective transition region of R_i.

2.3. Properties of DPLS

This subsection establishes some basic properties of DPLS with the final state-space partition.

Lemma 1: If R_j is the objective transition region of R_i, then for all $z\in R_i$, $A_iz\in R_j$.

Lemma 2: For all $z_1,z_2\in R_i$, the possible switching sequence set of z_1+z_2 is contained in the possible switching sequence set of either z_1 or z_2, i.e.,

$$\Sigma(z_1+z_2)\subseteq\Sigma(z_1)\cap\Sigma(z_2). \tag{6}$$

The proof of Lemma 1 and Lemma 2 can be found in [19]. Based on Lemma 1 and 2, some properties of the trajectories of DPLS are derived as follows:

Proposition 1: The system trajectories of DPLS (1) have the following properties:

1) (Homogeneity): For all $z\in R_i$ and $\sigma\in\Sigma(z)$, $x(t;kz,\sigma)=kx(t;z,\sigma)$.

2) (Piecewise-Additivity): For all $z_1,z_2\in R_i$ and $\sigma\in\Sigma(z_1+z_2)$, $x(t;z_1+z_2,\sigma)=x(t;z_1,\sigma)+x(t;z_2,\sigma)$.

Proof: 1) Assume the switch sequence $\sigma=(\sigma(0),\sigma(1)\cdots\sigma(t-1)\cdots)$, then we have

$$x(t;kz,\sigma)=A_{\sigma(t-1)}*\cdots A_{\sigma(0)}kz=kx(t;z,\sigma).$$

2) For all $z_1,z_2\in R_i$ and $\sigma\in\Sigma(z_1+z_2)$, it can be implied from Lemma 2 that $\sigma\in\Sigma(z_1)\cap\Sigma(z_2)$, then we have

$$\begin{aligned}x(t;z_1+z_2,\sigma)&=A_{\sigma(t-1)}*\cdots*A_{\sigma(0)}(z_1+z_2)\\&=x(t;z_1,\sigma)+x(t;z_2,\sigma).\end{aligned}$$

3. GENERATIONG FUNCTIONS OF DPLS

In this section, we present the generating function approach to the exponential stability analysis of DPLS.

Definition 3: We define generating function $G(\cdot,z)$ of the DPLS as:

$$G(\lambda,z)=\sup_{\sigma\in\Sigma(z)}\sum_{t=0}^{\infty}\lambda^t\|x(t;z,\sigma)\|^2. \tag{7}$$

For each fixed $\lambda \geq 0$, $G(\lambda, z)$ is a function of z only:

$$G_\lambda(z) := G(\lambda, z). \tag{8}$$

3.1. Properties of generating function

Based on Definition of generating function and Proposition 1, we obtain the following properties.

Proposition 1: $G_\lambda(z)$ has the following properties:

1) (Homogeneity): $I_\lambda(z)$ is homogeneous of degree two in z, i.e., $G_\lambda(\alpha z) = \alpha^2 G_\lambda(z)$, $\alpha > 0$.

2) (Sub-Additivity): For all $z_1, z_2 \in \mathrm{R}^n$,

$$\sqrt{G_\lambda(z_1+z_2)} \leq \sqrt{G_\lambda(z_1)} + \sqrt{G_\lambda(z_2)}. \tag{9}$$

3) (Convexity): For each $\lambda \geq 0$, $\sqrt{G_\lambda(z_1)}$ is a convex function, i.e., for all $z_1, z_2 \in \mathrm{R}^n$ and $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$,

$$\sqrt{G_\lambda(\alpha_1 z_1 + \alpha_2 z_2)} \leq \alpha_1 \sqrt{G_\lambda(z_1)} + \alpha_2 \sqrt{G_\lambda(z_2)}. \tag{10}$$

4) (Common-Bound): For each $\lambda \geq 0$, $I_\lambda(z) < \infty$ for all $z \in \mathrm{R}^n$ implies that $G_\lambda(z) < g_\lambda \|z\|^2$.

Proof: 1) The homogeneity property is a direct sequence of homogeneity property of system solution.

2) It is implied by Propostion1 and Cauchy-Schwartz inequality,

$$\begin{aligned} G_\lambda(z_1+z_2) &= \sup_{\sigma\in\Sigma(z_1+z_2)} \sum_{t=0}^{\infty} \lambda^t \|x(t;z_1+z_2,\sigma)\|^2 \\ &= \sup_{\sigma\in\Sigma(z_1+z_2)} \sum_{t=0}^{\infty} \lambda^t \|x(t;z_1,\sigma)+x(t;z_2,\sigma)\|^2 \\ &= \sup_{\sigma\in\Sigma(z_1+z_2)} \sum_{t=0}^{\infty} \lambda^t [\|x(t;z_1,\sigma)\|^2 + \|x(t;z_2,\sigma)\|^2 \\ &\quad + 2\|x(t;z_1,\sigma)\| \cdot \|x(t;z_2,\sigma)\|]dt \\ &\leq \sup_{\sigma\in\Sigma(z_1+z_2)} \sum_{t=0}^{\infty} \lambda^t \|x(t;z_1,\sigma)\|^2 dt \\ &\quad + \sup_{\sigma\in\Sigma(z_1+z_2)} \sum_{t=0}^{\infty} \lambda^t \|x(t;z_2,\sigma)\|^2 dt \\ &\quad + 2\sqrt{\sup_{\sigma\in\Sigma(z_1)} \sum_{t=0}^{\infty} \lambda^t \|x(t;z_1,\sigma)\|^2 dt} \\ &\quad \sqrt{\sup_{\sigma\in\Sigma(z_2)} \sum_{t=0}^{\infty} \lambda^t \|x(t;z_2,\sigma)\|^2 dt} \\ &= G_\lambda(z_1) + G_\lambda(z_2) + 2\sqrt{G_\lambda(z_1)} \cdot \sqrt{G_\lambda(z_2)} \\ &= \left(\sqrt{G_\lambda(z_1)} + \sqrt{G_\lambda(z_2)}\right)^2. \end{aligned}$$

This implies the result (9).

3) With the help of Sub-Additivity and Homogeneity for $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$, we have

$$\begin{aligned} \sqrt{G_\lambda(\alpha_1 z_1 + \alpha_2 z_2)} &\leq \sqrt{G_\lambda(\alpha_1 z_1)} + \sqrt{G_\lambda(\alpha_2 z_2)} \\ &= \alpha_1 \sqrt{G_\lambda(z_1)} + \alpha_2 \sqrt{G_\lambda(z_2)}. \end{aligned} \tag{11}$$

This proves the Convexity of $G_\lambda(z)$.

4) Assume there exist $\lambda \geq 0$ such that $G_\lambda(z) < \infty$. Let $\{z_i\}_{i=1}^n$ denote a standard basis of R^n, then for any $z \in \mathrm{S}^{n-1}$ (unit-ball of R^n), there exist $\alpha_j \geq 0$, $\sum_{j=1}^n \alpha_j{}^2 = 1$, such that

$$z = \sum_{j=1}^n \alpha_j z_j. \tag{12}$$

Apply the Sub-Additivity of $G_\lambda(z)$ and the Cauchy-Schwartz inequality in the summation form to get that, for all $z \in \mathrm{S}^{n-1}$

$$\begin{aligned} G_\lambda(z) &= \left[\sqrt{G_\lambda(\sum_{j=1}^n \alpha_j z_j)}\right]^2 \leq \left[\sum_{j=1}^n \alpha_j \sqrt{G_\lambda(z_j)}\right]^2 \\ &\leq \sum_{j=1}^n \alpha_j{}^2 . \sum_{j=1}^n G_\lambda(z_j) \leq g_\lambda, \end{aligned} \tag{13}$$

where

$$g_\lambda := n \cdot \max_{j\in\{1,\cdots,n\}} G_\lambda(z_j). \tag{14}$$

By homogeneity, we have $G_\lambda(z) < g_\lambda \|z\|^2$, $z \in \mathrm{R}^n$.

3.2. Exponential stability criterion of DPLS

In this subsection, a sufficient and necessary condition of exponential stability is presented for DPLS via the proposed generating functions.

Theorem 1: The DPLS (1) is strongly exponentially stable if and only if the corresponding generating functions satisfy that $G_1(z)$ is finite for all $z \in \mathrm{R}^n$.

Proof: For sufficiency, suppose for the DPLS (1), $G_1(z) < \infty, \forall z$. Then by property 4 in Proposition 2, there exists a constant g_λ such that $G_1(z) < g_\lambda \|z\|^2$. Thus from the definition of $G_1(z)$ we have

$$\sup_{\sigma\in\Sigma(z)} \sum_{t=0}^{\infty} \|x(t;z,\sigma)\|^2 \leq g_\lambda \|z\|^2,$$

which implies that $\|x(t;z,\sigma)\| \leq \sqrt{g_\lambda} \|z\|$, $\forall t \in Z^+$ and $\|x(t;z,\sigma)\| \to 0$ as $t \to \infty$ due to the convergence of infinite positive series. Consequently, the DPLS is asymptotically stable, hence exponentially stable.

For necessity, assume the DPLS is strongly exponentially stable, i.e., there exists constants $k \geq 1$ and $r \in (0,1)$ such that $\|x(t;z,\sigma)\| \leq kr^t \|z\|$, $\forall t \in Z^+$ and $\sigma \in \Sigma(z)$. Then we have,

$$\begin{aligned} G_1(z) &= \sup_{\sigma\in\Sigma(z)} \sum_{t=0}^{\infty} \|x(t;z,\sigma)\|^2 \\ &\leq \sum_{t=0}^{\infty} k^2 r^{2t} \|z\|^2 = \frac{k^2}{1-r^2} \|z\|^2 < \infty. \end{aligned}$$

Furthermore, using the piecewise-convexity of $G_1(z)$, the following corollary can be concluded to show that the exponential stability of DPLS is fully determined by the generating functions on some finite points, hence computationally tractable.

Corollary 1: The DPLS (1) is strongly exponentially stable if and only if the generating functions are finite on all the unit-vertexes, i.e.,

$$G_1(z_{ij}) < g_\lambda \|z\|^2, \quad i \in [1,m], \quad j \in [1,n_i]. \tag{15}$$

3.3. Characterization of maximum exponential growth rate

In the following subsection, we define the radius of strong convergence of the generating functions. It can be shown this quantity characterizes the maximum exponential growth rate of DPLS exactly.

Definition 4: The radius of convergence of the generating function, denoted by λ^*, is defined as

$$\lambda^* = \sup\left\{\lambda \ge 0 \mid G_\lambda(z) < \infty, z \in \mathrm{R}^n\right\}. \tag{16}$$

Then, the following theorem shows the relationship of the radius of convergence λ^* and the maximum exponential growth rate r^*.

Theorem 2: Given a DPLS with a radius of convergence of generating function λ^*, for any $r > (\lambda^*)^{-1/2}$, there exists a constant k_r such that $\|x(t;z,\sigma)\| \le k_r r^t \|z\|$, $\forall t \in Z^+$, $\sigma \in \Sigma(z)$. Furthermore, $(\lambda^*)^{-1/2}$ is also the smallest value for the previous statement to hold. In other words, the maximum exponential growth rate of DPLS is $r^* = (\lambda^*)^{-1/2}$.

The proof is identical to that in [[15], Corollary 1].

3.4. Computation of generating functions

All the analysis methods proposed in previous subsections require the computation of the generating functions of DPLS. In this subsection, we develop an algorithm for computing the truncations of generating functions as approximations of $G_1(z)$ defined as below.

Definition 5: For each $k \in Z^+$, define

$$G_\lambda^k(z) = \sup_{\sigma \in \Sigma(z)} \sum_{t=0}^{k} \lambda^t \|x(t;z,\sigma)\|^2, \ \forall z \in \mathrm{R}^n.$$

The following proposition lists properties of $G_\lambda^k(z)$.

Proposition 3: For all $\lambda \in (0,\lambda^*)$ and $k \in Z^+$, $G_\lambda^k(z)$ converges exponentially fast to $G_\lambda(z)$, $k \to \infty$, i.e.

$$\left\|G_\lambda(z) - G_\lambda^k(z)\right\| \le g_\lambda \left(1 - \frac{1}{g_\lambda}\right)^{k+1} \|z\|^2, \quad \forall z \in \mathrm{R}^n.$$

Proof: Let $x(t) := x(t;z,\sigma)$, where $\sigma \in \Sigma(z)$ denotes the switch sequence to achieve the supremum of $G_\lambda(z)$, thus we have

$$G_\lambda(x(k-1)) - \lambda G_\lambda(x(k)) = \|x(k-1)\|^2.$$

From property 5 in Proposition 2, this further implies,

$$G_\lambda(x(k-1)) - \lambda G_\lambda(x(k)) \ge \frac{G_\lambda(x(k-1))}{g_\lambda},$$

which is equivalent to

$$G_\lambda(x(k)) \le \frac{1-1/g_\lambda}{\lambda} G_\lambda(x(k-1)).$$

By induction on this inequality, we have

$$\begin{aligned} G_\lambda(z) - G_\lambda^k(z) &= \lambda^{k+1} \sum_{t=0}^{\infty} \|x(t+k+1)\|^2 \\ &= \lambda^{k+1} G_\lambda(x(k+1)) \le \lambda^{k+1} \frac{1-1/g_\lambda}{\lambda} G_\lambda(x(k)) \\ &\le \cdots \le \lambda^{k+1} \left(\frac{1-1/g_\lambda}{\lambda}\right)^{k+1} G_\lambda(z) \\ &\le \left(1-1/g_\lambda\right)^{k+1} g_\lambda \|z\|^2 = g_\lambda \left(1-1/g_\lambda\right)^{k+1} \|z\|^2. \end{aligned}$$

This completes the proof.

Next, to present the algorithm for computing the truncations of generating functions, we first provide the following Lemma.

Lemma 3: For any $k \in Z^+$, the $G_\lambda^k(z)$ satisfies the Bellman Equation, i.e.,

$$G_\lambda^{k+1}(z) = \|z\|^2 + \lambda \cdot \max_{p \in \partial(z)} G_\lambda^k(A_p z), \tag{17}$$

where $\partial(z) = \{i \mid z \in R_i\}$.

The proof can be directly implied by the definition of $G_\lambda^k(z)$, thus omitted.

Lemma 3 actually provides an accurate approach to compute $G_\lambda^k(z)$, which is summarized as Algorithm1. From Proposition 3 we learn that, $G_\lambda^k(z)$ converges exponentially fast to $G_\lambda(z)$ as $k \to \infty$, and the approximation error is bounded by formula (30). Therefore, Algorithm 2 can be applied to compute $G_\lambda(z)$ with any precision as permitted by the numerical computation errors. By repeatedly applying Algorithm 1 to a increasing sequence of λ, an underestimates of λ^* can be obtained.

4. ILLUSTRATIVE EXAMPLES

In this section, we will demonstrate the proposed approach through two numerical examples.

Example 1: Consider the following DPLS

$$A_1 = A_3 = \begin{bmatrix} 0.7 & 0.1 \\ -3 & 0.8 \end{bmatrix}, \quad A_2 = A_4 = \begin{bmatrix} 0.7 & 3 \\ -0.1 & 0.8 \end{bmatrix};$$

$$R_1 = \left\{x \mid x_1 \le 0 \,\&\, x_2 \ge 0\right\}, \quad R_2 = \left\{x \mid x_1 \ge 0 \,\&\, x_2 \ge 0\right\},$$
$$R_3 = \left\{x \mid x_1 \le 0 \,\&\, x_2 \le 0\right\}, \quad R_4 = \left\{x \mid x_1 \ge 0 \,\&\, x_2 \le 0\right\}.$$

First, we need to further refine the original state-space partition. The refinement process is shown in Fig. 1. At step (1), $R_1^{(1)}$ splits into $R_2^{(2)}$ and $R_1^{(2)}$ such that any state trajectory starting from the initial state $x_0 \in R_2^{(2)}$ will enter $R_1^{(2)}$ in one step; While any state trajectory starting from the initial state $x_0 \in R_1^{(2)}$ will remain in $R_1^{(1)}$ in one step. The same procedure is applied for $R_2^{(1)}$, $R_3^{(1)}$ and $R_4^{(1)}$. Thus, we obtain one-step state-space partition. At step (2), $R_2^{(1)}$ splits into $R_2^{(3)}$ and $R_1^{(3)}$, $R_3^{(3)} = R_2^{(2)}$. In the same way, we can obtain $(R_4^{(3)}, R_5^{(3)}, R_6^{(3)})(R_7^{(3)}, R_8^{(3)}, R_9^{(3)})(R_{10}^{(3)}, R_{11}^{(3)}, R_{12}^{(3)})(R_{10}^{(3)}, R_{11}^{(3)}, R_{12}^{(3)})$. These obtained regions consist of the two-step state-space partition. It is noted from the obtained region that, every state trajectory with the initial state $x_0 \in R_1^{(3)}$ transients to $R_2^{(3)}$ in one step. For convenience, we write this as $R_1^{(3)} \to R_2^{(3)}$. Similarly, we have $R_2^{(3)} \to R_3^{(3)} \cdots R_{12}^{(3)} \to R_1^{(3)}$. These observations imply that if running the third step of partition, we have $R_i^{(4)} = R_i^{(3)}$. In other words, the refining procedures terminate.

Algorithm 1 is used to compute generating function for different values of $\lambda: 0.2, 0.4, 0.6, 0.8, 1.0, 1.1$. Fig. 1(d) illustrates the plots of the computed $1/g_\lambda$. Since g_λ at $\lambda = 1$ is finite, the given DPLS is exponentially stable from Theorem 1. Actually, an estimated value of $\lambda = 1.161$ can be obtained by extrapolation method, which shows that the maximum exponential rate $r^* = (\lambda^*)^{-1/2} \approx 0.9285$ from Theorem 2.

Furthermore, the following example shows that the proposed approach is also useful for the stability analysis of DPLS with higher dimensions.

Algorithm 1: Computation of $G_\lambda^K(z)$

1. **Initialize** $k = 0, G_\lambda^k(z) = 1, z^0 = z$;
2. **Repeat** $k \leftarrow k+1$;
3. **for** each z^k, **do**

$$G_\lambda^{k+1}(z^k) = \|z\|^2 + \lambda \cdot \max_{p \in \partial(z^k)} G_\lambda^k(A_p z^k)$$

end for

4. **Set** $p^k = \arg\left(\max_{p \in \partial(z^k)} G_\lambda^k(A_p z^k)\right)$;
5. **Set** $z^{k+1} = A_p z^k$
6. $k = K$
7. **Return** $G_\lambda^K(z)$.

Example 2: Consider the following DPLS

$$A_1 = \begin{bmatrix} 0 & 0 & 1.5 \\ 0.5 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1.6 & 0 \\ 0 & 0 & 0.8 \\ 0.6 & 0 & 0 \end{bmatrix};$$

$$R_1 = \{x \mid x_3 \ge 0\}, \quad R_2 = \{x \mid x_3 \le 0\}.$$

Similar to Example 1, the original state-space partition can be further refined. Fig. 2 illustrate the refinement process. After two steps, the obtained eight subregions are exactly the eight quadrants in standard three-dimensional space. It is easy to verify that every subregion has a unique objective transition region. In other words, the refining process terminate. Therefore, the unit-vertexes of the final state-space partition are $(1,0,0)(-1,0,0)(0,1,0)(0,-1,0)(0,0,1)(0,0,-1)$.

Algorithm 1 is used to compute $G_1(z_{ij})$ for different values of $\lambda: 0.2, 0.4, 0.6, 0.8, 1.0$. Fig. 2(d) illustrates the plots of the computed $1/g_\lambda$. Since g_λ at $\lambda = 1$ is finite, the given DPLS is exponentially stable from

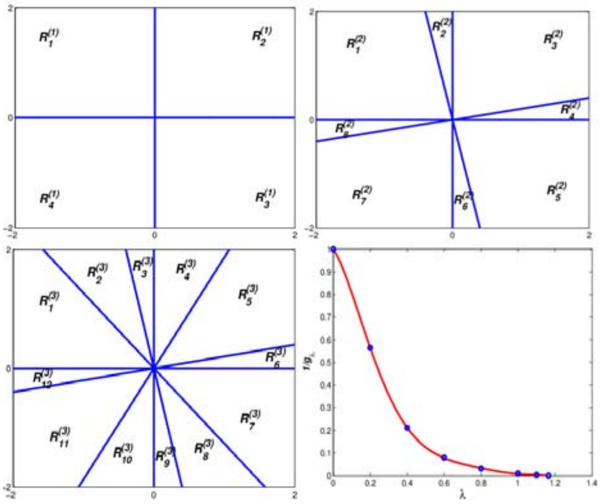

Fig. 1. Simulation results of Example 1.

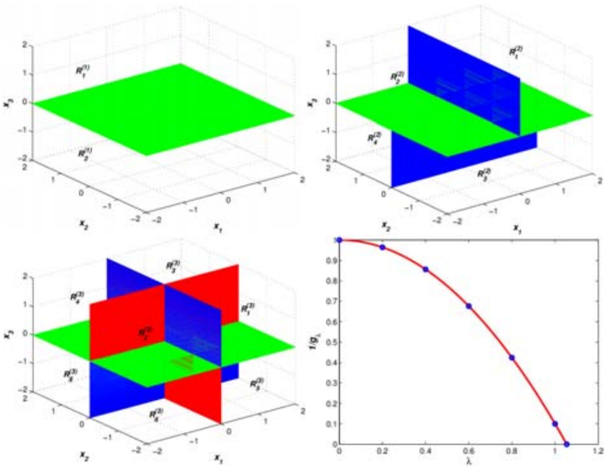

Fig. 2. Simulation results of Example 2.

Theorem 1. Actually, an estimated value of $\lambda = 1.054$ can be obtained by extrapolation method, which shows that the maximum exponential rate $r^* = (\lambda^*)^{-1/2} \approx 0.9740$ from Theorem 2.

5. CONCLUSION

A computational approach to analyzing the stability for a class of DPLS was presented via generating functions together with some quantities derived from generating functions, such as radii of convergence and quadratic bounds. These quantities fully characterize the exponential stability of the DPLS. Our future work will focus on how to extend the proposed approach to the robust stability analysis of DPLS.

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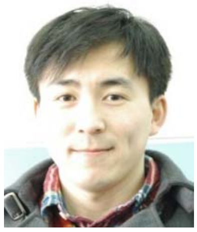

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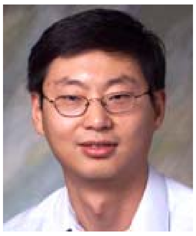

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