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A probabilistic measure of air traffic complexity in three-dimensional airspace

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SUMMARY

In this paper, we propose a new method to evaluate air traffic complexity in 3-D airspace through a probabilistic measure of the airspace occupancy. The key novelty of the approach is that uncertainty in the future aircraft positions is explicitly accounted for when evaluating complexity. Analytic –though approximate– expressions of the complexity measure are derived.
Prospective applications for the proposed complexity metric include the timely identification of those multi-aircraft conflict situations that would be difficult to solve because of limited maneuverability space, and the design of trajectories so as to avoid congested regions that would require many tactical maneuvers to pass them through. Numerical examples are provided to illustrate the approach.

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KEY WORDS: Air traffic management; air traffic complexity; probability of conflict

1. INTRODUCTION

The Air Traffic Management (ATM) system operates to coordinate aircraft and expedite the traffic flow in an airspace. Each aircraft seeks to improve its own performance in terms of factors such as passenger comfort, fuel consumption, and travel time, while sharing resources, namely airspace and runways, with other aircraft. Coordination among aircraft is essential to prevent situations where two or more aircraft come too close to each other, and in extreme cases collide. Hence the ATM system can be modeled as a coordinated multi-agent system with competing agents.

The current ATM system is ground-based and operates on two different time scales. The Air Traffic Control (ATC) function is responsible for ensuring the required separation of aircraft within a mid-term time horizon. The Traffic Flow Management (TFM) function operates to enable an efficient traffic flow pattern over a long-term time horizon [1]. For structured operation of ATM system, the airspace is divided into sectors with each sector having a team of two or three air traffic controllers. Each sector has its capacity constrained by the sustainable workload of the air traffic controllers. The TFM system accounts for this limit

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while optimizing traffic flow. Thus, the workload of the air traffic controllers is implicitly related to the functioning of the TFM system.

With the number of flights increasing, it is becoming more and more evident that the current ATM system is being pushed to the limit. For example, in 2007, there was a 5.3% growth in the air traffic over Europe over 2006, with a disproportionate increase of 17.4 % in the delay [2]. The traffic grew by 0.5% in 2008 compared to 2007 while the total delay increased by 10.4% over 2007 [3]. There is a need for the current capacity to be increased while ensuring that the safety requirements still hold. The currently adopted strategy consists in redistributing and reassigning human resources and reconfiguring sectors so as to maintain the air traffic controllers workload under sustainable levels. On a longer term perspective, the whole ATM system must be rethought and its degree of automation increased in order to adapt the capacity of the ATM system to the grown air traffic demand. In particular, the increase of the air traffic calls for a (partial) delegation of the ATM effort to the involved aircraft, endowed with some degree of autonomy in choosing their preferential routes but also sharing with the air traffic controllers part of the responsibility in maintaining the appropriate separation with other aircraft in the so-called self-separating airspace. Ground control will then assume a new role consisting in a higher level, possibly automated, supervisory function as opposed to lower level human-based control, which should allow an increase in the airspace capacity without compromising safety.

Air traffic complexity is a concept originally introduced to measure the difficulty and effort posed on air traffic controllers. In the current ground-based ATM system, complexity is ultimately related to the air traffic controller workload and complexity measures are generally conceived with the goal of supporting the TFM function in appropriately balancing workload between sectors. In the next generation ATM systems where the ATC function will be partially distributed on board of the aircraft, the availability of complexity metrics can help to predict air traffic situations that could over burden the distributed ATC system, and also benefit the trajectory management operations.

In this paper, we introduce a mid-term complexity measure that is based on the notion of probabilistic occupancy of the airspace: complexity is evaluated in terms of proximity in time and space of the aircraft present in the traffic as determined by their intent and current state, while taking into account uncertainty in the aircraft future position. Indeed, since complexity evaluation on some look-ahead time horizon relies on trajectory prediction, the uncertainty in the aircraft position plays a critical role in complexity evaluation. Specifically, air traffic complexity at a point x in an airspace region $\mathcal{S} \subset \mathbb{R}^3$ and at time t within some look-ahead time horizon T is evaluated as the probability that a certain buffer zone in the airspace surrounding x will be "congested" within $[t, t+\Delta]$, with $\Delta > 0$. By defining congestion as the simultaneous occupancy of the buffer zone by a certain number of aircraft and evaluating this complexity measure at all possible points in $\mathcal{S}$, a complexity map can be built. Forming the complexity maps associated with different consecutive time intervals allows to predict when the aircraft will enter and leave a particular zone in the airspace, and to identify regions of the airspace $\mathcal{S}$ with a limited inter-aircraft maneuverability space. Avoiding these high complexity areas will prevent an additional aircraft entering $\mathcal{S}$ from excessive tactical maneuvering.

According to a similar perspective, in the works on trajectory flexibility [4, 5] it is suggested that, to achieve the aggregate objective of avoiding excessive "air traffic complexity", aircraft should plan their trajectory so as to preserve maneuvering flexibility to accommodate possible disturbances stemming, for example, from additional traffic or from possible deviations of the aircraft from their intended trajectory. In contrast with the input-output approach in [6] and [7] where complexity is measured as the control effort required for implementing a specific optimal conflict resolution strategy to accommodate an additional aircraft, the attempt here is to measure complexity as the effort required for determining a feasible, not necessarily optimal, resolution maneuver. This makes complexity evaluation independent of the adopted optimality criterion and of the actual controller in place, which is accounted for indirectly, through its effect on the air traffic organization. As a consequence, the proposed complexity measure can

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be easily applied in the current ground-based ATM system as well as in the prospective next generation ATM systems. A preliminary version of this paper has appeared in [8].

The rest of the paper is organized as follows. In Section 2, a brief review of the various approaches proposed in the literature to complexity evaluation is given. In Section 3, measures of complexity from a global and a single-aircraft perspective are defined. Analytical – though approximated – expressions for the introduced complexity measures are derived in Section 4. A possible application to trajectory planning is described in Section 3.2. Some illustrative numerical examples are reported in Section 6. Finally, some concluding remarks are drawn in Section 7.

2. EXISTING APPROACHES TO DEFINING COMPLEXITY

Most of the existing approaches to air traffic complexity were conceived with a ground-based ATM system in mind. The work [9] is an early study of the relationship between perceived workload and air traffic characteristics.

It has been noted that aircraft density is only one of the factors that play a significant role in determining the ATC workload. Following inputs from qualified air traffic controllers, dynamic density was introduced as a complexity metric in [10]. Dynamic density is a weighted aggregate of the traffic density and other factors (such as number of heading changes and speed changes), determined through interviews of air traffic controllers. A model for the prediction of dynamic density in advance up to a specified look-ahead time horizon is given in [1]. Note that the problem with a measure of complexity incorporating workload is that workload is difficult to measure and depends on human factors, [11] and [12].

More recently, approaches to air traffic complexity focusing on the traffic structure itself rather than the ATC workload have been proposed, see e.g. [13], [14], and [15].

Workload independent metrics can be classified into control-dependent or control-independent, depending on whether the controller is explicitly accounted for or not. The input-output approach in [6], [7] evaluates complexity based on the control effort needed to safely introduce an additional aircraft into the airspace, and hence is control-dependent. Control-independent metrics, instead, account for the controller indirectly, through its effect on the air traffic organization. Measures of the intrinsic complexity of the traffic were discussed in [16], [14], [17], and [15]. The underlying idea of all these approaches is to interpret the observed traffic as the flow of some vector field and then derive some indicator of complexity based on the identified vector field. For example, a spatial map of (local) complexity is derived in [17] based on the Lyapunov exponents of the nonlinear dynamical system generating air traffic. A different type of intrinsic complexity measure is the fractal dimension of the air traffic pattern introduced in [13], which is a scalar measure of the geometric complexity of the whole set of trajectories observed over an infinite time horizon. The fact that timing information is lost is a main limitation of fractal dimension as a complexity measure for timely detecting critical encounter situations.

The problem of generating complexity metrics relevant to the prevention of critical encounter situations is strictly related to the Conflict Detection and Resolution (CD&R) problem. Conflict probability is one metric used in evaluating the criticality of aircraft encounters [18]. Approaches to evaluating the probability of conflict involve predicting aircraft trajectories into the future and computing the probability that two aircraft get closer than a minimum required distance based on these trajectories [19], [20], [21], [22]. A Petri net model combined with a Sequential Monte Carlo approach was used in [23], [24],[25] to estimate the criticality of a free flight operational concept. A Monte Carlo approach for conflict resolution in presence of uncertainty in aircraft dynamics is given in [26]. Uncertainty in aircraft position is a key factor in trajectory prediction and hence is closely related to the CD&R problem. Despite the extensive studies on uncertainty in the modeling and analysis of ATM systems by various researchers, its effect on air traffic complexity evaluation has not received adequate attention. Our work represents an attempt towards this direction.

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3. PROPOSED DEFINITION OF AIR TRAFFIC COMPLEXITY

Consider N aircraft $A_i, i = 1, \ldots, N$, flying in the 3-D airspace $\mathcal{S} \subset \mathbb{R}^3$ during the look-ahead time horizon $T = [0, \bar{t}]$, with $t = 0$ representing the current time instant and $\bar{t} > 0$ the time horizon length. Suppose that each aircraft is following a nominal trajectory with a velocity profile $u^{A_i} : T \to \mathbb{R}^3$, starting from the initial position $x_0^{A_i}$ at time $t = 0$. The aircraft future position during T is not known exactly, and we assume that the prediction error can be modeled through a Gaussian random perturbation whose variance grows not only linearly with time t but also faster in the along-track direction (namely the direction of u^{A_i}) than in the cross-track directions (i.e., directions orthogonal to u^{A_i}). The predicted position $x^{A_i}(t) \in \mathbb{R}^3$ at time $t \in T$ of aircraft A_i is then given by

$$x^{A_i}(t) = x_0^{A_i} + \int_0^t u^{A_i}(s)ds + Q^{A_i}(t)\Sigma^{A_i} B^{A_i}(t), \quad t \geq 0, \tag{1}$$

where $B^{A_i}(t)$ is a standard 3-D Brownian motion starting from the origin whose variance is modulated by the matrix $Q^{A_i}(t)\Sigma^{A_i} \in \mathbb{R}^{3\times3}$. More precisely, $\Sigma^{A_i} = \mathrm{diag}(\sigma_1^{A_i}, \sigma_2^{A_i}, \sigma_3^{A_i})$ is a diagonal matrix whose entries $\sigma_1^{A_i}$, $\sigma_2^{A_i}$, and $\sigma_3^{A_i}$ are the variance growth rates of the perturbation in the along-track direction and the two cross-track directions and satisfy $\sigma_1^{A_i} \geq \sigma_2^{A_i} = \sigma_3^{A_i} > 0$, whereas $Q^{A_i}(t) = \begin{bmatrix} q_1^{A_i}(t) & q_2^{A_i}(t) & q_3^{A_i}(t) \end{bmatrix} \in \mathbb{R}^{3\times3}$ is an orthogonal matrix whose first column $q_1^{A_i}(t)$ is aligned with $u^{A_i}(t)$: $q_1^{A_i}(t) = \frac{u^{A_i}(t)}{\|u^{A_i}(t)\|}$. Similar models have been proposed in [27], [20] and [19] for predicting aircraft trajectories over a mid-term look-ahead time horizon of tens of minutes.

For each $x \in \mathcal{S}$, let us consider the ellipsoidal region $\mathcal{M}(x)$ centered at x and defined as:

$$\mathcal{M}(x) = \left\{ \hat{x} \in \mathbb{R}^3 : \ (\hat{x} - x)^T M (\hat{x} - x) \leq 1 \right\}, \tag{2}$$

where $M \in \mathbb{R}^{3\times3}$ is a diagonal matrix given by

$$M = \mathrm{diag}\left(\frac{1}{{r_h}^2}, \frac{1}{{r_h}^2}, \frac{1}{{r_v}^2} \right),$$

with $r_h \geq r_v > 0$ defining the size of the ellipsoid in the horizontal plane and in the vertical direction. If $r_h = r_v$, then the ellipsoid reduces to a sphere of radius r_h, and proximity in the horizontal plane is weighted the same as that in the vertical direction. Typically, $r_h > r_v$ since vertical proximity between aircraft is considered in ATM to be less critical than horizontal proximity.

3.1. *Complexity from a global perspective*

The complexity of the air traffic within $\mathcal{S}$ can be evaluated through the following *first and second order complexity measures.*

Definition *1* (first order complexity)
The first order complexity $\mathrm{c}_1(x, [t_s, t_f])$ at position $x \in \mathcal{S}$ within the time interval $[t_s, t_f] \subseteq T$ is defined as

$$\mathrm{c}_1(x, [t_s, t_f]) := P\left(x^{A_i}(t) \in \mathcal{M}(x), \text{ for some } t \in [t_s, t_f] \text{ and } i \in \{1, 2, \ldots, N\} \right) \tag{3}$$

and represents the probability of at least one aircraft entering the ellipsoid $\mathcal{M}(x)$ within the time frame $[t_s, t_f]$.

Note that $\mathrm{c}_1(x, [t_s, t_f]) = 0$ means that none of the existing aircraft will be inside the ellipsoid $\mathcal{M}(x)$ during the time interval $[t_s, t_f]$. On the other hand, $\mathrm{c}_1(x, [t_s, t_f]) = 1$ implies that with certainty there will be at least one aircraft within $\mathcal{M}(x)$ at some time instant belonging to $[t_s, t_f]$.

Similarly, we can define the second order complexity measure.

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Definition *2* (second order complexity)
The second order complexity $\mathrm{c}_2(x,[t_s,t_f])$ at position $x \in \mathcal{S}$ within the time interval $[t_s,t_f] \subseteq T$ is defined as

$$\mathrm{c}_2(x,[t_s,t_f]) := P\left(x^{A_i}(t) \text{ and } x^{A_j}(t') \in \mathcal{M}(x) \text{ for some } t,t' \in [t_s,t_f] \text{ and } i \neq j \in \{1,2,\ldots,N\}\right) \tag{4}$$

and represents the probability of at least two aircraft entering the ellipsoid $\mathcal{M}(x)$ within the time frame $[t_s,t_f]$.

If $\mathrm{c}_2(x,[t_s,t_f]) = 0$, then there will be at most a single aircraft inside the ellipsoid $\mathcal{M}(x)$ within the time interval $[t_s,t_f]$. Hence, at any time $t \in [t_s,t_f]$, an aircraft passing through $\mathcal{M}(x)$ will not be sharing $\mathcal{M}(x)$ with any of the other N aircraft. If $\mathrm{c}_2(x,[t_s,t_f]) = 1$, then with probability 1, at least two aircraft will enter the ellipsoid $\mathcal{M}(x)$ during the time interval $[t_s,t_f]$, though possibly not at exactly the same time.

By letting x vary over $\mathcal{S}$, one can define the *first order and second order complexity maps* of the airspace region $\mathcal{S}$ within the time frame $[t_s,t_f]$ as follows:

$$\mathcal{C}_1(\cdot,[t_s,t_f]) : x \in \mathcal{S} \to \mathrm{c}_1(x,[t_s,t_f])$$
$$\mathcal{C}_2(\cdot,[t_s,t_f]) : x \in \mathcal{S} \to \mathrm{c}_2(x,[t_s,t_f]).$$

Evidently, at any point $x \in \mathcal{S}$, the $\mathcal{C}_2$ map has a value smaller than or equal to the $\mathcal{C}_1$ map, since the corresponding events are nested.

Higher order complexity measures and maps can also be defined according to a similar procedure.

The availability of these complexity maps could be a useful support for trajectory management and CD&R operations. Forming the complexity maps for different consecutive time intervals allows to predict when the aircraft enter and leave a certain zone in the airspace, and to define the occupancy of the airspace region $\mathcal{S}$. This information can be used for detecting congested areas (i.e., areas where multi-aircraft encounters with limited inter-aircraft spacing are likely to occur) in the time-space coordinates, and to identify surrounding areas where the traffic could be deviated. The presence of a region with a high value of the second order complexity implies a high likelihood that two or more aircraft will get close in time and space, hence having a conflict. The traffic flow should be designed so as to reduce second order complexity.

The introduced complexity measures can be easily adapted to provide a measure of complexity from the perspective of an additional aircraft that is entering $\mathcal{S}$. This can be useful for trajectory design purposes and appears particularly interesting for the prospective next generation ATM systems, where aircraft entering a self-separating airspace region are allowed to choose their preferential route while being responsible for maintaining the appropriate separation from the other traffic in $\mathcal{S}$.

3.2. Complexity from a single aircraft perspective

According to Definitions 3 and 4, complexity is evaluated as the probability of occupancy of a buffer zone surrounding a point by a certain number of the aircraft A_i, $i = 1,2,\ldots,N$, that are present in the airspace region $\mathcal{S}$ (at least one aircraft for the first order complexity and at least two for the second order complexity).
Suppose now that an additional aircraft, say aircraft B, is entering $\mathcal{S}$ at time 0 following a nominal trajectory $\bar{x}^B : T \to \mathbb{R}^3$. The idea is to evaluate the complexity encountered by aircraft B along its nominal trajectory by making the buffer zone move along the trajectory of aircraft B and computing the probability that some of the other aircraft A_i, $i = 1,2,\ldots,N$, will enter such a moving zone. This leads to the following definition of single-aircraft complexity.

Definition *3* (single-aircraft complexity)
The complexity experienced by aircraft B along its nominal trajectory $\bar{x}^B : T \to \mathcal{S}$ within the

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time interval $[t_s, t_f]$ is defined as:

$$c(t_s, t_f) := P\left(x^{A_i}(t) \in \mathcal{M}(\bar{x}^B(t)) \text{ for some } t \in [t_s, t_f] \text{ and } i \in \{1, 2, \dots, N\}\right) \quad (5)$$

Remark 1
Interestingly, if the time window $[t_s, t_f]$ extends to the whole look-ahead time horizon T and the buffer zone reproduces the protection zone surrounding each aircraft, the single-aircraft complexity measure can as well be interpreted as the probability of aircraft B getting in conflict with another aircraft A_i within T. CD&R then becomes an integrable task in complexity evaluation.

From an operational perspective, the introduced single-aircraft complexity measure can be used by aircraft B to evaluate the maneuverability space surrounding its nominal trajectory and to eventually redesign its trajectory. An approach that exploits this measure of complexity for trajectory design is described in Section 5.

4. COMPUTING THE COMPLEXITY MEASURES AND MAPS

In this section we address the issue of determining analytic –though approximate– expressions of the complexity measures $c_1(x, [t_s, t_f])$ and $c_2(x, [t_s, t_f])$ representing the the probability of multiple (at least one for c_1 and at least two for c_2) aircraft entering the same buffer zone $\mathcal{M}(x)$ within $[t_s, t_f]$. Approaches to computing this probability are usually computationally intensive as the required computing time typically grows exponentially with the number of aircraft, e.g. [21]. Here, analytical formulas approximating this probability will be derived, which results in a linear growth of computation time with the number of aircraft. These analytic formulas can be extended to estimate the single-aircraft complexity measure $c(t_s, t_f)$, as explained in Section 4.4.

Denote as $P^{A_i}(x, [t_s, t_f])$ the probability that aircraft A_i enters the ellipsoid $\mathcal{M}(x)$ centered at $x \in \mathcal{S}$ within the time frame $[t_s, t_f]$:

$$P^{A_i}(x, [t_s, t_f]) := P\left(x^{A_i}(t) \in \mathcal{M}(x) \text{ for some } t \in [t_s, t_f]\right). \quad (6)$$

If the Brownian motions affecting the future positions of the N aircraft are assumed to be independent, then the first order and second order complexity measures (3) and (4) satisfy:

$$c_1(x, [t_s, t_f]) = 1 - \prod_{i=1}^{N} (1 - P^{A_i}(x, [t_s, t_f])) \quad (7)$$

$$c_2(x, [t_s, t_f]) = 1 - \prod_{i=1}^{N} (1 - P^{A_i}(x, [t_s, t_f])) - \sum_{i=1}^{N} P^{A_i}(x, [t_s, t_f]) \prod_{j=1, j\neq i}^{N} (1 - P^{A_j}(x, [t_s, t_f])), \quad (8)$$

and the problem of evaluating complexity reduces to that of estimating the probability $P^{A_i}(x, [t_s, t_f])$.

Remark 2
Note that the assumption of independent Brownian motions is reasonable if the correlation due to the effect of wind is negligible and, in particular, if the N aircraft are flying far apart, [21], [28]. In presence of non-negligible wind correlation, the above expressions represent only approximations of the complexity measures c_1 and c_2.

We next determine an analytic approximation of the probability $P^{A_i}(x, [t_s, t_f])$ in (6). The obtained approximate expression is then used to estimate $c_1(x, [t_s, t_f])$ and $c_2(x, [t_s, t_f])$ through (7) and (8). Derivations refer to aircraft following straight line nominal trajectories with constant velocity. Possible extensions to the case of multi-legged nominal trajectories specified by a sequence of timed way points are discussed in the concluding section.

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4.1. Analytical approximation of $P^{A_i}(x, [t_s, t_f])$

For ease of notation, in this subsection we shall refer to aircraft A_i as aircraft A, dropping the subscript.

Under the assumption that aircraft A is following a straight line nominal trajectory with constant velocity, equation (1) can be rewritten as

$$x^A(t) = x_0^A + u^A t + Q^A \Sigma^A B^A(t), \quad t \geq 0.$$

From this equation, we have that the relative position $\Delta x(t) = x - x^A(t)$ of aircraft A with respect to the point x is given by

$$\Delta x(t) = \Delta x_0 + \Delta u\, t - n(t), \tag{9}$$

where we set $\Delta x_0 = x - x_0^A$, $\Delta u = -u^A$ and $n(t) = Q^A \Sigma^A B(t)$.

Equation (9) suggests that we can evaluate $P^A(x, [t_s, t_f])$ by determining the probability that the perturbation $n(t)$ hits an ellipsoid whose center is moving at a constant velocity Δu starting from Δx_0: $n(t) \in \mathcal{M}(\Delta x_0 + \Delta u\, t)$, for some $t \in [t_s, t_f]$.

Define the vector

$$v = \Gamma^{-1} \Delta u$$

with $\Gamma := Q^A \Sigma^A$. An orthogonal matrix $P = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} \in \mathbb{R}^{3\times 3}$ can be constructed whose first column $p_1 = -v/\|v\|$ is aligned with $-v$ (the choice of p_2 and p_3, hence P, is not unique). Its inverse $P^{-1} = P^T$ represents a rotation that makes the $-v/\|v\|$ direction coincide with the first coordinate axis direction:

$$P^{-1} \frac{-v}{\|v\|} = e_1 := \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T.$$

Using the coordinate transformation $\Delta z(t) = P^{-1}\Gamma^{-1}\Delta x(t)$, we transform (9) to the following:

$$\Delta z(t) = a + wt - \hat{B}(t), \tag{10}$$

where $\hat{B}(t) := P^{-1}B(t)$ is still a standard Brownian motion starting from the origin (rotation of a Brownian motion is still a Brownian motion), and $a \in \mathbb{R}^3, w \in \mathbb{R}^3$ are defined by

$$a = P^{-1}\Gamma^{-1}\Delta x_0, \qquad w = P^{-1}\Gamma^{-1}\Delta u = P^{-1}v = -\|v\| e_1.$$

From equation (10), we can again think of computing $P^A(x, [t_s, t_f])$ as determining the probability that the standard Brownian motion $\hat{B}(t)$ starting from the origin hits a moving ellipsoid E_t obtained by transforming $\mathcal{M}(\Delta x_0 + \Delta u t)$ in the new coordinate system, that is:

$$P^A(x, [t_s, t_f]) = P\left(\hat{B}(t) \in E_t \text{ for some } t \in [t_s, t_f]\right), \tag{11}$$

where

$$E_t = \left\{ z : \ (z - (a + wt))^T P^T \Gamma^T M \Gamma P \left(z - (a + wt)\right) \leq 1 \right\}.$$

The center of E_t is moving at the constant velocity w starting from a. From our choice of matrix P, the velocity w is directed along the negative Δz_1 axis (see Figure 1).

We next find an analytical approximation for the probability in (11). For a fixed time $t \in [t_s, t_f]$, define p_t as the plane that passes through the center $\Delta z^c = a + wt$ of the ellipsoid E_t and is orthogonal to its velocity w (hence orthogonal to the Δz_1-axis). Then p_t divides the ellipsoid E_t into two equal parts. We shall first find the projection of E_t onto the plane p_t and then the minimum bounding rectangle containing such a projection.

Let $\pi_t : \mathbb{R}^3 \to p_t$ be the orthogonal projection operator onto the two dimensional plane p_t which can be identified with $\mathbb{R}^2$. Then, the projection $\tilde{E}_t := \pi_t(E_t)$ of E_t onto p_t is itself an ellipse centered at $\tilde{z}^c := \pi_t(\Delta z^c) \in \mathbb{R}^2$. Specifically, a point $\tilde{z} \in \mathbb{R}^2$ belongs to $\tilde{E}_t$ if and only if

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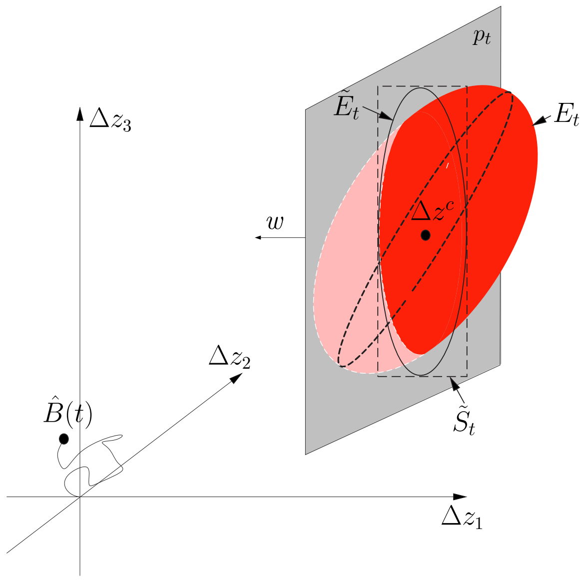

Figure 1. Ellipsoidal protection zone E_t in the new coordinates.

there exists some $z_1 \in \mathbb{R}$ such that $\Delta z = \begin{bmatrix} z_1 \\ \tilde{z} \end{bmatrix} \in E_t$ or, equivalently,

$$(\Delta z - \Delta z^c)^T \, P^T \Gamma^T M \Gamma P \, (\Delta z - \Delta z^c) \leq 1. \tag{12}$$

Write $P^T \Gamma^T M \Gamma P$ in block matrix form as:

$$P^T \Gamma^T M \Gamma P = \begin{bmatrix} \alpha & y^T \\ y & R \end{bmatrix},$$

where $R \in \mathbb{R}^{2\times 2}$, $y \in \mathbb{R}^2$, and $\alpha \in \mathbb{R}$. Since $P^T \Gamma^T M \Gamma P$ is positive definite, we must have that R is positive definite and $\alpha > 0$. Then condition (12) is equivalent to

$$\alpha (z_1 - \Delta z_1^c)^2 + 2 y^T \, (\tilde{z} - \tilde{z}^c) \, (z_1 - \Delta z_1^c) + (\tilde{z} - \tilde{z}^c)^T \, R \, (\tilde{z} - \tilde{z}^c) \leq 1, \tag{13}$$

for some $z_1 \in \mathbb{R}$. The left-hand-side of (13) is a quadratic function in z_1 whose minimum with respect to z_1 is given by

$$(\tilde{z} - \tilde{z}^c)^T \, R \, (\tilde{z} - \tilde{z}^c) - \frac{1}{\alpha} \left[y^T \, (\tilde{z} - \tilde{z}^c) \right]^2 = (\tilde{z} - \tilde{z}^c)^T \, \tilde{R} \, (\tilde{z} - \tilde{z}^c),$$

where $\tilde{R} \in \mathbb{R}^{2\times 2}$ is the positive definite matrix defined by

$$\tilde{R} := R - \frac{y y^T}{\alpha}.$$

Condition (13) is equivalent to that its left-hand-side minimum with respect to z_1 is smaller that its right-hand-side. This yields the exact expression of the projection ellipse $\tilde{E}_t$ as:

$$\tilde{E}_t = \left\{ \tilde{z} \in \mathbb{R}^2 \; : \; (\tilde{z} - \tilde{z}^c)^T \, \tilde{R} \, (\tilde{z} - \tilde{z}^c) \leq 1 \right\}.$$

We now determine the rectangle on p_t that encloses $\tilde{E}_t$ and has the smallest area, namely, the *minimum bounding rectangle* of $\tilde{E}_t$. This rectangle, denoted by $\tilde{S}_t$, will be used in the approximation of the probability $P^A(x, [t_s, t_f])$.

We decompose the positive definite matrix $\tilde{R}$ as

$$\tilde{R} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \operatorname{diag}(\lambda_1, \lambda_2) \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T, \tag{14}$$

where λ_1 and λ_2 are the eigenvalues of $\tilde{R}$ with $\lambda_1 \geq \lambda_2 > 0$, and v_1 and v_2 are the corresponding eigenvectors which can be assumed to be an orthonormal pair. Then v_2 identifies the direction of the major axis of the ellipse $\tilde{E}_t$, along which $\tilde{E}_t$ has length $\frac{2}{\sqrt{\lambda_2}}$, whereas v_1 identifies the minor axis direction, along which $\tilde{E}_t$ has length $\frac{2}{\sqrt{\lambda_1}}$. As a result, the minimum bounding rectangle $\tilde{S}_t$ of the ellipse $\tilde{E}_t$ is the one centered at $\tilde{z}^c$, with length $\frac{2}{\sqrt{\lambda_2}}$ along the v_2 direction and $\frac{2}{\sqrt{\lambda_1}}$ along the v_1 direction.

Recall that the probability of interest $P^A(x, [t_s, t_f])$ that aircraft A enters the ellipsoid $\mathcal{M}(x)$ within $[t_s, t_f]$ is expressed in (11) as the probability that the Brownian motion $\hat{B}(t)$ starting from the origin hits the ellipsoid E_t whose center moves in the negative Δz_1-axis direction starting from a at time 0. The analytical expression of such a probability is difficult to obtain. We then suggest to approximate it by the probability that, when $\hat{B}(t)$ first hits the moving plane p_t, the hitting location is inside the minimum bounding rectangle $\tilde{S}_t$ of the projected ellipse $\tilde{E}_t$.

Remark 3
The idea underlying this approximation scheme is that, since the velocity w of the ellipsoid is typically much larger than the growth rate of the variance of the Brownian motion, then, the only dimension of the ellipsoid that is relevant for the event of interest is that perpendicular to w. A similar approximation scheme was used in [19] with reference to the problem of computing the probability of conflict in the 2-D case. A formal discussion on the quality of the approximation is reported in [29]. The additional error that is introduced in the 3-D case, when the minimum bounding rectangle $\tilde{S}_t$ is used in place of the projected ellipse $\tilde{E}_t$, contributes to the over-approximation of $P^A(x, [t_s, t_f])$ and, hence, to the overestimation of the complexity measures.

Define $\tau := \inf\{t \geq 0 \;:\; \hat{B}(t) \in p_t\}$ to be the first time that the Brownian motion $\hat{B}(t)$ ever hits the moving plane p_t. Since the three coordinates of $\hat{B}(t)$ are independent one-dimensional Brownian motions, and the directions orthogonal to plane p_t and along which the plane p_t is moving are both aligned with the Δz_1-axis, it is easy to see that τ depends only on the first component $\hat{B}_1(t)$ of $\hat{B}(t)$. Specifically, τ is the first time that the one-dimensional Brownian motion $\hat{B}_1(t)$ starting from the origin hits a point $z_1(t) \in \mathbb{R}$ that moves according to the dynamics $z_1(t) = a_1 - \|v\|t$, where a_1 is the first component of $a \in \mathbb{R}^3$.

Note that $a_1 < 0$ implies that the aircraft A is moving away from the ellipsoid $\mathcal{M}(x)$ in the x-coordinates and results in approximately zero probability of entering $\mathcal{M}(x)$. For the purpose of complexity evaluation, we then set $P^A(x, [t_s, t_f]) = 0$ when $a_1 < 0$.

When $a_1 \geq 0$, the probability distribution of τ is characterized by the following lemma.

Lemma *1* (Bachelier-Levy, [30])
Define $\tau := \inf\{t \geq 0 : \; \hat{B}_1(t) = a_1 - \|v\|t\}$ to be the first time the 1-D Brownian motion $\hat{B}_1(t)$ starting from the origin reaches a point moving at the speed $\|v\|$ towards the origin starting from some $a_1 \geq 0$. Then, τ has the probability density function:

$$p_\tau(t) = \frac{a_1}{\sqrt{2\pi t^3}} e^{-\frac{(a_1 - \|v\|t)^2}{2t}}, \quad t \geq 0. \tag{15}$$

Lemma *2* ([29])
Let $p_\tau(t)$ be the probability density function of τ as given in (15) and assume $a_1 \geq 0$. Then

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for any $t_f \geq 0$,

$$\int_0^{t_f} p_\tau(t)\,dt = Q\left(a_1 t_f^{-1/2} - \|v\| t_f^{1/2}\right) + e^{2a_1\|v\|} Q\left(a_1 t_f^{-1/2} + \|v\| t_f^{1/2}\right), \tag{16}$$

$$\int_0^{t_f} t\,p_\tau(t)\,dt = \frac{a_1}{\|v\|} Q\left(a_1 t_f^{-1/2} - \|v\| t_f^{1/2}\right) - \frac{a_1}{\|v\|} e^{2a_1\|v\|} Q\left(a_1 t_f^{-1/2} + \|v\| t_f^{1/2}\right),$$

where $Q(x) := \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$ is the Q-function, which is related to the error function $\mathrm{erf}(\cdot)$ by: $Q(x) = \frac{1}{2} - \frac{1}{2}\mathrm{erf}(\frac{x}{\sqrt{2}})$. In particular, letting $t_f \to \infty$, we have $\int_0^\infty p_\tau(t)\,dt = 1$ and

$$E[\tau] = \int_0^\infty t\,p_\tau(t)\,dt = \frac{a_1}{\|v\|}.$$

Let $\pi_\tau(\hat{B}(t)) \in \mathbb{R}^2$ be the projection of the Brownian motion $\hat{B}(t)$ at the hitting time τ onto the plane p_t. Conditioning on $\tau = t$, the distribution of $\pi_\tau(\hat{B}(t))$ is a two-dimensional Gaussian random variable with zero mean and covariance tI_2; hence $\pi_\tau(\hat{B}(t)) \sim W_1 v_1 + W_2 v_2$, where v_1 and v_2 are the orthonormal pair given in (14), and $W_1, W_2 \sim N(0,t)$ are independent one-dimensional Gaussian random variables. Moreover, by our previous discussions, the minimum bounding rectangle $\tilde{S}_t$ can be expressed as the set of all $\alpha_1 v_1 + \alpha_2 v_2$ with $|\alpha_1 - v_1^T \tilde{a}| \leq \frac{1}{\sqrt{\lambda_1}}$ and $|\alpha_2 - v_2^T \tilde{a}| \leq \frac{1}{\sqrt{\lambda_2}}$ where $\tilde{a} = \pi(a)$ is the projection of a onto p_t, which coincides with $\tilde{z}^c$.

As a result, $g(t) := P\left(\pi_\tau(\hat{B}(t)) \in \tilde{S}_\tau \mid \tau = t\right)$ can be computed as follows

$$\begin{aligned} g(t) &= P\left(W_1 v_1 + W_2 v_2 \in \tilde{S}_t\right) \\ &= P\left(\left|W_1 - v_1^T \tilde{a}\right| \leq \frac{1}{\sqrt{\lambda_1}}\right) \cdot P\left(\left|W_2 - v_2^T \tilde{a}\right| \leq \frac{1}{\sqrt{\lambda_2}}\right) \\ &= \left[Q\left(\frac{v_1^T \tilde{a}\sqrt{\lambda_1} - 1}{\sqrt{\lambda_1 t}}\right) - Q\left(\frac{v_1^T \tilde{a}\sqrt{\lambda_1} + 1}{\sqrt{\lambda_1 t}}\right)\right]\left[Q\left(\frac{v_2^T \tilde{a}\sqrt{\lambda_2} - 1}{\sqrt{\lambda_2 t}}\right) - Q\left(\frac{v_2^T \tilde{a}\sqrt{\lambda_2} + 1}{\sqrt{\lambda_2 t}}\right)\right]. \end{aligned} \tag{17}$$

Finally, an approximated expression for $P^A(x, [t_s, t_f])$ can be computed as

$$\hat{P}^A(x, [t_s, t_f]) = \int_{t_s}^{t_f} P\left(\tilde{z}^A(\tau) \in \tilde{S}_\tau \mid \tau = t\right) p_\tau(t)\,dt. \tag{18}$$

Evaluating expression (18) involves an integration, which may be time-consuming. Thus, simplified expressions that are easier to compute are needed. One way to approximate (18) is to expand $g(t)$ around $t_e := E[\tau | t_s \leq \tau \leq t_f]$. If a zero-th order expansion is used, then

$$\hat{P}^A(x, [t_s, t_f]) \simeq g(t_e) \int_{t_s}^{t_f} p_\tau(t)\,dt, \tag{19}$$

which can then be evaluated using (16) in Lemma 2. The time instant t_e is the expected time that the Brownian motion $\hat{B}(t)$ hits the plane p_t conditioning on that it hits within the time interval $[t_s, t_f]$ and is given by

$$t_e = \frac{\int_{t_s}^{t_f} t\,p_\tau(t)\,dt}{\int_{t_s}^{t_f} p_\tau(t)\,dt},$$

where $\int_{t_s}^{t_f} t\,p_\tau(t)\,dt = \int_0^{t_f} t\,p_\tau(t)\,dt - \int_0^{t_s} t\,p_\tau(t)\,dt$ can be evaluated using Lemma 2.

Using a first order approximation of $g(t) \simeq g(t_e) + (t - t_e)\dot{g}(t_e)$ around $t = t_e$, we obtain

$$\hat{P}^A(x, [t_s, t_f]) \simeq [g(t_e) - t_e \dot{g}(t_e)] \int_{t_s}^{t_f} p_\tau(t)\,dt + \dot{g}(t_e) \int_{t_s}^{t_f} t\,p_\tau(t)\,dt, \tag{20}$$

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where $\dot{g}(t) = \frac{dg(t)}{dt}$ can be computed from (17) using the fact that $Q(x) = \int_x^\infty e^{-z^2/2}dz$ as

$$\dot{g}(t_e) = -\frac{\sqrt{2\pi}}{2t_e}\left\{[Q(u_1) - Q(u_2)](u_4 e^{-{u_4}^2} - u_3 e^{-{u_3}^2}) + [Q(u_3) - Q(u_4)](u_2 e^{-{u_2}^2} - u_1 e^{-{u_1}^2})\right\},$$

with

$$u_1 := \frac{v_1^T \tilde{a}\sqrt{\lambda_1} - 1}{\sqrt{\lambda_1}t},\ u_2 := \frac{v_1^T \tilde{a}\sqrt{\lambda_1} + 1}{\sqrt{\lambda_1}t},\ u_3 := \frac{v_2^T \tilde{a}\sqrt{\lambda_2} - 1}{\sqrt{\lambda_2}t} \text{ and } u_4 := \frac{v_2^T \tilde{a}\sqrt{\lambda_2} + 1}{\sqrt{\lambda_2}t}.$$

4.2. Analytic approximation of the complexity measures $c_1(x, [t_s, t_f])$ and $c_2(x, [t_s, t_f])$

An analytic approximation of $c_1(x, [t_s, t_f])$ and $c_2(x, [t_s, t_f])$ can be easily obtained by plugging the estimates $\hat{P}^{A_i}(x, [t_s, t_f])$ in (19) or (20) of $P^{A_i}(x, [t_s, t_f])$, $i = 1, 2, \ldots, N$, into the formulas (7) and (8), thus getting

$$\hat{c}_1(x, [t_s, t_f]) = 1 - \prod_{i=1}^{N}(1 - \hat{P}^{A_i}(x, [t_s, t_f])) \tag{21}$$

$$\hat{c}_2(x, [t_s, t_f]) = 1 - \prod_{i=1}^{N}(1 - \hat{P}^{A_i}(x, [t_s, t_f])) - \sum_{i=1}^{N}\hat{P}^{A_i}(x, [t_s, t_f]) \prod_{j=1, j\neq i}^{N}(1 - \hat{P}^{A_j}(x, [t_s, t_f])), \tag{22}$$

These expressions show that the computational effort involved in the evaluation of complexity at position $x \in \mathcal{S}$ scales linearly with the number of aircraft N and, hence, it does not radically increases when an additional aircraft is introduced. Suppose in fact that computing $\hat{P}^{A_i}(x, [t_s, t_f])$ for aircraft A_i takes a unit time. In an airspace region with N aircraft, a total of N time units is taken to compute $\hat{P}^{A_i}(x, [t_s, t_f])$ for all the N aircraft. Once all $\hat{P}^{A_i}(x, [t_s, t_f])$, $i = 1, 2, \ldots, N$, are obtained, both the $c_1(x, [t_s, t_f])$ and the $c_2(x, [t_s, t_f])$ metrics can be computed with a constant number of additional operations. Higher order complexity measures could be also estimated based on $\hat{P}^{A_i}(x, [t_s, t_f])$, $i = 1, 2, \ldots, N$, at no additional cost.

This is an important property, since real time computability is typically required in time-critical operations such as CD&R.

4.3. Construction of the complexity maps $\mathcal{C}_1(\cdot, [t_s, t_f])$ and $\mathcal{C}_2(\cdot, [t_s, t_f])$

In order to build the complexity maps, one has to evaluate $c_1(x, [t_s, t_f])$ and $c_2(x, [t_s, t_f])$ across $\mathcal{S}$. This calls for some discretization of $\mathcal{S}$. Using an uniform gridding of step size $\delta > 0$ along all axes will result in $\mathbf{O}(\delta^{-3})$ grid points. Halving the step size, for example, would then result in eight times more grid points. It then follows that evaluating the complexity maps $\mathcal{C}_1(\cdot, [t_s, t_f])$ and $\mathcal{C}_2(\cdot, [t_s, t_f])$ in an airspace region $\mathcal{S}$ with N aircraft would require a computational time proportional to $N\delta^{-3}$.

One possible way to alleviate the exponential growth of computation time as the grid size decreases would be to use a variable sized grid. A coarser grid could be used to evaluate the complexity in regions that do not require a significant accuracy (e.g. regions sufficiently far from the nominal trajectories of the aircraft), while a finer grid could be used in regions requiring higher accuracy. The identification of such regions might be done using the complexity maps from the previous time interval.

4.4. Analytical approximation of the single-aircraft complexity measure $c(t_s, t_f)$

The procedure for approximating the first order complexity $c_1(x, [t_s, t_f])$ can be easily adapted to compute an analytical approximation of the single-aircraft complexity measure $c(t_s, t_f)$ defined in (5). Indeed, $c(t_s, t_f)$ can be expressed as

$$c(t_s, t_f) = 1 - \prod_{i=1}^{N}(1 - P^{A_i}(\bar{x}^B(t), [t_s, t_f])),$$

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and to compute $P^{A_i}(\bar{x}^B(t),[t_s,t_f])$ one just needs to consider the relative position of aircraft A_i with respect to aircraft B rather than with respect to the fix position x. If aircraft B enters $\mathcal{S}$ at time 0 starting from x with a constant velocity u^B, then, this will lead to an equation of the same form of equation (9) with the only difference being that $\Delta u = u^B - u_i^A$.

5. APPLICATION TO TRAJECTORY DESIGN

Suppose that aircraft B has to enter the airspace region $\mathcal{S}$ at time 0 and reach some destination position at time $\bar{t}$. The intended trajectory of the aircraft is a straight line traveled at constant velocity between its entry point and destination. However, this trajectory is not guaranteed to be of low-complexity due to the presence of other aircraft. Aircraft B can then choose a fixed number m of velocity changes at specified points in time $0 < t_1 < t_2 < \ldots < t_m < t_f$ to reduce the complexity along its trajectory.

Note that the way points $X_1, X_2, \ldots, X_m$ at which aircraft B changes its velocity completely specify its nominal multi-legged trajectory. Since the flight time between successive way points is given, the velocity of aircraft B within each interval can be determined from the way points $X_1, X_2, \ldots, X_m$ and the starting and destination positions.

We seek to find an optimal trajectory in the sense that both the deviation from the intended trajectory and the complexity experienced by aircraft B within the flight time $[0,\bar{t}]$ are minimized. We take the sum of the distances of the way points $X_1, X_2, \ldots, X_m$ from the intended trajectory as a measure of the deviation d.

The complexity $\mathrm{c}(0,\bar{t})$ experienced by aircraft B along a multi-legged trajectory is not easy to compute since aircraft B does not have a constant velocity through out its flight, but only keeps its velocity constant during each interval $[t_i, t_{i+1}], i = 0, 1, \ldots, m$. However, we can over-approximate it by $\sum_{i=0}^{m} \mathrm{c}(t_i, t_{i+1})$, where we set $t_0 = 0$ and $t_{m+1} = \bar{t}$. The complexity $\mathrm{c}(t_i, t_{i+1})$ in the interval $[t_i, t_{i+1}]$ where aircraft B is flying at constant velocity v_i from X_i to X_{i+1} can be obtained by considering a fictitious straight line trajectory for aircraft B traveled at constant velocity v_i and passing through X_i at time t_i. This trajectory coincides with the original trajectory in the interval $[t_i, t_{i+1}]$ and can be used to evaluate the complexity encountered by aircraft B within that interval according to the computational procedure described in Section 4.4.

The problem of finding a suitable trajectory is then formulated as that of minimizing the cost:

$$J := d + \lambda \sum_{i=0}^{m} \mathrm{c}(t_i, t_{i+1}), \tag{23}$$

which is a weighted sum of the deviation measure d and the over-approximation of the complexity measure $\mathrm{c}(0,\bar{t})$. A higher value of the weighting coefficient $\lambda > 0$ attributes a greater priority to the low-complexity requirement, and results in a less conflict-prone trajectory for an appropriately chosen size of the buffer zone (see Remark 1).

6. NUMERICAL EXAMPLES

In all the examples, the uncertainty affecting the aircraft future positions is characterized through the spectral densities $\sigma_1^{A_i} = 0.25$ nmi $\cdot$ (min)$^{-1/2}$ in the along track direction, and $\sigma_2^{A_i} = \sigma_3^{A_i} = 0.2$ nmi $\cdot$ (min)$^{-1/2}$ in the cross track directions. The parameters r_h and r_v defining the ellipsoidal buffer region $\mathcal{M}(x)$ in (2) are set equal to $r_h = 5$ nmi and $r_v = 2000$ feet (0.3291 nmi), and the look-ahead time horizon is $T = [0, 10]$ min. The zero-th order approximation formula (19) is used when computing the complexity measures by (21) and (22).

6.1. Evaluating the airspace occupancy

Consider a 3-D airspace region with six aircraft. Each aircraft is moving at constant velocity along a straight line during the time interval T. The nominal trajectories of the aircraft are shown in Figure 2. Figure 3(a) shows the first order complexity map $\mathcal{C}_1(\cdot, [t, t+\Delta])$ for

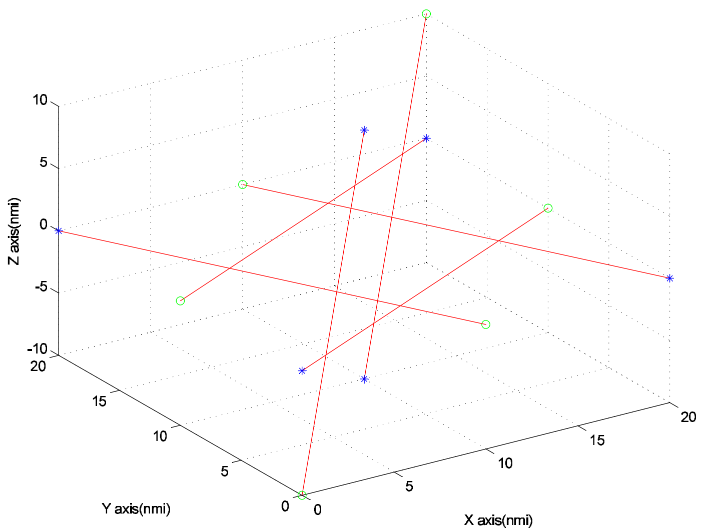
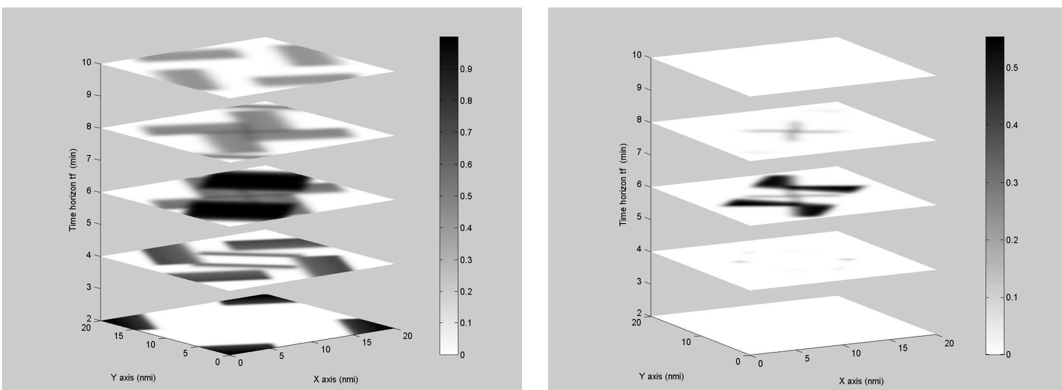


Figure 2. Initial positions and nominal trajectories of the aircraft. '*' denote starting points, and 'o' denote the nominal position of the aircraft at time t= 10 min.

five different consecutive time frames $[t, t+\Delta]$ of length $\Delta = 2$ min, covering the whole time horizon T. For each time frame, the complexity map is evaluated at uniformly sampled points in the horizontal plane XY with an uniform gridding of size $\delta_x = \delta_y = 0.2$ nmi. Similarly, the complexity maps $\mathcal{C}_2(\cdot, [t, t+\Delta])$, $t = 0, 2, 4, 6, 8$, are plotted in Figure 3(b).



(a) $\mathcal{C}_1$ complexity maps

(b) $\mathcal{C}_2$ complexity maps

Figure 3. Complexity maps over the XY plane corresponding to different time frames $[t_s, t_f]$ of length 2 min in the time horizon $[0, 10]$ min.

Figure 3(a) shows that the first order complexity is high initially in those zones that the aircraft are most likely to occupy in the XY plane. However, it can be seen from the second order complexity map that no two aircraft come close to each other in the first two time frames. During the time frame $[4, 6]$, there is a zone of high $\mathcal{C}_1$ and $\mathcal{C}_2$ complexity in the airspace. From

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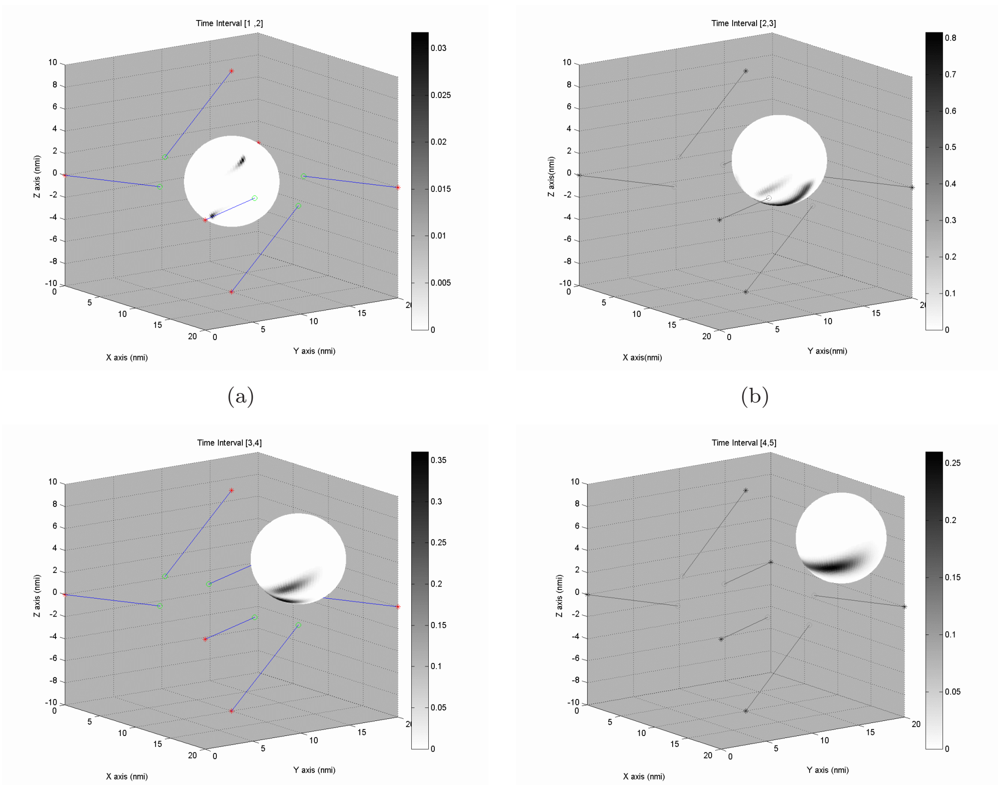
the $\mathcal{C}_2$ map, we can deduce that there will be more than one aircraft during this interval in that zone. This is to be expected considering that the nominal trajectories take the aircraft close to each other around this time. Also, the drastic decrease in the $\mathcal{C}_1$ complexity map in successive subintervals indicates that the aircraft then move away from each other. Additional traffic entering the airspace should then better avoid crossing the XY plane in the time frame $[2, 4]$.

6.2. *Evaluation of the maneuverability space*



(a) (b) (c) (d)

Figure 4. Complexity experienced by aircraft B entering an airspace region with other six aircraft as a function of its heading at a few points along its straight line trajectory.

Suppose that an additional aircraft B is introduced at time $t = 0$ at the point $[8, 8, -2]^T$ nmi in the airspace where the six aircraft are flying. Aircraft B is following a straight line trajectory at the constant velocity $u^B = [2, 2, 2]^T$ nmi/min. Due to the presence of the six aircraft, aircraft B is not free to change its heading arbitrarily during the flight. In Figure 4 we represent the complexity $c(t, t + \Delta)$ defined in (5) as a function of the heading of aircraft B over a time frame of length $\Delta = 1$ minute at a few sampled-points along the nominal trajectory of aircraft B. It can be observed that aircraft B faces a decrease in the amount of low-complexity prospective headings at some of these points, indicating that the airspace surrounding them is congested. This information might be used by aircraft B to find a minimal-complexity trajectory through the airspace.

6.3. *Trajectory design*

In Figure 5, another encounter situation is shown, where some aircraft B enters an airspace region at time 0 and aims at reaching a destination position at time $\bar{t} = 10$, while keeping at

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some constant altitude. Four aircraft are already present in that region.
Assume that aircraft B follows a level flight trajectory with one possible velocity change ($m = 1$) at $t_1 = 5$ out of a total flight time $\bar{t} = 10$.

Figure 5 shows the optimal trajectory of aircraft B obtained by minimizing the cost function (23) with $\lambda = 1500$. The minimization was done using the MATLAB function *fmincon* and with the intended straight trajectory as the initial guess for the solution. The color map in Figure 5 represents the sum of the complexity measures within the time intervals $[0, 5]$ and $[5, 10]$ evaluated for different choices of the intermediate way point. The original intended trajectory is also plotted for comparison. It can be observed that the sum of the complexity measures along this trajectory is greater than that of the optimal one. A larger value of λ places more emphasis on the low-complexity requirement and thus leads to more aggressive maneuvering.

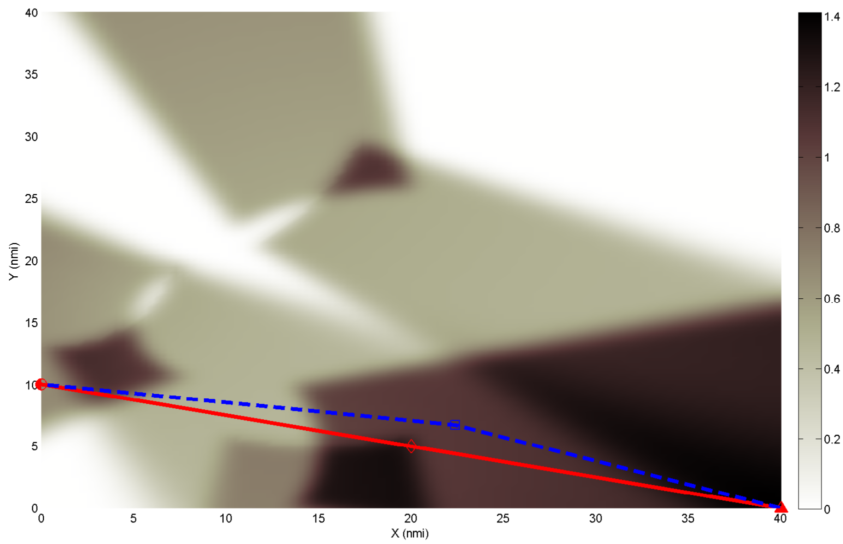

Figure 5. Originally intended trajectory (solid line) and optimal trajectory (dashed line) of aircraft B flying from the starting position on the left to the destination position on the right ($\lambda = 1500$). The color map in the background represents the complexity along aircraft B trajectory as a function of the intermediate way point position.

7. CONCLUSIONS

In this work, a probabilistic approach to mid-term complexity evaluation in 3-D airspace is proposed, where uncertainty in the aircraft future position is incorporated in the complexity measure. Possible applications have been illustrated through some simple numerical examples, which include using complexity maps for detecting congested airspace areas and the design of a trajectory for an additional aircraft crossing some airspace region. An analytical approximation of the complexity measure associated to each point in the airspace has been derived in the case when aircraft are following straight line trajectories traveled at constant velocity.

On-going work includes the extension of the analytical approximation to the case of multi-legged trajectories specified in terms of a sequence of timed way-points, as it is a common practice in ATM. Solutions inspired by that adopted in Section 3.2 for computing the complexity experienced by an aircraft following a piecewise linear trajectory will also be studied.

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