Probability of Conflict Analysis of 3D Aircraft Flight Based on Two-Level Markov Chain Approximation Approach

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***Abstract*— This paper introduces a new approach of Markov chain approximation for the reachability analysis of stochastic systems and particularly for computing the probability of conflict during aircraft flight. Two-level Markov chains are designed to achieve a better compromise between accuracy and computation time compared with the one-level Markov chain approximation. We used a simplified aircraft dynamics model of the aircraft-to-airspace conflict prediction problem. However, the new approach in this paper can also be applied to the aircraft-to-aircraft conflict prediction problem. Numerical example is studied to analyze the probability of conflict using the two-level Markov chain approximation. The results show that the approach obtains accurate estimation of the probability of conflict with an acceptable running time.**

I. Introduction

Conflict prediction and avoidance is a critical and challenging task in any transportation systems in general and in Air Traffic Management (ATM) systems in particular. The challenging part here is to meet the safety requirements of the transportation systems. For instance, vehicles must keep a minimum distance (safety zone) between them and from obstacles to avoid collision, namely, conflict avoidance. This paper will be devoted to studying such tasks in the ATM systems. Due to the rapidly increase in air traffic, guaranteeing the safety of air travel has become even more important and challenging. Usually, in the ATM systems the aircrafts are required to be separated by a safety zone from other aircrafts and from restricted areas, such as severe weather zones or Special Use Airspaces (SUA). Therefore, a significant task to be performed is to first detect the conflict -this is called conflict prediction, and then perform a conflict resolution or a conflict avoidance [1]–[3].

The analysis in this paper focuses only on the probabilistic conflict detection problem by modeling aircraft paths as the solutions to some stochastic differential equations. The idea behind this approach is to compute the probability of conflict, namely, the probability of the aircraft's position violating the safety requirement, such as entering some forbidden regions or being in some other aircraft's protection zone. The computed probability values will give an indication on the safeness of the assigned path of the aircraft. For instance, if the probability of conflict is high, then an urgent resolution action must be performed; otherwise the path is free of conflict and no resolution will be taken till further conflict detection.

The work in this paper follows [4], [5] in computing the probability of conflict based on the discretization of stochastic differential equations into Markov chains, whose state space is formed by gridding the region of interest in the airspace where the aircraft path is being analyzed with a fixed grid size. The Markov chain with a proper choice of transition probabilities will guarantee to converge weakly to the solution of the approximated stochastic system modeling the aircraft dynamics as the grid size approaches zero. However, the approach here differs from the previous work by using different gird sizes for the Markov chain approximation scheme.

The new approach of Markov chain approximation introduced in this paper is inspired by the Multi-Level (ML) methods for Markov chains used in [6]–[8] that achieved an excellent performance and an improved solution speed compared with the traditional One-Level method for their applications. The ML methods have been widely and successfully used to solve partial differential equations, however, this paper is the first to combine the ML techniques with the Markov chains to solve such transportation systems (ATM systems). The approach adopted in this paper uses two-level Markov chains for simplicity. Certainly, the general ML methods give a wider variety of choices for having ML Markov chains structure in different interested regions of the airspace; and may lead to more efficient approaches for dealing with multiple aircraft-to-aircraft prediction problem.

This paper is organized as follows. An aircraft dynamics model based on stochastic differential equations is presented in Section 2. The conflict prediction problem and the probability of conflict are formulated in Section 3. Section 4 introduces the two-level Markov chain approximation scheme and the iterative algorithm for computing the conflict probability based on this structure, followed by a numerical example in Section 5. Lastly, conclusions of this paper are stated in Section 6.

II. Aircraft dynamics model

This paper adopts a dynamics model of the aircraft given by the approximated stochastic differential equations for predicting the aircraft's future position during the interval time horizon T. For more information on stochastic approximation the reader can refer to [9], [10]. Here, we study the general case of the aircraft flight in which its altitude changes during resolution maneuvers to avoid collisions. Therefore the airspace is represented by $\mathbb{R}^3$ and the aircraft position at time $t \in T$ is given by $X(t) \in \mathbb{R}^3$. According to ATM practice, the aircraft is typically assigned a constant speed piecewise linear motion specified by a series of way-points; thus the aircraft velocity $u : T \rightarrow \mathbb{R}^3$ is a piecewise constant function. For simplicity, speeds with a constant altitude are assumed. In addition, we address the wind factor, which is one of the main contributors to uncertainty in the aircraft future positions during the time horizon T. Therefore, the aircraft velocity consists of the piecewise constant velocity $u(t)$ (called airspeed) and the wind-contributed velocity (called wind speed). The wind speed furthermore is composed of two terms i) a deterministic nominal wind speed term that is known to the Air Traffic Control (ATC) through measurements or forecast, and ii) a stochastic term representing the effect of air turbulence and errors in the wind speed measurements and forecast [11].

From the above discussions, we can now formulate the approximated stochastic system for modeling the aircraft dynamics. Consider $X(t)$ to be the position of the aircraft during the time horizon T. Then the aircraft dynamics model is given by the following stochastic differential equation:

$$dX(t) = u(t)dt + f(X,t)dt + \Sigma(X,t)dB(t), \qquad (1)$$

where $f : \mathbb{R}^3 \times T \rightarrow \mathbb{R}^3$ is a time-varying vector field on $\mathbb{R}^3$ representing the nominal wind speed at position x and time t, also called the wind field. The third term in the right-side of the above equation is the stochastic term, where $B(t)$ is a standard three-dimensional Brownian motion whose variance is modulated by the mapping $\Sigma : \mathbb{R}^3 \times T \rightarrow \mathbb{R}^{3\times 3}$.

The above stochastic differential equation can be simplified under the assumption that Σ is a constant diagonal matrix: $\Sigma(x,t) = \Sigma = \text{diag}(\sigma_h, \sigma_h, \sigma_v)$, for some constant $\sigma_h > \sigma_v > 0$ and that the nominal wind field f at any given $t \in T$ is uniform in the region of interest in the airspace, which allows it to act additively on the aircraft velocity. In this case, equation (1) can be rewritten as follows:

$$dX(t) = u(t)dt + \Sigma \; dB(t). \qquad (2)$$

Although the simplified aircraft dynamics model (2) is used in the rest of the paper, our analysis approach can be extended easily to the general case of equation (1), as well as the aircraft dynamics model in [12] with the presence of a correlated wind field for multiple aircraft conflict case.

III. Problem Formulation

In the literature on conflict detection and resolution (CDR) in the ATM systems, the conflict prediction problem studies an aircraft flying near a forbidden region of the airspace (aircraft-to-airspace conflict problem) or trying to keep a minimum separation from other aircraft by a horizontal distance r and a vertical distance H (aircraft-to-aircraft conflict problem). Based on the minimum operational performance standards for the air traffic alert, the minimum horizontal separation is 5 nautical miles (nmi), whereas within TRACON area it is reduced to 3 nmi, and the minimum vertical separation is 2000 ft or 1000 ft if the aircraft flies at an altitude above 29,000 ft or below 29,000 ft, respectively ([13], [14]).

In the study of the aircraft-to-airspace case, consider an aircraft flying in an open bounded region U with the assigned path velocity $u(t),\ t \in T = [0, t_f]$ where 0 represents the current time instant and t_f represents the look-ahead time horizon. Then, the evolution of $X(t)$ over the time interval is given by equation (2). Let a compact set $D \subset U$ be a restricted region to avoid. For example, D can have the shape of a closed cylinder of radius *r* and height *2H* centered at the origin, whose set is denoted in this paper by C. Now, to evaluate the safety of the aircraft path, we need to compute the probability of the aircraft's position entering the region D, namely, detecting the likelihood of the occurrence of conflict. Therefore, the probability of conflict is defined as follows:

$$P\left\{X(t) \in D \text{ for } t \in T\right\}. \qquad (3)$$

Observe that, if $X(t)$ escapes from U within T, then in this case we claim no conflict occurs. Therefore, the open bounded domain U must be large enough to assure safety of the aircraft flight whenever $X(t)$ is outside U ($X(t) \in U^c$). More precisely, the definition of the probability of conflict is as follows:

$$P_c \triangleq P\left\{X(t) \text{ hits } D \text{ before hitting } U^c, t \in T\right\}. \qquad (4)$$

In the study of the aircraft-to-aircraft case, generally speaking and without loss of generality, it can be considered as a special case of the aircraft-to-airspace conflict problem. Consider $X_1(t)$ and $X_2(t)$ to be the positions of aircraft 1 and aircraft 2, respectively, where $X_1(t),\ X_2(t) \in \mathbb{R}^3$. According to equation (2), the dynamics model of both aircrafts over time interval T are given by the following stochastic differential equations:

$$dX_1(t) = u_1(t)dt + \Sigma_1 \; dB_1(t),$$
$$dX_2(t) = u_2(t)dt + \Sigma_2 \; dB_2(t),$$

with initial condition $X_1(0)$ and $X_2(0)$, respectively.
Next, define the following:

$$Y(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}, \qquad v(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix},$$
$$\hat{\Sigma} = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \qquad \hat{B}(t) = \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix},$$

and

$$D = \left\{Y(t) \in \mathbb{R}^6 : \|X_1 - X_2\| \in C\right\},$$

where C is the set of the closed cylinder mentioned above. Then, the dynamics of $Y(t)$ is given by:

$$Y(t) = v(t)dt + \hat{\Sigma} \; d\hat{B}(t).$$

Therefore, the aircraft-to-aircraft problem becomes a stationary aircraft-to-airspace conflict problem. In this paper we focus on the aircraft-to-airspace case as a general conflict problem application.

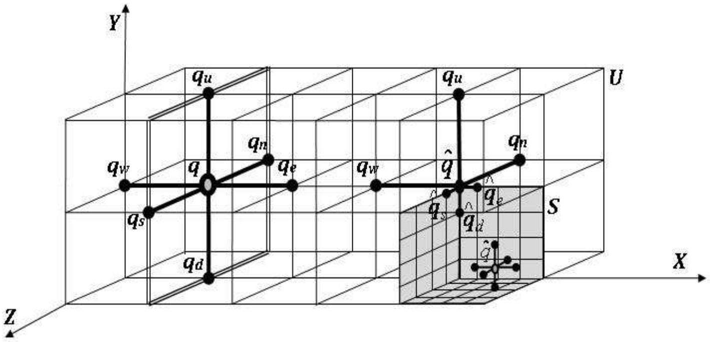

Fig. 1. Grid structure of the two-level Markov Chain Approximation

IV. Two-Level Markov Chain Approximation

We first briefly describe the procedure for Markov chain approximation method. First, discretize the interested domain $U\backslash D$ into grid points with step size δ that form the state space of the Markov chain $\{Q_{k\Delta t}, k \geq 0\}$, where Δt represents the time step between two successive jumps. Then, choose transition probabilities carefully so that the solution of the Markov chain will converge weakly to the solution of the stochastic differential equation (2) as the grid size δ approaches zero. Notice that, the smaller the grid size is, the better estimates of P_c can be obtained, though the running time for the algorithm will be longer. Consequently, a new approach using Multi-Level Markov chains has been developed in this paper to achieve a better compromise between the accuracy and computation time of P_c.

This paper proposes the use of the two-level Markov chains consisting of a Fine-Level Markov Chain (FLMC) and a Coarse-Level Markov Chain (CLMC) for the reachability analysis of the stochastic systems introduced in Section II. The FLMC will be used in a smaller subset $S\backslash D$ of the open domain U where we seek accurate estimates of P_c; and the CLMC will be used in the domain $U\backslash D$ (see Figure 1). Observe that, the CLMC is used over $U\backslash D$ instead just over $U\backslash(D \cup S)$. This will speed up the algorithm since the common grid points of the CLMC and the FLMC will be updated more frequently than other grid points of either the CLMC or the FLMC. Therefore, a faster solution is achieved in the interested region of the airspace.

Since the CLMC and the FLMC will be defined on two different grid spaces, a question arises on how to merge one to the another along with their boundaries. In other words, if some grid point q of either the CLMC or the FLMC lies on the boundary of the subset S with U, then in this case q has at least one neighbor at a distance of δ which is inside $U\backslash D$ and at least one neighbor at a distance of the fine step size $\hat{\delta} < \delta$ inside $S\backslash D$. Figure 1 also shows one of the corner point cases. Therefore, every grid point lying on the boundary of S with U has neighbor points in different levels.

The three cases of Markov chain approximation are discussed in the following: 1) Coarse-Level Markov chain approximation. 2) Fine-Level Markov chain approximation. 3) Boundary of Coarse- and Fine-Level Markov chain approximation.

A. Coarse-Level Markov Chain Approximation

The CLMC construction is adopted from the Markov chain approximation in [5]. The following is a quick review of the Markov chain definition:

- Markov chain state space [1]: $\mathcal{Q} = (U\backslash D) \cap \delta\mathbb{Z}^3$, a finite set obtained by gridding $U\backslash D$ with the step size δ;
- The six immediate neighbors of each grid point $q \in \delta\mathbb{Z}^3$: $q_e = q + (\delta, 0, 0),\ q_w = q + (-\delta, 0, 0),\ q_n = q + (0, \delta, 0),\ q_s = q + (0, -\delta, 0),\ q_u = q + (0, 0, \kappa\delta),\ q_d = q + (0, 0, -\kappa\delta)$, where $\kappa \triangleq \sigma_v/\sigma_h$;
- To satisfy the definition of the probability of conflict, every interior state of Markov chain ($q \in \mathcal{Q}^o$) is an *arbitrary* state and every boundary state ($q \in \partial\mathcal{Q}$) is an *absorbing* state, where $\partial\mathcal{Q} = \partial\mathcal{Q}_U \cup \partial\mathcal{Q}_D$;
- The transition probabilities for interior states ($q \in \mathcal{Q}^o$): $P\{Q_{k+1} = q'|Q_k = q\} =$
$$\begin{cases} p^k_{i_+}(q) = C(q)\ e^{+\delta\xi_i}, & q' = q_{i_+}, i = 1,2,3; \\ p^k_{i_-}(q) = C(q)\ e^{-\delta\xi_i}, & q' = q_{i_-}, i = 1,2,3; \\ p^k_o(q) = C(q)\ \xi_o, & q' = q, \end{cases} \quad (5)$$
where $q_e = q_{1_+},\ q_n = q_{2_+},\ q_u = q_{3_+},\ q_w = q_{1_-},\ q_s = q_{2_-},\ q_d = q_{3_-},\ \xi_i(q) = \frac{u_i}{\sigma_h^2},\ \xi_o(q) = \frac{2}{\lambda\sigma_h^2} - 6$, $C(q) = (\xi_o + 2\sum_{i=1}^{3}\cosh(\delta\xi_i(q)))^{-1}$, and $\Delta t = \lambda\delta^2$ where $0 < \lambda \leq (3\sigma_h^2)^{-1}$ so that ξ_o is positive for any q. See [5] for more details.

With the above choice of the Markov chain state space and the transition probabilities, the Markov chain converges weakly to the solution $X(t)$ of the stochastic differential equation (2) according to the following proposition:

Proposition 1: For a fixed $\delta > 0$, consider the Markov chain $\{Q_{k\Delta t}, k \geq 0\}$. Now Suppose that the state of the Markov chain is constant $Q_{k\Delta t}$ on each time interval $[k\Delta t, (k+1)\Delta t)$ between successive jumps. Then as $\delta \to 0$, the solution $\{Q_{k\Delta t}, k \geq 0\}$ converges weakly to the solution $\{X(t), t \geq 0\}$ of the diffusion equation (2).
For the proof see [5].

Consider $k = 0, ..., k_f$, where k_f is the largest integer not exceeding $t_f/\Delta t$. As a result of the weak convergence of $Q_{k\Delta t}$ to $X(t)$, we can approximate the probability of conflict in (4) for small δ as follows:

$$P^\delta_c \triangleq P\{Q_{k_f\Delta t} \in \partial\mathcal{Q}_D\} \cong P\{Q_{k\Delta t} \text{ hits } \partial\mathcal{Q}_D \text{ before hitting } \partial\mathcal{Q}_U, 0 \leq k \leq k_f\}. \quad (6)$$

Then, starting from the state $\bar{q} = (\bar{m}\delta, \bar{n}\delta, \bar{l}\kappa\delta) \in \mathcal{Q}$ closest to $X(0)$, the given initial aircraft's position, the estimated probability of conflict for the aircraft located at q

[1] Note that for notational convenience, we denote the integer grids of $\mathbb{R}^3$ by $\delta\mathbb{Z}^3$. However, the vertical axis and the horizontal axes have different grid levels of $\kappa\delta$ and δ, respectively.

and at time k is:

$$P_c^k(q) = \begin{cases} \sum_{i=1}^3 p_{i_+}^k(q) P_c^{k+1}(q_{i_+}) \\ + \sum_{i=1}^3 p_{i_-}^k(q) P_c^{k+1}(q_{i_-}) \\ + p_o^k(q) P_c^{k+1}(q) & q \in \mathcal{Q}^o \\ 1, & q \in \partial\mathcal{Q}_D \\ 0, & q \in \partial\mathcal{Q}_U, \end{cases} \tag{7}$$

where the initial condition is:

$$P_c^{k_f}(q) = \begin{cases} 1, & \text{if } q \in \partial\mathcal{Q}_D \\ 0, & \text{elsewhere.} \end{cases} \tag{8}$$

The procedure for computing $P_c^0(\bar{q})$, which is our aim in this paper, is illustrated later in *Algorithm 1* at the end of this section. Briefly speaking, the computation is performed by iterating equation (7) backward in times from k_f to $k = 0$.

B. Fine-Level Markov Chain Approximation

Consider a smaller subset $S \subset \mathbb{R}^3$ contained in the open domain U to be the domain of the FLMC structure. Usually, S can be chosen in any shape and in any location inside U. However, for simplicity we choose S to be a cubic region around D where we seek accurate estimates of P_c. For less computation complexity, we choose S to be aligned with the CL points; thus the eight corner points of the FL grids of $S \backslash D$ are also the coarse grid points in $U \backslash D$.

The FLMC is defined similarly as the CLMC, except for the step size and the boundary of S with the open domain U. Define the FLMC as $\{\hat{Q}_{\hat{k}\widehat{\Delta t}}, \hat{k} \geq 0\}$, where $0 < \widehat{\Delta t} < \Delta t$; the fine step size $\hat{\delta} = s\ \delta$, with $s = 1/m$ for some $m \in \mathbb{Z}$; and the state space is $\hat{\mathcal{Q}} = S \backslash D \cap \hat{\delta}\mathbb{Z}^3$. The FLMC is then obtained by replacing δ with $\hat{\delta}$ and q with $\hat{q}$ in the CLMC definition in Section IV-A.

The set of the interior states of the FLMC is denoted by $\hat{\mathcal{Q}}^o$, which consists of those gird points in $\hat{\delta}\mathbb{Z}^3$ with all their six immediate neighbors in $\hat{\mathcal{Q}}$. The boundary of $\hat{\mathcal{Q}}$ is given by $\partial\hat{\mathcal{Q}} = \partial\hat{\mathcal{Q}}_S \cup \partial\hat{\mathcal{Q}}_D$, where $\partial\hat{\mathcal{Q}}_D$ is the boundary of S with the forbidden region D and every state belongs to it ($\hat{q} \in \partial\hat{\mathcal{Q}}_D$) is considered to be an absorbing state. The set $\partial\hat{\mathcal{Q}}_S$, which is to be elaborated in Section IV-C below, is the boundary of S with the open domain U -the boundary of FL with CL. Every state $\hat{q} \in \partial\hat{\mathcal{Q}}_S$ is considered to be an arbitrary state, unless the boundary $\partial\hat{\mathcal{Q}}_S$ lies on $\partial\mathcal{Q}_U$ where the Markov chain is about to jump outside U then, it is considered to be an absorbing state. Based on example of the grid structure in Figure 1, three sides of the boundary of the subset S (south, east, and bottom side) are laying on the boundary of U ($\partial\mathcal{Q}_U$) and the other three sides (north, west, and top side of the boundary of S) are inside U. Therefore, any state belongs to the south, east, and bottom side of the boundary of S is considered to be an absorbing state, while those on other sides are arbitrary states.

C. Boundary of Coarse- and Fine-Level Markov Chain Approximation

In this section, we discuss how to merge two different levels of Markov chain. According to our assumption of the location of the subset S, the eight corner grid points of the FL are also the CL grid points and the interior grid points among those points will have three neighbors inside the FL ($\hat{\mathcal{Q}}^o \backslash D$) and three neighbors inside the CL ($\mathcal{Q}^o \backslash D$). Take the example of one of the arbitrary corner state $\hat{q}$ shown in Figure 1, where $\hat{q}$ is the top-west-north corner grid point $\in \hat{\delta}\mathbb{Z}^3$ that has the following six neighbors:

$$\begin{aligned} \hat{q}_e &= q + (\hat{\delta}, 0, 0), & q_w &= q + (-\delta, 0, 0), \\ \hat{q}_s &= q + (0, -\hat{\delta}, 0), & q_n &= q + (0, \delta, 0), \\ \hat{q}_d &= q + (0, 0, -\kappa\hat{\delta}), & q_u &= q + (0, 0, \kappa\delta). \end{aligned}$$

Starting from any interior corner point $\hat{q} \in \partial\hat{\mathcal{Q}}_S$, the Markov chain jumps to one of the six neighbors above or stays at the same state corresponding to the following transition probabilities:
$P\{Q_{k+1} = q' | Q_k = q\} =$

$$\begin{cases} \hat{p}_e^{\hat{k}}(q) = s\ \exp(\hat{\delta}\xi_q^{\hat{k}})/C(q), & q' = \hat{q}_e; \\ p_w^k(q) = \exp(-\delta\xi_q^k)/C(q), & q' = q_w; \\ \hat{p}_s^{\hat{k}}(q) = s\ \exp(-\hat{\delta}\eta_q^{\hat{k}})/C(q), & q' = \hat{q}_s; \\ p_n^k(q) = \exp(\delta\eta_q^k)/C(q), & q' = q_n; \\ \hat{p}_d^{\hat{k}}(q) = s\ \exp(-\hat{\delta}\zeta_q^{\hat{k}})/C(q), & q' = \hat{q}_d; \\ p_u^k(q) = \exp(\delta\zeta_q^k)/C(q), & q' = q_u; \\ p_o^k(q) = \chi_{\hat{q}}^k/C(q), & q' = q. \end{cases} \tag{9}$$

where the parameters in the above expression are similar to those defined in [5, eq (13)] except for the following:

$$\begin{aligned} \widehat{\Delta t} &= \lambda\hat{\delta}^2 = \lambda s^2 \delta^2, \\ \chi_q^k &= \frac{s+1}{\lambda\sigma_h^2} - 3 - 3s, \\ C(q) &= \chi_q^k + \exp(-\delta\xi_q^k) \\ &\quad + \exp(\delta\eta_q^k) + \exp(\delta\zeta_q^k) \\ &\quad + s\ \exp(\hat{\delta}\xi_q^{\hat{k}}) + s\ \exp(-\hat{\delta}\eta_q^{\hat{k}}) \\ &\quad + s\ \exp(-\hat{\delta}\zeta_q^{\hat{k}}). \end{aligned}$$

Here, we also have $0 < \lambda \leq (3\sigma_h)^{-1}$ to guarantee the positivity of χ_q^k for all k and q.

In closing, the two constraints of weak convergence stated in Theorem 8.7.1 in [9] and Proposition 1 for the corner points of the boundary ($q \in \partial\hat{\mathcal{Q}}_S$) are proved as follows:

Consider the mean and the covariance of the Markov chain jumps as follows,

$$\begin{aligned}
m_q^k &= m_{\hat{q}}^{\hat{k}\widehat{\Delta t}} + m_q^{k\Delta t}\\
&= \frac{1}{\widehat{\Delta t}} E\left\{\hat{Q}_{(\hat{k}+1)\widehat{\Delta t}} - \hat{Q}_{\hat{k}\widehat{\Delta t}} | \hat{Q}_{\hat{k}\widehat{\Delta t}} = \hat{q}\right\}\\
&+ \frac{1}{\Delta t} E\left\{Q_{(k+1)\Delta t} - Q_{k\Delta t} | Q_{k\Delta t} = q\right\},\\
V_q^k &= V_{\hat{q}}^{\hat{k}\widehat{\Delta t}} + V_q^{k\Delta t}\\
&= \frac{1}{\widehat{\Delta t}} E\left\{\left(\hat{Q}_{(\hat{k}+1)\widehat{\Delta t}} - \hat{Q}_{\hat{k}\widehat{\Delta t}}\right)\right.\\
&\quad \left.\left(\hat{Q}_{(\hat{k}+1)\widehat{\Delta t}} - \hat{Q}_{\hat{k}\widehat{\Delta t}}\right)^T | \hat{Q}_{\hat{k}\widehat{\Delta t}} = \hat{q}\right\}\\
&+ \frac{1}{\Delta t} E\left\{\left(Q_{(k+1)\Delta t} - Q_{k\Delta t}\right)\right.\\
&\quad \left.\left(Q_{(k+1)\Delta t} - Q_{k\Delta t}\right)^T | Q_{k\Delta t} = q\right\}.
\end{aligned}$$

After performing some simple manipulation, we obtain:

$$\begin{aligned}
m_q^k &= \frac{\delta}{C(q)\ \Delta t}\begin{bmatrix} \exp(\hat{\delta}\xi_q^{\hat{k}}) - \exp(-\delta\xi_q^k) \\ \exp(\hat{\delta}\eta_q^{\hat{k}}) - \exp(-\delta\eta_q^k) \\ \kappa\exp(\hat{\delta}\zeta_q^{\hat{k}}) - \kappa\exp(-\delta\zeta_q^k) \end{bmatrix},\\
V_q^k &= \frac{\delta^2}{C(q)\ \Delta t}\begin{bmatrix} \exp(\hat{\delta}\xi_q^{\hat{k}}) + \exp(-\delta\xi_q^k) \\ \exp(\hat{\delta}\eta_q^{\hat{k}}) + \exp(-\delta\eta_q^k) \\ \kappa^2\exp(\hat{\delta}\zeta_q^{\hat{k}}) + \kappa^2\exp(-\delta\zeta_q^k) \end{bmatrix}^T I_3.
\end{aligned}$$

Next, for any $\delta > 0$ we choose q to be closest to a fixed point $x \in \partial\hat{\mathcal{Q}}_S$. Then it can be verified that as $\delta \to 0$.

$$m_q^k \to v(k), \text{ and } \qquad V_q^k \to \Sigma^2.$$

Notice that, not all of the grid points of the FL boundary ($\hat{q} \in \partial\hat{\mathcal{Q}}_S$) belong to the state space of the Markov chain, only those common points on the boundary of the FL grid and the CL grid that have six neighbors in both $S\backslash D$ and $U\backslash D$. Starting from those points, the Markov chain can jump to their neighbors or stay at the same state according to the transition probabilities. The other FL boundary grid points not on the CL grid do not have neighbors inside the CL. Therefore, Markov chain approximation approach can not be used on such grid points; instead, an approximation approach will be adopted by interpolating the conflict probability of both fine neighbor points and two or more of the closest coarse points.

Finally, the estimated probability of conflict for the corner grid points is defined as follows:

$$\hat{P}_c^k(q) = \begin{cases} p_o^k(q)P_c^{k+1}(q) + \hat{p}_e^{\hat{k}}(q)\hat{P}_c^{\hat{k}+1}(\hat{q}_e) & \\ +p_w^k(q)P_c^{k+1}(q_w) + \hat{p}_n^{\hat{k}}(q)P_c^{\hat{k}+1}(\hat{q}_n) & \\ +p_s^k(q)P_c^{k+1}(q_s) + \hat{p}_u^{\hat{k}}(q)P_c^{\hat{k}+1}(\hat{q}_u) & \\ +p_d^k(q)P_c^{k+1}(q_d), & q \in \partial\hat{\mathcal{Q}}_S \\ 1, & q \in \partial\mathcal{Q}_D \\ 0, & q \in \partial\mathcal{Q}_U, \end{cases} \tag{10}$$

where the initial condition is:

$$P_c^{\hat{k}_f}(q) = \begin{cases} 1, & \text{if } q \in \partial\hat{\mathcal{Q}}_D \\ 0, & \text{elsewhere.} \end{cases} \tag{11}$$

Algorithm 1: Given X(0) as the initial position of the aircraft.

1) Fix δ and $\hat{\delta} > 0$ over $U\backslash D$ and $S\backslash D$ respectively, let $\Delta t = \lambda\delta^2$ and $\widehat{\Delta t} = \lambda\hat{\delta}^2$ with $\lambda \in (0, 1/3\sigma_h^2]$, and define the CLMC $\{Q_{k\Delta t}, k \geq 0\}$, the FLMC $\left\{\hat{Q}_{\hat{k}\Delta\hat{t}}, \hat{k} \geq 0\right\}$, and the boundary between them along with their state spaces and transition probabilities,
2) Set $\hat{k}_f = t_f/\widehat{\Delta t}$, $k_f = t_f/\Delta t$, and $i = 0$. Then, initialize the FL $P_c^{\hat{k}_{\text{final}}}(\hat{q})$,
3) **for** $K = \hat{k}_{\text{final}} - 1, ..., 0$ **do**
   $i = i + 1$
   if $K = k_{\text{final}}$ **then**
   Initialize $P_c^{k_{\text{final}}}(q)$ according to equation (8)
   else if $i = \Delta t/\widehat{\Delta t} = 1/s^2$ **then**
   Compute CL $P_c^k(q)$ according to equation (7)
   $i = 0$
   else
   if $q = \hat{q}$ (corner boundary points) **then**
   Compute $\hat{P}_c^k(q)$ according to equation (10)
   else
   Compute FL $P_c^{\hat{k}}(\hat{q})$ then interpolate $P_c^k(\hat{q})$ for the boundary points ($\hat{q} \in \partial\hat{\mathcal{Q}}_U\backslash q$)
   end if
   end if
   end for
4) Finally, choose any point $\bar{q}$ closest to $X(0)$, then the estimated probability of conflict is given as follows:

$$P_c^0(\bar{q}) = \begin{cases} P_c^0(\bar{q}), & \text{if } \bar{q} \in \mathcal{Q}^o, \\ \hat{P}_c^0(\bar{q}), & \text{if } \bar{q} \in \hat{\mathcal{Q}}^o. \end{cases}$$

V. Numerical Example and Results

In this section we apply the above algorithm of the two-level Markov chain approximation approach to determine the probability of conflict for the general case of aircraft-to-airspace conflict problem. The following example is adopted from [5].

Consider an aircraft flying in the open domain U of the airspace during the time interval $T = [0, 30]$ with a nominal velocity as follows:

$$u(t) = \begin{cases} (2, 0, 0), & 0 \leq t < 10; \\ (0, 1, 0), & 10 \leq t < 20; \\ (2, 0, 0), & 20 \leq t \leq 30. \end{cases}$$

We choose the forbidden region D to be the closed cylinder of radius $r = 3$ and hight $H = 2$; the opened domain U to be $U = (-40, 5) \times (-20, 7) \times (-5, 5)$; and the fine subset S to be $S = (-12, 5) \times (-6, 7) \times (-3, 5)$. Then, we set the coarse step size to be $\delta = 1$, the fine step size $\hat{\delta} = 0.5$, and $\sigma_h = \sigma_v = 1$. Hence, we have $\lambda = 1/3$,

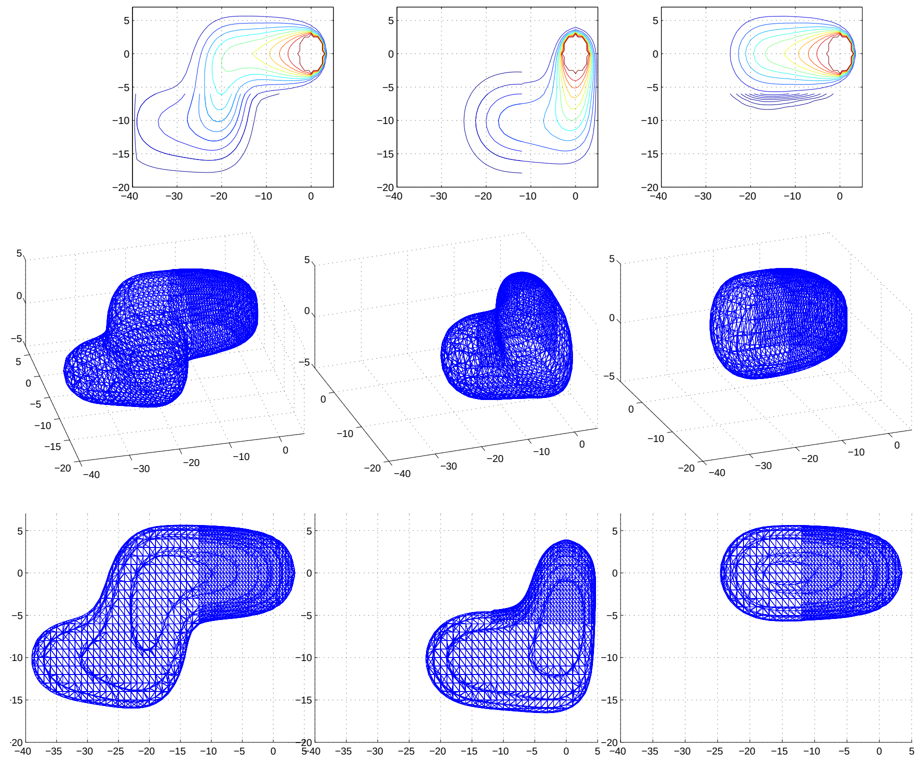

Fig. 2. Probability of conflict $\hat{P}_c$ over time horizon $T = [t, 30]$, where $t = 0, 10,$ and 20 from left to right

$\Delta t = 1/3$, and $\Delta \hat{t} = 1/12$.

Figure 2 shows the probability of conflict computed using Algorithm 1. The top row of Figure 2 shows the contour of $\hat{P}_c$ at zero altitude; the second row is the three dimensional plot of $\hat{P}_c$ with isosurface of value 0.1; and the last row is the top view of the same isosurface. Notice that, in the plot of the $\hat{P}_c$ contour, there are a few discontinued contours. This is because of the lag in merging the two-level Markov chains during the very first time segments.

VI. Conclusions

The two-level Markov chains designed in this paper achieve a better performance and significantly improved the solution speed comparing with the traditional one-level Markov chain method. A simplified dynamics model of the aircraft flight in three dimensional approximated by stochastic differential equations was presented here. Furthermore, an aircraft-to-airspace conflict problem has been studied to analyze the probability of conflict, and iterative algorithm has been designed for this purpose. A numerical example of the aircraft-to-airspace conflict problem was studied based on the two-level Markov chain approximation method. The results showed that faster and more accurate solution of the probability of conflict can be achieved.

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