Stabilization of Switched Linear Systems Using Continuous Control Input against Known Adversarial Switching

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***Abstract*— The problem of designing continuous control input to stabilize switched linear control systems against adversarial switching is studied. It is assumed that the continuous controller has access to the current switching mode and can be of the form of an ensemble of mode-dependent state feedback controllers. The fastest stabilizing rate under the given information structure is proposed as a quantitative metric of the system's stabilizability, and its bounds are derived using seminorms. Computation algorithms for the stabilizing rate are developed and illustrated through examples.**

I. Introduction

In this paper, we study the stabilization problem of switched linear control systems (SLCS):

$$x(t+1) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad t = 0, 1, \dots, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ is the control input, $\sigma(t) \in \mathcal{M} := \{1, \dots, m\}$ is the switching mode, and $(A_i, B_i)_{i\in\mathcal{M}}$ are the subsystem matrices in different modes. Different from the many existing work (e.g. [1], [2], [3], [4]) where both the control input u and the switching signal σ are utilized to stabilize the SLCS, the problem we study here assumes that only u can be controlled by the user to stabilize the SLCS, while σ is controlled by an adversary to destablize the system. Thus, the problem becomes a two-player dynamic game between the user and the adversary.

A similar perspective is adopted in [5], where the *mode-resilient stabilization problem* of SLCS is studied under the assumption that, at any time t, the user decides $u(t)$ first and the adversary decides $\sigma(t)$ with the knowledge of $u(t)$ (i.e., the user moves first in the two-player game). Due to the disadvantageous position of the user, stabilizing the SLCS can be very challenging: the user needs to design the stabilizing $u(t)$ without any knowledge of the current mode $\sigma(t)$. In contrast, this paper studies the *mode-conscious stabilization problem*, where the decision of the adversary on $\sigma(t)$ is known to the user when deciding $u(t)$ at each time t (i.e., the adversary moves first). This new information structure affords the user the luxury of adopting better control strategies that are impossible in the previous setting, e.g., mode-dependent linear state feedback control with different feedback gains for different modes. As a consequence, less stringent stabilizing conditions can be expected.

Despite this relative ease, the presence of an adversary capable of producing destabilizing $\sigma(t)$ ensures that the mode-conscious stabilization problem remains a challenging one. For instance, it is possible that each subsystem (A_i, B_i) is stabilizable, say, by the linear state feedback controller $u(t) = K_i x(t)$, but the SLCS cannot be stabilized by the user using the mode-dependent feedback controller $u(t) = K_{\sigma(t)}x(t)$ or by any admissible control strategy $u(t) = f_t(x(t), \sigma(t))$ even with the full knowledge of $\sigma(t)$. A simple example is given by a two-mode SLCS:

$$A_1 = \begin{bmatrix} 0.5 & 2 \\ 0 & 0.5 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix};\ A_2 = \begin{bmatrix} 0.5 & 0 \\ 2 & 0.5 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{2}$$

Suppose at each t, the adversary chooses $\sigma(t) = 1$ if $|x_1(t)| \le |x_2(t)|$, and $\sigma(t) = 2$ if $|x_1(t)| > |x_2(t)|$. Regardless of the user's choice on $u(t)$, $\|x(t+1)\|_\infty \ge \frac{3}{2}\|x(t)\|_\infty$ for all t. This implies that the SLCS is not σ_*-stabilizable, even though each individual subsystem is stabilizable. The latter is a necessary but not sufficient condition for σ_*-stabilizability.

Another distinguishing feature of this paper is that we will adopt a quantitative perspective: instead of just deriving conditions characterizing stabilizability, we will *quantify* it by calculating the stabilizing rate ρ_*, which is defined as the tightest upper bound on the exponential growth rate of the state trajectories. This rate allows us to compare and measure the robustness of the stabilizability of different SLCS. A similar quantitative perspective has been adopted in the study of the stability of autonomous switched linear systems (SLS), namely, the joint spectral radius [6], which provides the inspiration for many of the concepts proposed in this paper.

Stabilization of SLS and SLCS has been well studied [7], [8], [9]. Many of the existing work (e.g., [10], [11], [12]) focuses on the switching stabilization problem, namely, stabilizing SLS using σ. For SLCS, their stabilization using both u and σ has been studied extensively [1], [13], [2], [3], [4]. Stabilization of SLCS using u against σ has also been studied in the user-move-first setting in [14], [15], [16], [17], [5], and in the adversary-move-first setting by using parameter-dependent quadratic Lyapunov functions method [18], [19], [20], [21], multiple Lyapunov function (norm) method [22], Lyapunov-like function method for discrete-time SLCSs with average dwell time constraints [23], and time-varying quadratic Lyapunov function method for continuous-time SLCSs with dwell time constraints [24].

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Compared to these previous work, the main contributions of this paper consist of the following: (i) conditions for marginal stabilizability using the notions of defectiveness and reducibility (Section III); (ii) analytical bounds on the stabilizing rate using (semi)norms and conditions on when such bounds are tight (Section IV); (iv) numerical algorithms for computing the stabilizing rate (Section V). To illustrate the results, some numerical examples are provided in Section VI. Section VII contains some concluding remarks.

II. σ_*-Stabilizating Rate of SLCS

For the SLCS (1), assume that at each time t, the user's decision on the continuous input $u(t)$ is made with the full knowledge of the current mode $\sigma(t)$ chosen by the adversary, specifically, $u(t) = \mathbf{u}_t(\sigma(t), x(t))$ for some feedback law $\mathbf{u}_t$. A sequence of such laws, $\mathbf{u} = (\mathbf{u}_0, \mathbf{u}_1, \ldots)$, is called a (feasible user) control policy. Denoted by $\mathcal{U}$ the set of all feasible user control policies, and by $\mathcal{S} := \mathcal{M}^\infty$ the set of all switchings sequences $\sigma = (\sigma(0), \sigma(1), \ldots)$. Let $x(\cdot; \sigma, \mathbf{u}, x(0))$ be the solution of the SLCS starting from the initial state $x(0)$ under the control policy $\mathbf{u} \in \mathcal{U}$ and the switching sequence $\sigma \in \mathcal{S}$.

Definition 1: The *σ_*-stabilizing rate* of the SLCS, denoted by $\rho_* \in [0, \infty)$, is defined to be the infimum of all ρ for which the following holds for some $\mathbf{u} \in \mathcal{U}$ and $K \in [0, \infty)$:

$$\|x(t; \sigma, \mathbf{u}, x(0))\| \leq K\rho^t \|x(0)\|, \quad \forall\, t, \forall\, x(0), \forall\, \sigma. \tag{3}$$

The SLCS is called *σ_*-exponentially stabilizable* if $\rho_* < 1$, i.e., (3) holds for some $\mathbf{u} \in \mathcal{U}$, $K \in [0, \infty)$, and $\rho \in [0, 1)$.

The SLCS is called *σ_*-asymptotically stabilizable* if there exists $\mathbf{u} \in \mathcal{U}$ such that $x(t; \sigma, \mathbf{u}, x(0)) \to 0$ as $t \to \infty$, $\forall\, x(0)$, $\forall\, \sigma \in \mathcal{S}$. The following theorem shows that the two notions of stabilizability are equivalent. Thus, we will refer to either of them as σ_*-stabilizability in the rest of this paper.

Theorem 1 ([25]): A SLCS is σ_*-exponential stabilizable if and only if it is σ_*-asymptotically stabilizable.

A related but different notion of stabilizability is defined in [17]. The SLCS is called *σ^*-stabilizable* if a user control policy without the knowledge of the current mode $\sigma(t)$, $u(t) = \mathbf{u}_t(x(t))$, can stabilize the system against arbitrary switching sequence $\sigma \in \mathcal{S}$. The σ^*-stabilizing rate ρ^* can be defined accordingly. As shown in the following example, σ^*-stabilizability is a stronger notion than σ_*-stabilizability and $\rho^* \geq \rho_*$ in general, with the gap representing the information premium of the knowledge of the current mode.

Example 1: Consider a SLCS with two modes and a one-dimensional state space: $A_1 = 1, B_1 = 1$; $A_2 = -1, B_2 = 1$. Given the state $x(t) \neq 0$ at any time t, without knowing $\sigma(t)$, any user decision $u(t) \neq 0$ will be countered by an adversarial decision $\sigma(t) = 1$ if $x(t)u(t) > 0$ and $\sigma(t) = 2$ if $x(t)u(t) < 0$. The resulting $x(t+1)$ satisfies $|x(t+1)| > |x(t)|$. Thus, the best control strategy without knowing $\sigma(t)$ is $u(t) = 0$, which results in $|x(t+1)| = |x(t)|$. This implies that the SLCS is not σ^*-stabilizable and its σ^*-stabilizing rate is $\rho^* = 1$. In comparison, if the user knows $\sigma(t)$, then $u(t)$ can be chosen to be $u(t) = -x(t)$ if $\sigma(t) = 1$ and $u(t) = x(t)$ if $\sigma(t) = 2$, which results in $x(t+1) = 0$. Hence, the SLCS is σ_*-stabilizable and its σ_*-stabilizing rate is $\rho_* = 0$ (since $x(t)$ is brought to the origin in one step).

An obvious necessary condition for the SLCS to be σ_*-stabilizable is that each subsystem (A_i, B_i) is a stabilizable LTI system, for otherwise the adversary will keep choosing the unstabilizable mode. This condition, however, is not sufficient as demonstrated by the example in (2). When all $B_i = 0$, the SLCS becomes an autonomous SLS, its σ_*-stabilizability is equivalent to its absolute stability [26], and its σ_*-stabilizing rate ρ_* is exactly the joint spectral radius (JSR) of the matrix set $\{A_i\}_{i\in\mathcal{M}}$ ([6]).

The following result states that ρ_* is positively homogeneous of degree one w.r.t the collective scale of $\{A_i\}_{i\in\mathcal{M}}$ but is independent of the scale of any individual B_i. The latter is not surprising due to the lack of penalty on control input in our problem formulation.

Lemma 1: Let ρ_* be the σ_*-stabilizing rate of the SLCS $\{(A_i, B_i)\}_{i\in\mathcal{M}}$. Given constants α and β_i, $i \in \mathcal{M}$, in $\mathbb{R}$ with $\beta_i \neq 0$, the scaled SLCS $\{(\alpha A_i, \beta_i B_i)\}_{i\in\mathcal{M}}$ has the σ_*-stabilizing rate $|\alpha|\, \rho_*$.

Proof: The conclusion is trivial if $\alpha = 0$. Suppose $\alpha \neq 0$. If the SLCS $\{(A_i, B_i)\}_{i\in\mathcal{M}}$ has the solution $x(t; \sigma, \mathbf{u}, x(0))$ under a control policy $\mathbf{u} \in \mathcal{U}$, then under the control policy $\hat{\mathbf{u}} \in \mathcal{U}$ such that $\hat{\mathbf{u}}_t(i, x) = \alpha^{t+1}(\beta_i)^{-1}\mathbf{u}_t(i, x)$, $\forall\, i \in \mathcal{M}$, $x \in \mathbb{R}^n$, the solution to the SLCS $\{(\alpha A_i, \beta_i B_i)\}_{i\in\mathcal{M}}$ is $\hat{x}(t; \sigma, \hat{\mathbf{u}}, \hat{x}(0)) = \alpha^t x(t; \sigma, \mathbf{u}, x(0))$. This proves the desired conclusion. ■

III. Defectiveness and Reducibility

Definition 2: The SLCS is called *nondefective* if there exist $\mathbf{u} \in \mathcal{U}$ and constants $K \in [0, \infty)$ such that $\|x(t; \sigma, \mathbf{u}, x(0))\| \leq K(\rho_*)^t \|x(0)\|$, $\forall\, t$, for all $x(0)$ and $\sigma \in \mathcal{S}$. Otherwise, it is called *defective*.

In other words, the SLCS is nondefective if the infimum of ρ in (3) can be exactly achieved by the stabilizing rate ρ_*. A simple example of defective SLCS is given by a (non-switched) LTI system (A_1, B_1) with A_1 and B_1 defined in (2). Its σ_*-stabilizing rate $\rho_* = 0.5$ is the spectral radius of A_1; however, starting from $x(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, the exponential growth rate of $\|x(t)\|$ cannot be exactly 0.5.

If $\rho_* = 0$, then the SLCS is nondefective if and only if each subsystem (A_i, B_i) is controllable to the origin in one time step, i.e., $A_i = B_i K_i$ for some matrix K_i. Hence, the LTI system (A, B), $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, is defective.

For general SLCS, nondefectiveness is very difficult to verify. A useful sufficient condition is given below. A subspace $\mathcal{V}$ of $\mathbb{R}^n$ is called *control σ_*-invariant* if for each $z \in \mathcal{V}$ and each $i \in \mathcal{M}$, there exists $v_i \in \mathbb{R}^p$ such that $A_i z + B_i v_i \in \mathcal{V}$. Two trivial control σ_*-invariant subspaces are $\{0\}$ and $\mathbb{R}^n$.

Definition 3: The SLCS is called *irreducible* if it has no nontrivial control σ_*-invariant subspaces. Otherwise, it is called *reducible*.

The following result which shows that, to check if a SLCS is nondefective, a sufficient condition is to check if it is irreducible (an easier task in general).

Theorem 2 ([25]): If the SLCS with $\rho_* > 0$ is irreducible, then it is nondefective.

IV. Bounds via Seminorms

In this section, we will derive various bounds on the σ_*-stabilizing rate ρ_*. Recall that a *seminorm* on $\mathbb{R}^n$ is a nonnegative function $\xi : \mathbb{R}^n \to \mathbb{R}_+$ that is convex and positively homogeneous of degree one [27]. A seminorm that is further positive definite, i.e., $\xi(x) = 0$ only if $x = 0$, becomes a norm on $\mathbb{R}^n$.

For any seminorm ξ on $\mathbb{R}^n$, define the operator

$$\mathcal{F}[\xi](z) := \max_{i\in\mathcal{M}} \inf_{v\in\mathbb{R}^p} \xi(A_i z + B_i v), \quad \forall z \in \mathbb{R}^n. \tag{4}$$

The operator $\mathcal{F}$ is similar to the operator $\mathcal{T}$ in [5] with the crucial difference that the order of $\max$ and $\min$ are switched due to the different information structure in this paper. In general, $\mathcal{F}(\xi) \le \mathcal{T}(\xi)$ with strict inequality possible.

It is easily verified that $\mathcal{F}$ maps the seminorm ξ to another seminorm $\mathcal{F}(\xi)$, which we denote by $\xi_\sharp$. In particular, if $\xi = \|\cdot\|$ is a norm, then $\xi_\sharp$, which we denoted by $\|\cdot\|_\sharp$, is a seminorm but not necessarily a norm on $\mathbb{R}^n$.

Proposition 1: Let $\alpha, \beta \ge 0$ be constants.

(i) If a nonzero seminorm ξ satisfies $\xi_\sharp \ge \alpha\xi$, then $\rho_* \ge \alpha$.
(ii) If a norm $\|\cdot\|$ satisfies $\alpha\|\cdot\| \le \|\cdot\|_\sharp \le \beta\|\cdot\|$, then $\alpha \le \rho_* \le \beta$.

Proof: Let ξ be a nonzero seminorm satisfying $\xi_\sharp \ge \alpha\xi$. Assume that the adversary adopts the switching policy

$$\sigma(t) = \arg\max_{i\in\mathcal{M}} \inf_v \xi(A_i x(t) + B_i v).$$

Then, for any choice of $u(t) = \mathbf{u}_t(\sigma(t), x(t))$,

$$\begin{aligned}\xi(x(t+1)) &= \xi(A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t)) \\ &\ge \inf_v \xi(A_{\sigma(t)}x(t) + B_{\sigma(t)}v) = \xi_\sharp(x(t)) \ge \alpha\,\xi(x(t)),\ \forall t.\end{aligned}$$

Thus, starting from $x(0)$ such that $\xi(x(0)) > 0$, we have $\xi(x(t)) \ge \alpha^t \xi(x(0))$ for any $\mathbf{u} \in \mathcal{U}$. This proves statement (i) as well as the first inequality of the statement (ii). For the second inequality of statement (ii), assume $\|\cdot\|$ is a norm satisfying $\|\cdot\|_\sharp \le \beta\|\cdot\|$. The user can then adopt the policy

$$u(t) = \mathbf{u}_t(\sigma(t), x(t)) := \arg\min_v \xi(A_{\sigma(t)}x(t) + B_{\sigma(t)}v).$$

Note that the minimizer v exists and is finite due to ξ being a seminorm [5], although it may not be unique. Under this control policy, for any $x(0)$ and any $\sigma \in \mathcal{S}$, we have

$$\begin{aligned}\xi(x(t+1)) &= \xi(A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t)) \\ &= \inf_v \xi(A_{\sigma(t)}x(t) + B_{\sigma(t)}v) \\ &\le \xi_\sharp(x(t)) \le \beta\xi(x(t)),\ \forall\, t.\end{aligned}$$

Hence, $\xi(x(t)) \le \beta^t \xi(x(0))$, which implies that $\rho_* \le \beta$. ∎

Definition 4: A seminorm ξ on $\mathbb{R}^n$ is called a *lower extremal seminorm* of the SLCS if $\xi_\sharp \ge \rho_*\xi$. A norm $\|\cdot\|$ on $\mathbb{R}^n$ is called an *(upper) extremal norm* if $\|\cdot\|_\sharp \le \rho_*\|\cdot\|$, and a *Barabanov norm* if $\|\cdot\|_\sharp = \rho_*\|\cdot\|$.

Proposition 1 shows that the task of finding tight lower and upper bounds of ρ_* can be reduced to finding lower and upper extremal (semi)norms of the SLCS. Based on this observation, we will develop a number of algorithms in Section V for the computation of ρ_*. In particular, if the SLCS has a Barabanov norm, then ρ_* can be precisely characterized. An example of such SLCS is given below.

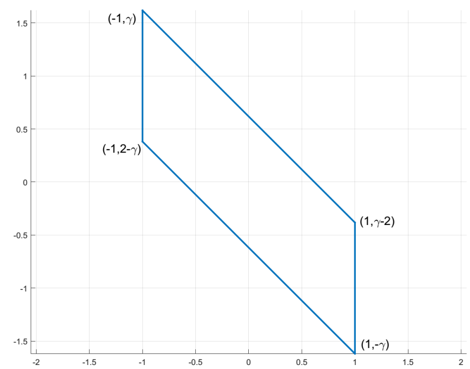

Fig. 1. Barabanov norm of the SLCS in Example 2.

Example 2: Consider the following SLCS:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix};\ A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Let $\gamma = (\sqrt{5}+1)/2 \approx 1.6180$, which satisfies $\gamma(\gamma-1) = 1$.

Define a norm on $\mathbb{R}^2$ as

$$\|z\| = \max\{|z_1|, \gamma|z_1 + z_2|\}, \quad \forall z = (z_1, z_2) \in \mathbb{R}^2. \tag{5}$$

The unit ball of $\|\cdot\|$ is shown in Fig. 1. We claim that $\|\cdot\|$ is a Barabanov norm of the SLCS: $\|\cdot\|_\sharp = \gamma^{-1}\|\cdot\|$. This would imply that $\rho_* = \gamma^{-1}$. Due to homogeneity, we only need to check the condition $\|z\|_\sharp = \gamma^{-1}\|z\|$ for $z = (1, 0)$ and $z = (z_1, 1)$ where $z_1 \in \mathbb{R}$.

Suppose $z = (1, 0)$. Then $\|z\| = \max\{1, \gamma\} = \gamma$, and

$$\begin{aligned}&\|z\|_\sharp \\ &= \max\left\{\min_v \max\{1, \gamma|v+2|\}, \min_v \max\{|v|, \gamma|v-1|\}\right\} \\ &= \max\{1, \gamma/(\gamma+1)\} = 1 = \gamma^{-1}\|z\|,\end{aligned}$$

which satisfies the desired condition.

Suppose $z = (z_1, 1)$. Then $\|z\| = \max\{|z_1|, \gamma|z_1 + 1|\}$,

$$\begin{aligned}\|z\|_\sharp = \max\Big\{&\min_v \max\{|z_1+1|, \gamma|2z_1 + v|\}, \\ &\min_v \max\{|v+1|, \gamma|v - z_1 + 1|\}\Big\} \\ = \max&\left\{|z_1+1|, \gamma^{-1}|z_1|\right\} = \gamma^{-1}\|z\|,\end{aligned} \tag{6}$$

which also satisfies the desired condition. As a result, the norm $\|\cdot\|$ constructed in (5) is indeed a Barabanov norm of the SLCS and the σ_*-stabilizing rate is $\rho_* = \gamma^{-1} \approx 0.6180$. In contrast, the σ^*-stabilizing rate ρ^* of the SLCS is shown to lie in the interval $[1.2183, 1.2239]$ (see [5]). This means that, although it is possible to design a user control policy to stabilize the SLCS against adversarial switchings if the user knows the current mode, the same task becomes impossible if the user has no knowledge of the current mode.

The optimal controls policy can also be constructed from (6). Suppose $x(t) = z = (z_1, 1)$. If $\sigma(t) = 1$, then $u^*(t) = \arg\min_v \max\{|z_1+1|, \gamma|2z_1+v|\}$, which can take any value between the two values $-2z_1 \pm (z_1+1)/\gamma$. If $\sigma(t) = 2$, then $u^*(t) = \arg\min_v \max\{|v+1|, \gamma|v - z_1 + 1|\} = (\gamma-1)z_1 - 1$. On the other hand, if $z = (1, 0)$, then $u^*(t)$ can be of arbitrary value between $-2 \pm \gamma^{-1}$ if $\sigma(t) = 1$; and $u^*(t) = \gamma - 1$ if $\sigma^*(t) = 2$. Via homogeneity, the above control policy can be extended to arbitrary $z \in \mathbb{R}^2$. In particular, it is noted that the following static mode-dependent linear controller is optimal in that it achieves the σ_*-stabilizing rate: $u^*(t) = K^*_{\sigma(t)} x(t)$, where

$$K_1^* = \begin{bmatrix} -2 & 0 \end{bmatrix}, \quad K_2^* = \begin{bmatrix} (\gamma-1) & 1 \end{bmatrix}.$$

By replacing A_1 with $\frac{1}{2}A_1$ and keep B_1, A_2, and B_2 unchanged, it can be verified that the new SLCS has the Barabanov norm $\|z\| = \max\{|z_1|, |z_1+z_2|\}$ and the σ_*-stabilizing rate $\rho_* = \frac{1}{2}$. In [5], it is founded that $\rho^* \in [0.8660, 0.8732]$, which is again strictly larger than ρ_*.

For general SLCS, Barabanov norms may not exist. A simple example is given by the (nonswitched) LTI system (A_1, B_1) with A_1 and B_1 given in (2). As $B_1 = 0$, the σ_*-stabilizing rate $\rho_* = \rho(A_1) = 0.5$. However, since A_1 is defective, there exists no norm $\|\cdot\|$ on $\mathbb{R}^n$ satisfying $\|\cdot\|_\sharp = 0.5\|\cdot\|$, i.e., (A_1, B_1) has no Barabanov norm.

The following theorem desribes a family of SLCS whose Barabanov norms exist.

Theorem 3: An irreducible SLCS has a Barabanov norm.

By the discussion in Section III, a generic SLCS (e.g., with the matrices A_i and B_i randomly generated) is irreducible. Thus, generic SLCS have Barabanov norms.

A. Lifting Method

Let h be a positive integer. For a seminorm ξ on $\mathbb{R}^n$, define

$$\mathcal{F}^{(h)}[\xi](z) := \max_{i_0 \in \mathcal{M}} \inf_{v_0 \in \mathbb{R}^p} \cdots \max_{i_{h-1} \in \mathcal{M}} \inf_{v_{h-1} \in \mathbb{R}^p} \xi\left(A_{i_{h-1}} \cdots A_{i_0} z + \sum_{j=0}^{h-1} A_{i_{h-1}} \cdots A_{i_{j+1}} B_{i_j} v_j \right)$$

for $z \in \mathbb{R}^n$. Then, it is easily verified that $\mathcal{F}^{(h)}(\xi)$ is also a seminorm. In other words, $\mathcal{F}^{(h)}$ is self mapping of the set of seminorms on $\mathbb{R}^n$. In particular, when $h = 1$, $\mathcal{F}^{(h)}$ is reduced to the operator $\mathcal{F}$ defined in (4).

By using a proof similar to that of Proposition 1, we can establish the following bounds of ρ_*.

Proposition 2: Let h be a positive integer and let $\alpha, \beta \geq 0$ be constants.

(i) If a seminorm ξ satisfies $\mathcal{F}^{(h)}(\xi) \geq \alpha\xi$, then $\rho_* \geq \sqrt[h]{\alpha}$.
(ii) If a norm $\|\cdot\|$ satisfies $\alpha\|\cdot\| \leq \mathcal{F}^{(h)}(\|\cdot\|) \leq \beta\|\cdot\|$, then $\sqrt[h]{\alpha} \leq \rho_* \leq \sqrt[h]{\beta}$.

V. Computation Algorithms

We now use the results in Section IV to develop algorithms for computing the σ_*-stabilizing rate ρ_* of SLCS.

A. Ellipsoidal Norms

Denote $\mathbb{P}_{\succ 0} = \{P \in \mathbb{R}^{n\times n} \mid P = P^T \succ 0\}$ the set of positive definite matrices. Similarly $\mathbb{P}_{\succeq 0}$ is the set of positive semidefinite matrices. Each $P \in \mathbb{P}_{\succeq 0}$ defines a seminorm $\|z\|_P := (z^T P z)^{1/2}$. When $P \in \mathbb{P}_{\succ 0}$, $\|\cdot\|_P$ is a norm, called the ellipsoidal norm.

Given an ellipsoidal norm $\|\cdot\|_P$ where $P \succ 0$, we have

$$\|z\|_{P\sharp} = \max_{i\in\mathcal{M}} z^T \left(A_i^T P A_i - A_i^T P B_i (B_i^T P B_i)^\dagger B_i^T P A_i \right) z,$$

where $\dagger$ stands for matrix pseudo-inverse. The condition that $\|\cdot\|_{P\sharp} \leq \beta\|\cdot\|_P$ is equivalent to

$$A_i^T P A_i - A_i^T P B_i (B_i^T P B_i)^\dagger B_i^T P A_i \preceq \beta^2 P, \quad \forall i \in \mathcal{M}.$$

The smallest β^* for the above to hold can be obtained by solving the above (nonconvex) problem in β^2 and P. For an easier but more conservative upper bound, we can focus on linear controllers $u(i,x) = K_i x$ and write

$$\begin{aligned} \|z\|_{P\sharp} &= \max_i \inf_v (A_i z + B_i v)^T P (A_i z + B_i v) \\ &\leq \max_i \inf_{K_i} z^T (A_i + B_i K_i)^T P (A_i + B_i K_i) z. \end{aligned}$$

Thus, a sufficient condition for $\|\cdot\|_{P\sharp} \leq \beta\|\cdot\|_P$ is given by

$$(A_i + B_i K_i)^T P (A_i + B_i K_i) \preceq \beta^2 P, \quad \forall i \in \mathcal{M}.$$

By letting $Q = P^{-1}$, $F_i = K_i P^{-1}$, and using Schur complement, we can rewrite the above as:

$$\begin{bmatrix} \beta Q & A_i Q + B_i F_i \\ Q A_i^T + F_i^T B_i^T & \beta Q \end{bmatrix} \succeq 0, \ \forall i \in \mathcal{M}, \tag{7}$$

for some $Q \succ 0$ and F_i, $i \in \mathcal{M}$. An upper bound of ρ_* is obtained by solving the LMI (7) with decreasing β until it becomes infeasible.

For lower bounds of ρ_*, a sufficient condition for $\|\cdot\|_{P\sharp} \geq \alpha\|\cdot\|_P$ is given by

$$\sum_{i\in\mathcal{M}} \theta_i \left(A_i^T P A_i - A_i^T P B_i (B_i^T P B_i)^\dagger B_i^T P A_i \right) \succeq \alpha^2 P,$$

for some $\theta \in \Delta_m$, where $\Delta_m := \{\theta \in \mathbb{R}^m \mid \theta_i \geq 0, \forall i, \sum_{i\in\mathcal{M}} \theta_i = 1\}$ is the m-simplex. Let $Q_i = Q_i^T \in \mathbb{R}^{n\times n}$. Then the above condition is equivalent to

$$\begin{aligned} &\left\{ \begin{array}{l} \sum_{i\in\mathcal{M}} \theta_i \left(A_i^T P A_i - Q_i \right) \succeq \alpha^2 P \\ Q_i \succeq A_i^T P B_i (B_i^T P B_i)^\dagger B_i^T P A_i, \ \forall i \in \mathcal{M} \end{array} \right. \\ \Leftrightarrow &\left\{ \begin{array}{l} \sum_{i\in\mathcal{M}} \theta_i \left(A_i^T P A_i - Q_i \right) \succeq \alpha^2 P \\ \begin{bmatrix} Q_i & A_i^T P B_i \\ B_i^T P A_i & B_i^T P B_i \end{bmatrix} \succeq 0, \ \forall i \in \mathcal{M}. \end{array} \right. \end{aligned}$$

The last condition is a BMI problem in $(P, Q_i, \theta_i, \alpha^2)$ and can be solved by using, e.g., the path following algorithm [28].

B. Polytopic Norm

Let $C \in \mathbb{R}^{n\times\ell}$ be such that its columns span $\mathbb{R}^n$. Then

$$\|z\|_C = \max_{j=1,\ldots,\ell} |c_j z|, \quad \forall z \in \mathbb{R}^n,$$

defines a norm on $\mathbb{R}^n$. Applying the operator (4), we have

$$\|z\|_{C\sharp} = \max_i \inf_v \max_j \left|c_j^T(A_i z + B_i v)\right|.$$

For each $i \in \mathcal{M}$, $\inf_v \max_j \left|c_j^T(A_i z + B_i v)\right|$ is the optimal value of the following linear programs:

$$\begin{aligned} \min_{v,y} \quad & y \\ \text{s.t.} \quad & \pm\, c_j^T(A_i z + B_i v) \le y,\ j = 1,\ldots,\ell. \end{aligned} \tag{8}$$

By introducing the multipliers $\theta_{ij}^+ \ge 0$ and $\theta_{ij}^- \ge 0$, $j = 1,\ldots,\ell$, the dual problem of problem (8) is

$$\begin{aligned} \max_{\theta_{ij}^+,\theta_{ij}^-} \quad & \sum_{j=1}^r (\theta_{ij}^+ - \theta_{ij}^-) c_j^T A_i z \\ \text{s.t.} \quad & \sum_{j=1}^r (\theta_{ij}^+ + \theta_{ij}^-) = 1, \quad \sum_{j=1}^r (\theta_{ij}^+ - \theta_{ij}^-) c_j^T B_i = 0, \\ & \theta_{ij}^+ \ge 0,\ \theta_{ij}^- \ge 0, \end{aligned} \tag{9}$$

whose optimal value is of the form $\max_{c\in\Omega_i} c^T z$, where

$$\Omega_i = \left\{ \sum_{j=1}^r (\theta_{ij}^+ - \theta_{ij}^-) A_i^T c_j \,\middle|\, \sum_{j=1}^r (\theta_{ij}^+ + \theta_{ij}^-) = 1, \right. \left. \sum_{j=1}^r (\theta_{ij}^+ - \theta_{ij}^-) c_j^T B_i = 0,\ \theta_{ij}^+ \ge 0,\ \theta_{ij}^- \ge 0 \right\}$$

is a bounded, centrally symmetric polytope in $\mathbb{R}^n$. Since the primal and dual problems have the same optimal value for each $i \in \mathcal{M}$, and $\|z\|_{C\sharp}$ is the maximum of these optimal values for $i \in \mathcal{M}$, we have

$$\|z\|_{C\sharp} = \max_{c\in\Omega} c^T z, \quad \text{where } \Omega := \mathrm{Co}\left(\cup_{i\in\mathcal{M}} \Omega_i\right).$$

Here, Co denotes the convex hull operation. We note that Ω is characterized by

$$\Omega = \left\{ \sum_{i,j} (\theta_{ij}^+ - \theta_{ij}^-) A_i^T c_j \,\middle|\, \sum_{i,j} (\theta_{ij}^+ + \theta_{ij}^-) = 1, \right. \left. \sum_{j} (\theta_{ij}^+ - \theta_{ij}^-) c_j^T B_i = 0,\ \forall i,\ \theta_{ij}^+ \ge 0,\ \theta_{ij}^- \ge 0 \right\}, \tag{10}$$

where the ranges of the indices are $i \in \mathcal{M}$ and $j = 1,\ldots,\ell$.

The condition that $\|\cdot\|_{C\sharp} \ge \alpha\|\cdot\|_C$ is equivalent to $\mathrm{Co}(\alpha c_1,\ldots,\alpha c_\ell) \subset \Omega$, i.e., $\alpha c_k \in \Omega$ for all $k = 1,\ldots,\ell$. The largest α for this to hold, $\alpha^* = \sup\{\alpha \ge 0 \,|\, \|\cdot\|_{C\sharp} \ge \alpha\|\cdot\|_C\}$, is equal to $\alpha^* = \min_{k=1,\ldots,\ell} \alpha_k^*$, where

$$\alpha_k^* := \sup\{\alpha \ge 0 \,|\, \alpha c_k \in \Omega\}$$

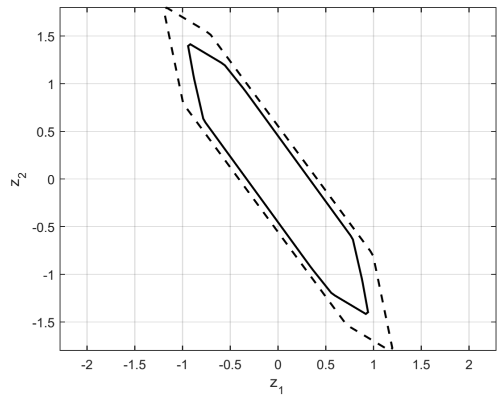

Fig. 2. Polytopic norm of the 2D SLCS computed by Algorithm 1. Solid line: unit sphere of $\|\cdot\|_C$; dashed line: unit sphere of $\|\cdot\|_{C\sharp}$.

is the solution of the following linear program in view of (10):

$$\begin{aligned} \max_{\theta_{ij}^+,\theta_{ij}^-,\alpha} \quad & \alpha \\ \text{subject to} \quad & \alpha \ge 0,\ \theta_{ij}^+ \ge 0,\ \theta_{ij}^- \ge 0,\ \forall i \in \mathcal{M},\ j = 1,\ldots,\ell \\ & \alpha c_k = \sum_{i,j} (\theta_{ij}^+ - \theta_{ij}^-) A_i^T c_j, \quad \sum_{i,j} (\theta_{ij}^+ + \theta_{ij}^-) = 1, \\ & \sum_j (\theta_{ij}^+ - \theta_{ij}^-) c_j^T B_i = 0,\ \forall\, i \in \mathcal{M}. \end{aligned} \tag{11}$$

An algorithm to compute the best lower bound of ρ_* using the polytopic norm $\|\cdot\|_C$ is given in Algorithm 1.

Algorithm 1

1: Initialize $C \in \mathbb{R}^{n\times\ell}$ with columns c_j, $j = 1,\ldots,\ell$
2: **repeat**
3: **for** $k = 1,\ldots,\ell$ **do**
4: Solve the linear program (11) to obtain α_k^*
5: **end for**
6: $k_1 \leftarrow \arg\max_k \alpha_k^*$, $k_2 \leftarrow \arg\min_k \alpha_k^*$
7: $c_{k_1} \leftarrow \sqrt{\alpha_{k_1}^*/\alpha_{k_2}^*} \cdot c_{k_1}$, $c_{k_2} \leftarrow \sqrt{\alpha_{k_2}^*/\alpha_{k_1}^*} \cdot c_{k_2}$
8: **until** $(\max_k \alpha_k^*)/(\min_k \alpha_k^*) \le 1 + \varepsilon$ or maximum number of iterations is reached
9: **return** $\alpha^* = \min_k \alpha_k^*$

Using the polytopic norm $\|\cdot\|_C$ obtained by Algorithm 1, one can solve a set of linear programs similar to (11) to find the smallest β such that $\|\cdot\|_{C\sharp} \le \beta\|\cdot\|$ holds. The results β^* will then yield an upper bound of ρ_*.

VI. Numerical Examples

Consider first the following SLCS on $\mathbb{R}^2$:

$$A_1 = \begin{bmatrix} -1.68 & -1.58 \\ -0.22 & 1.84 \end{bmatrix},\ B_1 = \begin{bmatrix} -0.07 \\ 0.41 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1.98 & 1.26 \\ 1.10 & 1.48 \end{bmatrix},\ B_2 = \begin{bmatrix} -0.32 \\ -0.24 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1.66 & -0.96 \\ -0.40 & 1.20 \end{bmatrix},\ B_3 = \begin{bmatrix} -0.35 \\ -0.36 \end{bmatrix}.$$

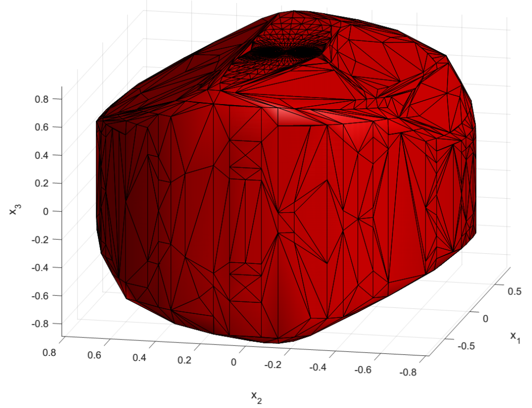

Fig. 3. Unit ball of the potytopic norm of the 3D SLCS computed by Algorithm 1.

By using Algorithm 1, a lower bound of ρ_* is computed as $\rho_* > 0.7862$. The unit spheres of the computed polytopic norm $\|\cdot\|_C$ and the corresponding sharp norm $\|\cdot\|_{C\sharp}$ are plotted in Fig. 2 in solid and dashed lines, respectively. Using this polytopic norm in Proposition 1, we conclude that the value of ρ_* lies in the interval $[0.7862, 0.8253]$.

Next consider the 3D SLCS given by

$$A_1 = \exp\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{\pi}{\sqrt{2}} \\ 0 & \frac{\pi}{\sqrt{2}} & 0 \end{bmatrix}\right),\ B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$A_2 = \exp\left(\begin{bmatrix} 0 & -\frac{\pi}{\sqrt{2}} & 0 \\ \frac{\pi}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right),\ B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$A_3 = \exp\left(\begin{bmatrix} 0 & 0 & -0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0 \end{bmatrix}\right),\ B_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The polytopic norm computed by Algorithm 1 is plotted in Figure 3. Using this norm and Proposition 1, it is found that the value of ρ_* lies in the interval $[0.9965, 1.0845]$.

VII. Conclusions

The stabilizability of switched linear control systems using continuous input against adversarial (but known) switchings is studied. Theoretical and practical techniques based on seminorms are proposed to derive bounds of the stabilizing rate, which provides a quantitative metric of the systems' stabilizability.

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