Stabilization of Discrete-Time Switched Linear Systems: A Control-Lyapunov Function Approach*

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Abstract. This paper studies the exponential stabilization problem for discrete-time switched linear systems based on a control-Lyapunov function approach. A number of versions of converse control-Lyapunov function theorems are proved and their connections to the switched LQR problem are derived. It is shown that the system is exponentially stabilizable if and only if there exists a finite integer N such that the N-horizon value function of the switched LQR problem is a control-Lyapunov function. An efficient algorithm is also proposed which is guaranteed to yield a control-Lyapunov function and a stabilizing strategy whenever the system is exponentially stabilizable.

1 Introduction

One of the basic problems for switched systems is to design a switched-control feedback strategy that ensures the stability of the closed-loop system [1]. The stabilization problem for switched systems, especially autonomous switched linear systems, has been extensively studied in recent years [2]. Most of the previous results are based on the existence of a switching strategy and a Lyapunov or Lyapunov-like function with decreasing values along the closed-loop system trajectory [3, 4]. These existence results have also led to some constructive ways to find the stabilizing switching strategy [5, 6]. The main idea is to parameterize the switching strategy and the Lyapunov function in terms of some matrices and then translate the Lyapunov theorem to some matrix inequalities. The solution of these matrix inequalities, when existing, will define a stabilizing switching strategy. However, these matrix inequalities are usually NP-hard to solve and relaxations and heuristic methods are often required. A similar idea is used to study the stabilization problem of nonautonomous switched linear systems [7, 8]. By assuming a linear state-feedback form for the continuous control of each mode, the problem is also formulated as a matrix inequality problem, where the feedback-gain matrices are part of the design variables. Although some sufficient and necessary conditions are derived for quadratic stabilizability [4, 9, 10], most

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of the previous stabilization results are far from necessary in the sense that the system may be asymptotically or exponentially stabilizable without satisfying the proposed conditions or the derived matrix inequalities.

In this paper, we study the exponential stabilization problem for discrete-time switched linear systems. Our goal is to develop a computationally appealing way to construct both a switching strategy and a continuous control strategy to exponentially stabilize the system when none of the subsystems is stabilizable but the switched system is exponentially stabilizable. Unlike most previous methods, we propose a controller synthesis framework based on the control-Lyapunov function approach which embeds the controller design in the design of the Lyapunov function. The control-Lyapunov function approach has been widely used for studying the stabilization problem of general nonlinear systems [11, 12]. However, its application in switched linear systems has not been adequately investigated. Another novelty of this paper is the derivation of some nice connections between the stabilization problem and the switched LQR problem. In particular, we show that the switched linear system is exponentially stabilizable if and only if there exists a finite integer N such that the N-horizon value function of the switched LQR problem is a control-Lyapunov function. This result not only serves as a converse control-Lyapunov function theorem, but also transforms the stabilization problem into the switched LQR problem. Motivated by the results of the switched LQR problem recently developed in [13–15], an efficient algorithm is proposed which is guaranteed to yield a control-Lyapunov function and a stabilizing strategy whenever the system is exponentially stabilizable. A numerical example is also carried out to demonstrate the effectiveness of the proposed algorithm.

2 Problem Formulation

We consider the discrete-time switched linear systems described by:

$$x(t+1) = A_{v(t)}x(t) + B_{v(t)}u(t),\ t \in \mathbb{Z}^+, \tag{1}$$

where $\mathbb{Z}^+$ denotes the set of nonnegative integers, $x(t) \in \mathbb{R}^n$ is the continuous state, $v(t) \in \mathbb{M} \triangleq \{1, \ldots, M\}$ is the discrete mode, and $u(t) \in \mathbb{R}^p$ is the continuous control. The integers n, M and p are all finite and the control u is unconstrained. The sequence of pairs $\{(u(t), v(t))\}_{t=0}^{\infty}$ is called the *hybrid control sequence*. For each $i \in \mathbb{M}$, A_i and B_i are constant matrices of appropriate dimensions and the pair (A_i, B_i) is called a subsystem. This switched linear system is time invariant in the sense that the set of available subsystems $\{(A_i, B_i)\}_{i=1}^{M}$ is independent of time t. We assume that there is no internal forced switchings, i.e., the system can stay at or switch to any mode at any time instant. At each time $t \in \mathbb{Z}^+$, denote by $\xi_t \triangleq (\mu_t, \nu_t) : \mathbb{R}^n \to \mathbb{R}^p \times \mathbb{M}$ the *hybrid control law* of system (1), where $\mu_t : \mathbb{R}^n \to \mathbb{R}^p$ is called the *continuous control law* and $\nu_t : \mathbb{R}^n \to \mathbb{M}$ is called the *switching control law*. A sequence of hybrid control laws constitutes an *infinite-horizon feedback policy*: $\pi \triangleq \{\xi_0, \xi_1, \ldots, \ldots\}$. If system (1) is driven by a feedback policy π, then the closed-loop dynamics is governed by

$$x(t+1) = A_{\nu_t(x(t))}x(t) + B_{\nu_t(x(t))}\mu_t(x(t)), \quad t \in \mathbb{Z}^+. \tag{2}$$

In this paper, the policy π is allowed to be time-varying and the feedback law $\xi_t = (\mu_t, \nu_t)$ at each time step can be an arbitrary function of the state. The special policy $\pi = \{\xi, \xi, \dots\}$ with the same feedback law $\xi_t = \xi$ at each time t is called a *stationary policy.*

Definition 1. *The origin of system (2) is exponentially stable if there exist constants $a > 0$ and $0 < c < 1$ such that the system trajectory starting from any initial state x_0 satisfies:*

$$\|x(t)\| \leq ac^t \|x_0\|.$$

Definition 2. *The system (1) is called* exponentially stabilizable *if there exists a feedback policy $\pi = \{(\mu_t, \nu_t)\}_{t \geq 0}$ under which the closed-loop system (2) is exponentially stable.*

Clearly, system (1) is exponentially stabilizable if one of the subsystems is stabilizable. A nontrivial problem is to stabilize the system when none of the subsystems are stabilizable. The main purpose of this paper is to develop an efficient and constructive way to solve the following stabilization problem.

Problem 1 (Stabilization Problem). Suppose that (A_i, B_i) is not stabilizable for any $i \in \mathbb{M}$. Find, if possible, a feedback policy π under which the closed-loop system (2) is exponentially stable.

3 A Control-Lyapunov Function Framework

We first recall a version of the Lyapunov theorem for exponential stability.

Theorem 1 (Lyapunov Theorem [16]). *Suppose that there exist a policy π and a nonnegative function $V : \mathbb{R}^n \to \mathbb{R}^+$ satisfying:*

1. *$\kappa_1 \|z\|^2 \leq V(z) \leq \kappa_2 \|z\|^2$ for some finite positive constants κ_1 and κ_2;*
2. *$V(x(t)) - V(x(t+1)) \geq \kappa_3 \|x(t)\|^2$ for some constant $\kappa_3 > 0$, where $x(t)$ is the closed-loop trajectory of system (2) under policy π.*

Then system (2) is exponentially stable under π.

To solve the stabilization problem, one usually needs to first propose a valid policy and then construct a Lyapunov function that satisfies the conditions in the above theorem. A more convenient way is to combine these two steps together, resulting in the control-Lyapunov function approach.

Definition 3 (ECLF). *The nonnegative function $V : \mathbb{R}^n \to \mathbb{R}^+$ is called an* exponentially stabilizing control Lyapunov function *(ECLF) of system (1) if*

1. *$\kappa_1 \|z\|^2 \leq V(z) \leq \kappa_2 \|z\|^2$ for some finite positive constants κ_1 and κ_2;*

2. $V(z) - \inf_{\{v\in\mathbb{M}, u\in\mathbb{R}^p\}} V(A_v z + B_v u) \geq \kappa_3 \|z\|^2$ *for some constant* $\kappa_3 > 0$.

The ECLF, if exists, represents certain abstract energy of the system. The second condition of Definition 3 guarantees that by choosing proper hybrid controls, the abstract energy decreases by a constant factor at each step. This together with the first condition implies the exponential stabilizability of system (1).

Theorem 2. *If system (1) has an ECLF, then it is exponentially stabilizable.*

Proof. Follows directly from Theorem 1 and Definition 3. □

If $V(z)$ is an ECLF, then one can always find a feedback law ξ that satisfies the conditions of Theorem 1. Such a feedback law is exponentially stabilizing, but may result in a large control action. A systematic way to stabilize the system with a reasonable control effort is to choose the hybrid control (u, v) that minimizes the abstract energy at the next step $V(A_v z + B_v u)$ plus certain kind of control energy expense. Toward this purpose, we introduce the following feedback law:

$$\xi_V(z) = (\mu_V(z), \nu_V(z)) = \arg\inf_{u\in\mathbb{R}^p, v\in\mathbb{M}} \left[V(A_v z + B_v u) + u^T R_v u\right], \tag{3}$$

where for each $v \in \mathbb{M}$, $R_v = R_v^T \succ 0$ characterizes the penalizing metric for the continuous control u. Since the quantity inside the bracket is bounded from below and grows to infinity as $\|u\| \to \infty$, the minimizer of (3) always exists in $\mathbb{R}^p \times \mathbb{M}$. Furthermore, if we have

$$V(z) - V(A_{\nu_V(z)} z + B_{\nu_V(z)} \mu_V(z)) \geq \kappa_3 \|z\|^2, \tag{4}$$

for some constant $\kappa_3 > 0$, we know that system (1) is exponentially stabilizable by the stationary policy $\{\xi_V, \xi_V, \ldots\}$. The challenge is how to find an ECLF that satisfies (4).

In the rest of this paper, we will focus on a particular class of piecewise quadratic functions as candidates for the ECLFs of system (1). Each of these functions can be written as a pointwise minimum of a finite number of quadratic functions as follows:

$$V_{\mathcal{H}}(z) = \min_{P\in\mathcal{H}} z^T P z, \tag{5}$$

where $\mathcal{H}$ is a finite set of positive definite matrices, hereby referred to as the *FPD set*. The main reason that we focus on functions of the form (5) is that this form is sufficiently rich in terms of characterizing the ECLFs of system (1). It will be shown in Section 5 that the system is exponentially stabilizable if and only if there exists an ECLF of the form (5).

With the particular structure of the candidate ECLFs (5), the feedback law defined in (3) can be derived in closed form. Its expression is closely related to the Riccati equation and the Kalman gain of the classical LQR problem. To derive this expression, we first define a few notations. Let $\mathcal{A}$ be the *positive* *semidefinite cone*, namely, the set of all symmetric positive semidefinite (p.s.d.) matrices. For each subsystem $i \in \mathbb{M}$, define a mapping $\rho_i^0 : \mathcal{A} \to \mathcal{A}$ as:

$$\rho_i^0(P) = A_i^T P A_i - A_i^T P B_i (R_i + B_i^T P B_i)^{-1} B_i^T P A_i. \tag{6}$$

It will become clear in Section 4 that the mapping ρ_i^0 is the difference Riccati equation of subsystem i with a zero state-weighting matrix. For each subsystem $i \in \mathbb{M}$ and each p.s.d. matrix P, the Kalman gain is defined as

$$K_i(P) \triangleq (R_i + B_i^T P B_i)^{-1} B_i^T P A_i. \tag{7}$$

Lemma 1. *Let $\mathcal{H}$ be an arbitrary FPD set. Let $V_{\mathcal{H}} : \mathbb{R}^n \to \mathbb{R}^+$ be defined by $\mathcal{H}$ through (5). Then the feedback law defined in (3) is given by*

$$\xi_{V_{\mathcal{H}}}(z) = \left(-K_{i_{\mathcal{H}}(z)}\left(P_{\mathcal{H}}(z)\right) z, i_{\mathcal{H}}(z)\right), \tag{8}$$

where $K_i(\cdot)$ is the Kalman gain defined in (7) and

$$(P_{\mathcal{H}}(z), i_{\mathcal{H}}(z)) = \underset{P \in \mathcal{H}, i \in \mathbb{M}}{\arg\min}\, z^T \rho_i^0(P) z. \tag{9}$$

Proof. By (3), to find ξ_V, we need to solve the following optimization problem:

$$\begin{aligned} f(z) &\triangleq \inf_{u \in \mathbb{R}^p, i \in \mathbb{M}} \left[\min_{P \in \mathcal{H}} (A_i z + B_i u)^T P (A_i z + B_i u) + u^T R_i u \right] \\ &= \min_{i \in \mathbb{M}, P \in \mathcal{H}} \left\{ \inf_{u \in \mathbb{R}^p} \left[(A_i z + B_i u)^T P (A_i z + B_i u) + u^T R_i u \right] \right\}. \end{aligned} \tag{10}$$

For each $i \in \mathbb{M}$ and $P \in \mathcal{H}$, the quantity inside the square bracket is quadratic in u. Thus, the optimal value of u can be easily computed as $u^* = -K_i(P)z$, where $K_i(P)$ is the Kalman gain defined in (7). Substituting u^* into (10) and simplifying the resulting expression yields $f(z) = z^T \rho^0_{i_{\mathcal{H}}(z)}(P_{\mathcal{H}}(z))z$, where $P_{\mathcal{H}}(z)$ and $i_{\mathcal{H}}(z)$ are defined in (9). □

To check whether a function $V_{\mathcal{H}}$ defined by a FPD set $\mathcal{H}$ is an ECLF, it is convenient to introduce another FPD set $\mathcal{F}_{\mathcal{H}}$ defined as:

$$\mathcal{F}_{\mathcal{H}} = \{\rho_i^0(P) : i \in \mathbb{M} \text{ and } P \in \mathcal{H}\}. \tag{11}$$

In other words, $\mathcal{F}_{\mathcal{H}}$ contains all the possible images of the mapping $\rho_i^0(P)$ as i ranges over $\mathbb{M}$ and P ranges over $\mathcal{H}$.

Theorem 3. *Let $\mathcal{H}$ be an arbitrary FPD set. Let $V_{\mathcal{H}} : \mathbb{R}^n \to \mathbb{R}^+$ and $V_{\mathcal{F}_{\mathcal{H}}} : \mathbb{R}^n \to \mathbb{R}^+$ be defined by $\mathcal{H}$ and $\mathcal{F}_{\mathcal{H}}$, respectively, by (5). Then the stationary policy $\pi_{V_{\mathcal{H}}} = \{\xi_{V_{\mathcal{H}}}, \xi_{V_{\mathcal{H}}}, \ldots\}$ is exponentially stabilizing if*

$$V_{\mathcal{H}}(z) - V_{\mathcal{F}_{\mathcal{H}}}(z) \geq \kappa_3 \|z\|^2, \tag{12}$$

for all $z \in \mathbb{R}^n$ and some constant $\kappa_3 > 0$.

Proof. Obviously, $V_{\mathcal{H}}$ satisfies the first condition of Definition 3. By (8), it can be easily verified that (12) implies (4). Thus, $V_{\mathcal{H}}$ is an ECLF satisfying (4) and the desired result follows. □

For a given function $V_{\mathcal{H}}$ of the form (5), to see whether it is an ECLF, we should check condition (12). Since both $V_{\mathcal{H}}$ and $V_{\mathcal{F}_{\mathcal{H}}}$ are homogeneous, we only need to consider the points on the unit sphere in $\mathbb{R}^n$ to verify (12). In $\mathbb{R}^2$, a practical way of checking (12) is to plot the functions $V_{\mathcal{H}}(z)$ and $V_{\mathcal{F}_{\mathcal{H}}}(z)$ along the unit circle to see whether $V_{\mathcal{H}}(z)$ is uniformly above $V_{\mathcal{F}_{\mathcal{H}}}(z)$. In higher dimensional state spaces, there is no general way to efficiently verify this condition. Nevertheless, a sufficient convex condition can be obtained using the S-procedure.

Theorem 4 (Convex Test). *With the same notations as in Theorem 3, the stationary policy $\pi_{V_{\mathcal{H}}} = \{\xi_{V_{\mathcal{H}}}, \xi_{V_{\mathcal{H}}}, \ldots\}$ is exponentially stabilizing if for each $P_{\mathcal{H}} \in \mathcal{H}$, there exists nonnegative constants α_j, $j = 1, \ldots, k$, such that*

$$\sum_{j=1}^{k} \alpha_j = 1, \text{ and } P_{\mathcal{H}} \succ \sum_{j=1}^{k} \alpha_j P_{\mathcal{F}_{\mathcal{H}}}^{(j)}, \tag{13}$$

where $k = |\mathcal{F}_{\mathcal{H}}|$ and $\{P_{\mathcal{F}_{\mathcal{H}}}^{(j)}\}_{j=1}^{k}$ is an enumeration of $\mathcal{F}_{\mathcal{H}}$.

Proof. See [17]. □

4 A Converse ECLF Theorem Using Dynamic Programming

By focusing on the ECLFs of the form (5) and the feedback laws of the form (3), the stabilization problem becomes a quadratic optimal control problem. The main purpose of this section is to prove that system (1) is exponentially stabilizable if and only if there exists an ECLF that satisfies (4). Our approach is based on the theory of the switched LQR problem recently developed in [13, 15].

4.1 The Switched LQR Problem

Let $Q_i = Q_i^T \succ 0$ and $R_i = R_i^T \succ 0$ be the weighting matrices for the state and the control, respectively, for subsystem $i \in \mathbb{M}$. Define the running cost as

$$L(x, u, v) = x^T Q_v x + u^T R_v u, \quad \text{for } x \in \mathbb{R}^n, u \in \mathbb{R}^p, v \in \mathbb{M}. \tag{14}$$

Denote by $J_\pi(z)$ the total cost, possibly infinite, starting from $x(0) = z$ under policy π, i.e.,

$$J_\pi(z) = \sum_{t=0}^{\infty} L(x(t), \mu_t(x(t)), \nu_t(x(t))). \tag{15}$$

Denote by Π the set of all admissible policies, i.e., the set of all sequences of functions $\pi = \{\xi_0, \xi_1, \ldots\}$ with $\xi_t : \mathbb{R}^n \to \mathbb{R}^p \times \mathbb{M}$ for $t \in \mathbb{Z}^+$. Define $V^*(z) = \inf_{\pi \in \Pi} J_\pi(z)$. Since the running cost is always nonnegative, the infimum always exists. The function $V^*(z)$ is usually called the *infinite-horizon value function*. It will be infinite if $J_\pi(z)$ is infinite for all the policies $\pi \in \Pi$. As a natural extension of the classical LQR problem, the *Discrete-time Switched LQR problem* (DSLQR) is defined as follows.

Problem 2 (DSLQR problem). For a given initial state $z \in \mathbb{R}^n$, find the infinite-horizon policy $\pi \in \Pi$ that minimizes $J_\pi(z)$ subject to equation (2).

4.2 The Value Functions of the DSLQR Problem

Dynamic programming solves the DSLQR problem by introducing a sequence of value functions. Define the N-horizon value function $V_N : \mathbb{R}^n \to \mathbb{R}$ as:

$$V_N(z)=\inf_{\substack{u(t)\in\mathbb{R}^p, v(t)\in\mathbb{M}\\ 0\le t\le N-1}} \left\{ \sum_{t=0}^{N-1} L(x(t),u(t),v(t)) \Big| x(0)=z \right\}. \tag{16}$$

For any function $V : \mathbb{R}^n \to \mathbb{R}^+$ and any feedback law $\xi = (\mu, \nu) : \mathbb{R}^n \to \mathbb{R}^p \times \mathbb{M}$, denote by $\mathcal{T}_\xi$ the operator that maps V to another function $\mathcal{T}_\xi[V]$ defined as:

$$\mathcal{T}_\xi[V](z) = L(z,\mu(z),\nu(z)) + V(A_{\nu(z)}z + B_{\nu(z)}\mu(z)),\ \forall z \in \mathbb{R}^n. \tag{17}$$

Similarly, for any function $V : \mathbb{R}^n \to \mathbb{R}^+$, define the operator $\mathcal{T}$ by

$$\mathcal{T}[V](z) = \inf_{u\in\mathbb{R}^p, v\in\mathbb{M}} \{L(z,u,v) + V(A_v z + B_v u)\},\ \forall z \in \mathbb{R}^n. \tag{18}$$

The equation defined above is called the *one-stage value iteration* of the DSLQR problem. We denote by $\mathcal{T}^k$ the composition of the mapping $\mathcal{T}$ with itself k times, i.e., $\mathcal{T}^k[V](z) = \mathcal{T}\left[\mathcal{T}^{k-1}[V]\right](z)$ for all $k \in \mathbb{Z}^+$ and $z \in \mathbb{R}^n$. Some standard results of Dynamic Programming are summarized in the following lemma.

Lemma 2 ([18]). *Let $V_0(z) = 0$ for all $z \in \mathbb{R}^n$. Then (i) $V_N(z) = \mathcal{T}^N[V_0](z)$ for all $N \in \mathbb{Z}^+$ and $z \in \mathbb{R}^n$; (ii) $V_N(z) \to V^*(z)$ pointwise in $\mathbb{R}^n$ as $N \to \infty$. (iii) The infinite-horizon value function satisfies the Bellman equation, i.e., $\mathcal{T}[V^*](z) = V^*(z)$ for all $z \in \mathbb{R}^n$.*

To derive the value function of the DSLQR problem, we introduce a few definitions. Denote by $\rho_i : \mathcal{A} \to \mathcal{A}$ the *Riccati Mapping* of subsystem $i \in \mathbb{M}$, i.e.,

$$\rho_i(P) = Q_i + A_i^T P A_i - A_i^T P B_i (R_i + B_i^T P B_i)^{-1} B_i^T P A_i. \tag{19}$$

Definition 4. *Let $2^{\mathcal{A}}$ be the power set of $\mathcal{A}$. The mapping $\rho_{\mathbb{M}} : 2^{\mathcal{A}} \to 2^{\mathcal{A}}$ defined by: $\rho_{\mathbb{M}}(\mathcal{H}) = \{\rho_i(P) : i \in \mathbb{M} \text{ and } P \in \mathcal{H}\}$ is called the* Switched Riccati Mapping *associated with Problem 2.*

Definition 5. *The sequence of sets $\{\mathcal{H}_k\}_{k=0}^N$ generated iteratively by $\mathcal{H}_{k+1} = \rho_{\mathbb{M}}(\mathcal{H}_k)$ with initial condition $\mathcal{H}_0 = \{0\}$ is called the* Switched Riccati Sets *of Problem 2.*

The switched Riccati sets always start from a singleton set $\{0\}$ and evolve according to the switched Riccati mapping. For any finite N, the set $\mathcal{H}_N$ consists of up to M^N p.s.d. matrices. An important fact about the DSLQR problem is that its value functions are completely characterized by the switched Riccati sets.

Theorem 5 ([13]). *The N-horizon value function for the DSLQR problem is given by*

$$V_N(z) = \min_{P \in \mathcal{H}_N} z^T P z. \tag{20}$$

4.3 A Converse ECLF Theorem

The main purpose of this subsection is to show that if system (1) is exponentially stabilizable, then an ECLF must exist and can be chosen to be the infinite-horizon value function V^* of the DSLQR problem. Denote by $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ the smallest and the largest eigenvalue of a p.s.d. matrix, respectively. Let

$$\sigma_A^+ = \max_{i \in \mathbb{M}} \left\{ \sqrt{\lambda_{\max}(A_i^T A_i)} \right\}, \ \lambda_Q^- = \min_{i \in \mathbb{M}} \{\lambda_{\min}(Q_i)\},$$
$$\lambda_Q^+ = \max_{i \in \mathbb{M}} \{\lambda_{\max}(Q_i)\}, \lambda_R^- = \min_{i \in \mathbb{M}} \{\lambda_{\min}(R_i)\} \text{ and } \lambda_R^+ = \max_{i \in \mathbb{M}} \{\lambda_{\max}(R_i)\}.$$

We first prove some important properties of V^*.

Lemma 3. *If system (1) is exponentially stabilizable, then (i) there exists a constant $\beta < \infty$ such that $\lambda_Q^- \|z\|^2 \leq V^*(z) \leq \beta \|z\|^2$; (ii) there exists a stationary optimal policy.*

Proof. (i) The proof of the first part is rather technical and is thus omitted here. Interested readers may refer to [17] for the detailed proof. (ii) By Lemma 2, $V^*(z)$ satisfies the Bellman equation, i.e.,

$$V^*(z) = \inf_{u \in \mathbb{R}^p, v \in \mathbb{M}} \{L(z,u,v) + V^*(A_v z + B_v u)\}, \ \forall z \in \mathbb{R}^n. \tag{21}$$

Let z be arbitrary and fixed. If $V^*(z)$ is infinite, then an arbitrary $\xi^*(z) \in \mathbb{R}^p \times \mathbb{M}$ achieves the infimum of (21) which is infinite. Now suppose $V^*(z)$ is finite. Then there exists a hybrid control (u, v) under which the quantity inside the bracket of (21) is finite. Denote by $\hat{V}$ this finite number. Since $R_v \succ 0$ for all $v \in \mathbb{M}$, there must exists a compact set $\mathcal{U}$ such that $L(z, u, v) \geq \hat{V}$ as long as $u \notin \mathcal{U}$. This implies that

$$V^*(z) = \inf_{u \in \mathcal{U}, v \in \mathbb{M}} \{L(z,u,v) + V^*(A_v z + B_v u)\}.$$

Since $\mathcal{U}$ is compact, there always exists a hybrid control that achieves the infimum of (21). Therefore, in any case, there must exist a feedback law $\xi^*(z) = (\mu^*(z), \nu^*(z))$ such that $\mathcal{T}_{\xi^*}[V^*](z) = V^*(z)$ for each $z \in \mathbb{R}^n$. □

The following theorem relates the exponential stabilizability with the infinite-horizon value function V^*.

Theorem 6 (Converse ECLF Theorem I). *System (1) is exponentially stabilizable if and only if $V^*(z)$ is an ECLF of system (1) that satisfies condition (4).*

Proof. The "only if" part follows directly from Theorem 2. Now suppose that system (1) is exponentially stabilizable. By part (i) of Lemma 3, $V^*(z)$ satisfies the first condition of Definition 3. Furthermore, by part (ii) of Lemma 3, there exists a feedback law $\xi^* = (\mu^*, \nu^*)$ such that $V^*(z) = \mathcal{T}_{\xi^*}[V^*](z)$. This implies that

$$V^*(z) - V^*(A_{\nu^*(z)}z + B_{\nu^*(z)}\mu^*(z)) - [\mu^*(z)]^T R_{\nu^*(z)}[\mu^*(z)] \geq \lambda_Q^- \|z\|^2.$$

Let $\xi_{V^*} = (\hat{\mu}, \hat{\nu})$ be defined as in (3) with V replaced by V^*. Then we have

$$\begin{aligned}
&V^*(z) - V^*\left(A_{\hat{\nu}(z)}z + B_{\hat{\nu}(z)}\hat{\mu}(z)\right) \\
\geq &V^*(z) - V^*\left(A_{\hat{\nu}(z)}z + B_{\hat{\nu}(z)}\hat{\mu}(z)\right) - [\hat{\mu}(z)]^T R_{\hat{\nu}(z)}[\hat{\mu}(z)] \\
\geq &V^*(z) - V^*\left(A_{\nu^*(z)}z + B_{\nu^*(z)}\mu^*(z)\right) - [\mu^*(z)]^T R_{\nu^*(z)}[\mu^*(z)] \geq \lambda_Q^- \|z\|^2,
\end{aligned}$$

where the last step follows from the definition of ξ_{V^*} in (3). Thus, V^* also satisfies condition (4). Hence, V^* is an ECLF satisfying (4). □

By this theorem, whenever system (1) is exponentially stabilizable, $V^*(z)$ can be used as an ECLF to construct an exponentially stabilizing feedback law ξ_{V^*}. However, from a design view point, such an existence result is not very useful as V^* can seldom be obtained exactly. In the next section, we will develop an efficient algorithm to compute an approximation of V^* which is also guaranteed to be an ECLF of system (1).

5 Efficient Computation of ECLFs

In this section, we will find an approximation of V^* which can be efficiently computed yet close enough to V^* so that it remains a valid ECLF of system (1). To find such an approximation, we need the following convergence result.

Theorem 7 ([14]). *If $V^*(z) \leq \beta \|z\|^2$ for some $\beta < \infty$, then*

$$|V_{N_1}(z) - V_N(z)| \leq \alpha \gamma^N \|z\|^2, \tag{22}$$

for any $N_1 \geq N \geq 1$, where $\gamma = \frac{1}{1+\lambda_Q^-/\beta} < 1$ and $\alpha = \max\{1, \frac{\sigma_A^+}{\gamma}\}$.

By this theorem, the N-horizon value function V_N approaches V^* exponentially fast as $N \to \infty$. Therefore, as we increase N, V_N will quickly become an ECLF of system (1).

Theorem 8 (Converse ECLF Theorem II). *If system (1) is exponentially stabilizable, then there exists an integer $N_0 < \infty$ such that $V_N(z)$ is an ECLF satisfying condition (4) for all $N \geq N_0$.*

Proof. Define

$$\xi_N^*(z) = (\mu_N^*, \nu_N^*) \triangleq \underset{u \in \mathbb{R}^p, v \in \mathbb{M}}{\arg\inf} \{L(z,u,v) + V_N(A_v z + B_v u)\}. \tag{23}$$

By Lemma 2 and equation (23), we know that

$$V_{N+1}(z) = \mathcal{T}[V_N](z) = \mathcal{T}_{\xi_N^*}(z)[V_N](z), \forall z \in \mathbb{R}^n.$$

We now fix an arbitrary $z \in \mathbb{R}^n$ and let $u^* = \mu_N^*(z)$, $v^* = \nu_N^*(z)$ and $x^*(1) = A_{v^*}z + B_{v^*}u^*$. Therefore, $V_{N+1}(z) - V_N(x^*(1)) - (u^*)^T R_{v^*}(u^*) \geq \lambda_Q^- \|z\|^2$. By Theorem 7, $V_{N+1}(z) \leq V_N(z) + \alpha\gamma^N \|z\|^2$. Hence,

$$V_N(z) - V_N(x^*(1)) - (u^*)^T R_{v^*}(u^*) \geq (\lambda_Q^- - \alpha\gamma^N)\|z\|^2.$$

Thus, there must exist an $N_0 \leq \infty$ such that $(\lambda_Q^- - \alpha\gamma^N) > \lambda_Q^-/2$ for all $N \geq N_0$. Then, by a similar argument as in the proof of Theorem 6, we can conclude that V_N is an ECLF satisfying (4) for all $N \geq N_0$. □

Theorem 8 implies that when the system is exponentially stabilizable, the ECLF not only exists but also can be chosen to be a piecewise quadratic function of the form (5). Furthermore, as N increases, the N-horizon value function V_N will eventually become an ECLF. Therefore, to solve the stabilization problem, we only need to compute the switched Riccati set $\mathcal{H}_N$. However, this method may not be computationally feasible as the size of $\mathcal{H}_N$ grows exponentially fast as N increases. Fortunately, if we allow a small numerical relaxation, an approximation of V_N can be efficiently computed [15].

Definition 6 (Numerical Redundancy). *A matrix $\hat{P} \in \mathcal{H}_N$ is called* (numerically) ϵ-redundant *with respect to $\mathcal{H}_N$ if*

$$\min_{P \in \mathcal{H}_N \setminus \hat{P}} z^T P z \leq \min_{P \in \mathcal{H}_N} z^T (P + \epsilon I_n) z, \text{ for any } z \in \mathbb{R}^n.$$

Definition 7 (ϵ-ES). *The set $\mathcal{H}_N^\epsilon$ is called an ϵ-Equivalent-Subset (ϵ-ES) of $\mathcal{H}_N$ if $\mathcal{H}_N^\epsilon \subset \mathcal{H}_N$ and for all $z \in \mathbb{R}^n$,*

$$\min_{P \in \mathcal{H}_N} z^T P z \leq \min_{P \in \mathcal{H}_N^\epsilon} z^T P z \leq \min_{P \in \mathcal{H}_N} z^T (P + \epsilon I_n) z.$$

Removing the ϵ-redundant matrices may introduce some error for the value function; but the error is no larger than ϵ for $\|z\| \leq 1$. To simplify the computation, for a given tolerance ϵ, we want to prune out as many ϵ-redundant matrices as possible. The following lemma provides a sufficient condition for testing the ϵ-redundancy for a given matrix.

Lemma 4 (Redundancy Test). *$\hat{P}$ is ϵ-redundant in $\mathcal{H}_N$ if there exist non-negative constants $\{\alpha_1\}_{i=1}^{k-1}$ such that $\sum_{i=1}^k \alpha_i = 1$ and $\hat{P} + \epsilon I_n \succeq \sum_{i=1}^k \alpha_i P^{(i)}$, where $k = |\mathcal{H}_N|$ and $\{P^{(i)}\}_{i=1}^{k-1}$ is an enumeration of $\mathcal{H}_N \setminus \{\hat{P}\}$.*

Algorithm 1

1. Denote by $P^{(i)}$ the i^{th} matrix in $\mathcal{H}_N$. Specify a tolerance ϵ and set $\mathcal{H}_N^{(1)} = \{P^{(1)}\}$.
2. For each $i = 2, \ldots, |\mathcal{H}_N|$, if $P^{(i)}$ satisfies the condition in Lemma 4 with respect to $\mathcal{H}_N$, then $\mathcal{H}_N^{(i)} = \mathcal{H}_N^{(i-1)}$; otherwise $\mathcal{H}_N^{(i)} = \mathcal{H}_N^{(i-1)} \cup \{P^{(i)}\}$.
3. Return $\mathcal{H}_N^{(|\mathcal{H}_N|)}$.

The condition in Lemma 4 can be easily verified using various existing convex optimization algorithms [19]. To compute an ϵ-ES of $\mathcal{H}_N$, we only need to remove the matrices in $\mathcal{H}_N$ that satisfy the condition in Lemma 4. The detailed procedure is summarized in Algorithm 1. Denote by $Algo_\epsilon(\mathcal{H}_N)$ the ϵ-ES of $\mathcal{H}_N$ returned by the algorithm. To further reduce the complexity, we can remove the ϵ-redundant matrices after every switched Riccati mapping. To this end, we define the *relaxed switched Riccati sets* $\{\mathcal{H}_k^\epsilon\}_{k=0}^N$ iteratively as:

$$\mathcal{H}_0^\epsilon = \mathcal{H}_0 \text{ and } \mathcal{H}_{k+1}^\epsilon = Algo_\epsilon(\rho_{\mathbb{M}}(\mathcal{H}_k^\epsilon)), \text{ for } k \leq N-1. \quad (24)$$

The function defined based on $\mathcal{H}_N^\epsilon$ is very close to V_N but much easier to compute as $\mathcal{H}_N^\epsilon$ usually contains much fewer matrices than $\mathcal{H}_N$. We now use the following example to demonstrate the simplicity of computing the set $\mathcal{H}_N^\epsilon$.

$$A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 1.5 & 1 \\ 0 & 1.5 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, Q_i = I_2, R_i = 1, i = 1, 2. \quad (25)$$

Clearly, neither subsystem is stabilizable. As shown in Fig. 1, a direct computation of $\{\mathcal{H}_k\}_{k=0}^N$ results in a combinatorial complexity of the order 10^9 for

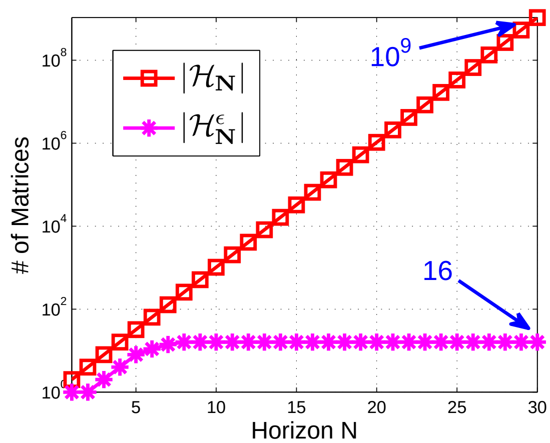

Fig. 1. Evolution of $|\mathcal{H}_N^\epsilon|$ with $\epsilon = 10^{-3}$.

Algorithm 2 (Computation of ECLF)

Specify proper values for ϵ, ϵ_{min} and N_{max}.
while $\epsilon > \epsilon_{\min}$ **do**
  for $N = 0$ to $N_{\max}$ **do**
    $\mathcal{H}_{N+1} = Algo_\epsilon(\rho_{\mathbb{M}}(\mathcal{H}_N))$
    if $\mathcal{H}^\epsilon_{N+1}$ satisfies the condition of Theorem 4 **then**
      stop and return V^ϵ_N as an ECLF
    end if
  end for
  $\epsilon = \epsilon/2$
end while

$N = 30$. However, if we use the relaxed iteration (24) with $\epsilon = 10^{-3}$, eventually $\mathcal{H}^\epsilon_N$ contains only 16 matrices. This example shows that the numerical relaxation can dramatically simplify the computation of $\mathcal{H}_N$. Our next task is to show that this relaxation does not change the value function too much. Define $V^\epsilon_N(z) = \min_{P\in\mathcal{H}^\epsilon_N} z^T P z$. It is proved in [15] that the total error between $V^\epsilon_N(z)$ and $V_N(z)$ can be bounded uniformly with respect to N.

Lemma 5 ([15]). *If $V^*(z) \le \beta\|z\|^2$ for some $\beta < \infty$, then*

$$V_N(z) \le V^\epsilon_N(z) \le V_N(z) + \epsilon\eta\|z\|^2, \tag{26}$$

where $\eta = \frac{1+(\beta/\lambda^-_Q - 1)\gamma}{1-\gamma}$.

The above lemma indicates that by choosing ϵ small enough, V^ϵ_N can approximate V_N with arbitrary accuracy. This warrants V^ϵ_N as an ECLF for large N and small ϵ.

Theorem 9 (Converse ECLF Theorem III). *If system (1) is exponentially stabilizable, then there exists an integer $N_0 < \infty$ and a real number $\epsilon_0 > 0$ such that $V^\epsilon_N(z)$ is an ECLF of system (2) satisfying condition (4) for all $N \ge N_0$ and all $\epsilon < \epsilon_0$.*

Proof. Similar to the proof of Theorem 8.

In summary, if the system is exponentially stabilizable, we can always find an ECLF of the form (5) defined by $\mathcal{H}^\epsilon_N$. To compute such an ECLF, we can start from a reasonable guess of ϵ and perform the relaxed switched Riccati mapping (24). After each iteration, we need to check whether the condition of Theorem 4 are met. If so, an ECLF is found; otherwise we should continue iteration (24). If the maximum iteration number N_{max} is reached, we should reduce ϵ and restart iteration (24). Since V^ϵ_N converges exponentially fast, N_{max} can usually be chosen rather small. The above procedure of constructing an ECLF is summarized in Algorithm 2. This algorithm is computationally efficient and guarantees to yield an ECLF provided that ϵ_{min} is sufficiently small and N_{max} is sufficiently large.

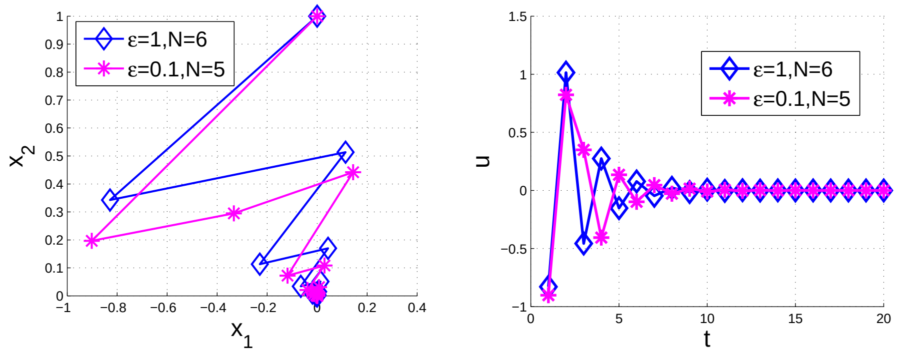

Fig. 2. Simulation Results. Left figure: phase-plane trajectories generated by the ECLFs V_6^1 and $V_5^{0.1}$ starting from the same initial condition $x_0 = [0, 1]^T$. Right figure: the corresponding continuous controls

.

6 Numerical Example

Consider the same two-mode switched system as defined in (25). Neither of the subsystems is stabilizable by itself. However, this switched system is stabilizable through a proper hybrid control. The stabilization problem can be easily solved using Algorithm 2. If we start from $\epsilon = 1$, then the algorithm terminates after 5 steps which results in an ECLF V_6^1 defined by the relaxed switched Riccati set $\mathcal{H}_6^1$. We have also tried a smaller relaxation $\epsilon = 0.1$. In this case, the algorithm stops after 4 steps resulting in an ECLF $V_5^{0.1}$ defined by the relaxed switched Riccati set $\mathcal{H}_5^{0.1}$. It is worth mentioning that $\mathcal{H}_6^1$ contains only two matrices and $\mathcal{H}_5^{0.1}$ contains 3 matrices. With these matrices, starting from any initial position x_0, the feedback laws corresponding to $\mathcal{H}_6^1$ and $\mathcal{H}_5^{0.1}$ can be easily computed using equation (3). The closed-loop trajectories generated by these two feedback laws starting from the same initial position $x_0 = [0, 1]^T$ are plotted on the left of Fig. 2. On the right of the same figure, the continuous control signals associated with the two trajectories are plotted. In both cases, the switching signals jump to the other mode at every time step and are not shown in the figure. It can also be seen that the ECLF $V_5^{0.1}$ stabilizes the system with a faster convergence speed and a smaller control energy than V_6^1. This is because it has a smaller relaxation ϵ which makes the resulting trajectory closer to the optimal trajectory of the DSLQR problem.

7 Conclusion

This paper studies the exponential stabilization problem for the discrete-time switched linear system. It has been proved that if the system is exponentially stabilizable, then there must exist a piecewise quadratic ECLF. More importantly, this ECLF can be chosen to be a finite-horizon value function of the switched LQR problem. An efficient algorithm has been developed to compute such an ECLF and the corresponding stabilizing policy whenever the system is exponentially stabilizable. Indicated by a numerical example, the ECLF and the stabilizing policy can usually be characterized by only a few p.s.d. matrices which can be easily computed using the relaxed switched Riccati mapping. Future research will focus on extending the algorithm to solve the robust stabilization problem for uncertain switched linear systems.

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