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Stabilizing Switched Linear Systems under Adversarial Switching*

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Abstract— **The problem of stabilizing discrete-time switched linear control systems using continuous input by the user and against adversarial switching by an adversary is studied. It is assumed that the adversary has the advantage in that at each time it knows the user's decision on the continuous control input but not vice versa. Stabilizability conditions and bounds on the fastest stabilizing rates are derived. Examples are given to illustrate the results.**

I. INTRODUCTION

Switched control systems as a family of hybrid control systems have dynamics that are subject to two input signals, namely, the (continuous) control input and the (discrete) switching signal (or mode sequence in the discrete-time case). Stabilization of switched control systems is the problem of designing control law for the control input and possibly also the switching signal so that the closed-loop switched systems are asymptotically or exponentially stable.

Stabilizability of switched control systems, especially switched linear control systems, has been a well studied problem [1], [2], [3], [4], [5], [6], [7]. Existing approaches can be roughly grouped into two categories. In the first category, both the continuous control input and the switching signal are utilized to stabilize the systems. Under this assumption, even if none of the subsystems is stabilizable by itself, properly designed switching laws and continuous controllers could still render the switched systems stable. Work in this category includes, e.g., [1], [8], [5], [6], [7]. In the second category, only the continuous input is under control, while the switching signal is unknown or a disturbance subject to some constraints on, e.g., switching frequency, dwell time, time delayed observability, etc. Typically, it is assumed that the continuous controller is aware of the current mode and can thus be of the form of a collection of mode-dependent state feedback controllers. Examples of prior work in the second category include [3], [9], [4], [10], [11].

The stabilization problem studied in this paper assumes that the user designs the continuous control to stabilize a discrete-time switched linear control system, while an adversary counters the user's effort with the most destabilizing switching sequence. This formulation differs from existing work in the second category above in that it has a different information structure: at any time the continuous input is first decided by the user without knowledge of the mode to be deployed; the mode is then chosen by the adversary with full knowledge of the user's decision. This information structure gives advantage to the adversary and makes the stabilization task much more difficult compared to the existing formulations. For example, even if each subsystem can be stabilized from any initial state to zero in one step, it is still possible that the switched system is not stabilizable under adversarial switching (see Example 1). A family of application examples of the problem studied in this paper can be found in the stabilization (or consensus, rendezvous) of networked control systems where the network is subject to the attack of an informed saboteur (see Example 2).

This paper is organized as follows. The σ-resilient stabilization problem is formulated in Section II. The concepts of irreducible and nondefective systems are introduced in Section III. In Section IV, lower and upper bounds on the σ-resilient stabilizing rate are obtained. The notions of generating functions are introduced in Sections V . Finally, some concluding remarks are given in Section VI.

II. RESILIENT STABILIZATION PROBLEM

Consider a switched linear controlled system (SLCS)

$$x(t+1) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \tag{1}$$

for $t \in \mathbb{N} = \{0, 1, \ldots\}$, where $x(\cdot) \in \mathbb{R}^n$ is the state, $u(\cdot) \in \mathbb{R}^p$ (or simply u) is the control input, and $\sigma(\cdot) \in \mathcal{M} = \{1, \ldots, m\}$ (or simply σ) is the switching sequence. Denote by $x(\cdot; \sigma, u, z) \in \mathbb{R}^n$ the SLCS solution starting from the initial state $x(0) = z$ under u and σ.

In this paper, we assume that the user specifies the control input u for the purpose of stabilizing the system, while an adversary specifies the switching sequence σ to prevent the user from stabilizing the system.

Assumption 1 (Information Structure): Denote by $\mathcal{F}_t := (x_{0:t}, u_{0:t-1}, \sigma_{0:t-1})$ the causal information available at time $t \in \mathbb{N}$, where $x_{0:t}$ denotes $x(0), \ldots, x(t)$, $u_{0:t-1}$ denotes $u(0), \ldots, u(t-1)$, and similarly for $\sigma_{0:t-1}$, with the understanding that $\mathcal{F}_0 = (x(0))$. At each time $t \in \mathbb{N}$, assume the user and the adversary determine the control input $u(t)$ and mode $\sigma(t)$ according to the functional forms $u(t) = u_t(\mathcal{F}_t)$ and $\sigma(t) = \sigma_t(\mathcal{F}_t, u(t))$, respectively.

Thus, the user and the adversary both have access to all the past information including the opponent's decisions when making their decisions at time t; and the adversary has the additional advantage of knowing the user's decision at time t as well. Both decisions are causal as no future information is utilized. Although the policies $u_t(\mathcal{F}_t)$ and $\sigma_t(\mathcal{F}_t, u(t))$ are in general of the feedback type, the subscript t in both of them allows for open-loop polices, i.e., dependence on t only. The set of all user control policies u and adversary switching

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policies σ compatible with the assumed information structure are denoted by $\mathcal{U}$ and $\mathcal{S}$, respectively.

Definition 1: The SLCS is called *σ-resiliently stabilizable* if there exists a user control policy $u \in \mathcal{U}$ such that $x(t;\sigma,u,x(0)) \to 0$ as $t \to \infty$ for all $x(0) \in \mathbb{R}^n$ and $\sigma \in \mathcal{S}$. It is called *σ-resiliently exponentially stabilizable* if we can find finite constants $K \geq 0$, $\rho \in [0,1)$, and a user control policy $u \in \mathcal{U}$ so that for all $x(0) \in \mathbb{R}^n$ and $\sigma \in \mathcal{S}$,

$$\|x(t;\sigma,u,x(0))\| \leq K\rho^t \|x(0)\|, \ \forall t \in \mathbb{N}. \qquad (2)$$

The *σ-resilient (exponential) stabilizing rate* ρ^* is the infimum of all ρ for which (2) holds.

The two stability notions above are equivalent.

Theorem 1 ([12]): The SLCS is σ-resiliently stabilizable if and only if it is σ-resiliently exponentially stabilizable.

The σ-resilient stabilizing rate provides a quantitative metric of the σ-resilient (exponential) stabilizability of the SLCS. The SLCS being σ-resiliently exponentially stabilizable is equivalent to $\rho^* < 1$. Note that ρ^* does not depend on the choice of the norm $\|\cdot\|$ in (2) as all such norms are equivalent.

The following result follows immediately.

Lemma 1: For any $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$, the scaled SLCS $\{(\alpha A_i, \beta B_i)\}_{i\in\mathcal{M}}$ has the σ-resilient stabilizing rate $|\alpha| \cdot \rho^*$.

Proof: We only need to prove for the nontrivial case $\alpha \neq 0$. First note that the SLCS $\{(\tilde{A}_i = \alpha A_i, \tilde{B}_i = \alpha B_i)\}_{i\in\mathcal{M}}$ has the σ-resilient stabilizing rate $|\alpha| \cdot \rho^*$ since its solutions $\tilde{x}(t;\sigma,\tilde{u},z) = \alpha^t \cdot x(t;\sigma,u,z)$, where $\tilde{u}(t) := \alpha^t u(t)$. Second, $\forall \beta \neq 0$, the SLCS $\{(\hat{A}_i = A_i, \hat{B}_i = \beta B_i)\}_{i\in\mathcal{M}}$ has the σ-resilient stabilizing rate ρ^* since its solutions $\hat{x}(t;\sigma,\beta^{-1}u,z) = x(t;\sigma,u,z)$, $\forall t$. Combining the above two results yields the desired conclusion. ■

Example 1: Consider a 1D SLCS with two subsystems: $A_1 = a_1$, $B_1 = b_1$, $A_2 = a_2$, $B_2 = b_2$, with $b_1^2 + b_2^2 \neq 0$. Thus, at least one subsystem (a_i, b_i) has $b_i \neq 0$ and is controllable hence stabilizable. To characterize σ-resilient stabilizability, suppose at any time $t \in \mathbb{N}$ we have $x(t) = z$. Applying a control $u(t) = v$ leads to two possible outcomes of $x(t+1)$: $\{a_1 z + b_1 v, a_2 z + b_2 v\}$. To achieve the slowest state growth rate for $\sigma \in \mathcal{S}$, $u(t)$ should be chosen to minimize $\max\{|a_1 z + b_1 v|, |a_2 z + b_2 v|\}$. We claim that

$$\min_v \max_{i=1,2} |a_i z + b_i v| = \frac{|a_1 b_2 - a_2 b_1|}{|b_1| + |b_2|} \cdot |z|. \qquad (3)$$

This can be proved by considering the following three cases:

(i) Suppose $b_1 b_2 < 0$. Then the minimizing v satisfies $a_1 z + b_1 v = a_2 z + b_2 v$, i.e., $v^* = -[(a_1 - a_2)/(b_1 - b_2)]z$. By choosing such v^*, $x(t+1) = [(a_2 b_1 - a_1 b_2)/(b_1 - b_2)]z$ regardless of $\sigma(t)$.

(ii) Suppose $b_1 b_2 > 0$. Then the minimizing v satisfies $a_1 z + b_1 v = -(a_2 z + b_2 v)$, i.e., $v^* = -[(a_1 + a_2)/(b_1 + b_2)]z$. This results in $x(t+1) = \pm[(a_1 b_2 - a_2 b_1)/(b_1 + b_2)]z$, with the sign depending on $\sigma(t) \in \{1,2\}$.

(iii) If $b_1 = 0$, then any v between $(a_1 - a_2)z/b_2$ and $-(a_1 + a_2)z/b_2$ is a minimizer of $\max\{|a_1 z|, |a_2 z + b_2 v|\}$, with the minimum being $|a_1 z|$.

The claim (3) implies that the σ-resilient stabilizing rate is

$$\rho^* = |a_1 b_2 - a_2 b_1|/(|b_1| + |b_2|), \qquad (4)$$

which satisfies the scaling property predicted by Lemma 1. If $a_1 a_2 b_2 b_2 \leq 0$, then $\rho^* = \nu|a_1| + (1-\nu)|a_2|$ with $\nu = |b_2|/(|b_1| + |b_2|)$ is between $|a_1|$ and $|a_2|$, the stabilizing rates of the two individual autonomous subsystems. However, if $a_1 a_2 b_2 b_2 > 0$, then ρ^* can be smaller than both $|a_1|$ and $|a_2|$. For example, suppose $a_1/b_1 = a_2/b_2$, i.e., the two subsystems are scaled versions of each other. Then $\rho^* = 0$. Indeed, from any $x(0) = z$, the control $u^*(0) = -(a_1/b_1)z = -(a_2/b_2)z$ ensures that $x(1) = 0$ regardless of $\sigma(0)$.

Remark 1: An observation from the above example is that the adversary will not gain any advantage if the user adopts the optimal *state-feedback* policy $u^*(\cdot)$ for all of its future control inputs and reveals it to the adversary at time 0. On the other hand, if the user adopts an *open-loop* control policy by implementing a fixed control input sequence, then the adversary by knowing such a sequence in advance will have a much greater advantage. In fact, it would be impossible to stabilize the system in Example 1 in the second setting. This observation remains valid for general SLCS's.

Example 2: A family of problems is given by the distributed stabilization of networked systems. In such systems, a number of linear subsystems are interconnected, can exchange information, and have dynamics couplings, via network links. Under a distributed control that uses only local (network neighbors') information, the overall system dynamics can be written as $x(t+1) = A_{G(t)}x(t) + B_{G(t)}u(t)$, where $x(t)$ and $u(t)$ are the concatenation of subsystems' state and control and $G(t)$ is the network topology at time t. Assume the network is hacked by an adversary, which may disable, e.g., up to a certain number of network connections. Then, the problem of stabilizing the networked systems, or its variant such as consensus/rendezvous [13] can be formulated as a σ-resilient stabilization problem.

III. Reducibility and Defectiveness

Recall that the σ-resilient stabilizing rate ρ^* is defined to be the infimum of all ρ satisfying (2). We now study the class of SLCSs for which the infimum can be exactly achieved.

Definition 2: A subset $\mathcal{V} \subset \mathbb{R}^n$ is called a *control σ-invariant set* of the SLCS (1) if for any $z \in \mathcal{V}$ there exists a control $u \in \mathbb{R}^p$ such that $A_i z + B_i u \in \mathcal{V}$ for all $i \in \mathcal{M}$. If $\mathcal{V}$ is further a subspace of $\mathbb{R}^n$, then it is called a *control σ-invariant subspace*.

Two trivial control σ-invariant subspaces are $\{0\}$ and $\mathbb{R}^n$.

Definition 3: The SLCS (1) is called

- *irreducible* if it does not have any nontrivial control σ-invariant subspaces. Otherwise, it is called *reducible*.
- *nondefective* if there exists a finite $K \geq 0$ and $u \in \mathcal{U}$ such that, for all $\sigma \in \mathcal{S}$ and z, $\|x(t;\sigma,u,z)\| \leq K(\rho^*)^t \|z\|$, $\forall t \in \mathbb{N}$. Otherwise, it is called *defective*.

Thus, nondefective SLCSs are those for which the minimal exponential growth rate in (2) is exactly achieved at ρ^*.

If there exists a common coordinate change so that all the subsystem matrices have the same block structure $A_i = \begin{bmatrix} \star & \star \\ 0 & \star \end{bmatrix}$ and $B_i = \begin{bmatrix} \star \\ \star \end{bmatrix}$, then $\left\{ \begin{bmatrix} \star \\ 0 \end{bmatrix} \right\}$ is a nontrivial control

σ-invariant subspace and the SLCS is reducible. For an example of defective SLCSs, consider the one with a single subsystem (A,B), where $A = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ and $B = \begin{bmatrix}0\\ 0\end{bmatrix}$. As B is zero, $\rho^* = 1$ is the spectral radius of A. However, $x(t) = A^t x(0) = \begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix} x(0)$ is unbounded for some $x(0)$. For another example, consider the SLCS with a single subsystem (A,B) with $A = \begin{bmatrix}2 & 1\\ 3 & 1\end{bmatrix}$ and $B = \begin{bmatrix}0\\ 1\end{bmatrix}$. As the LTI system is controllable to the origin in two steps, we have $\rho^* = 0$. The system is not controllable to the origin in one time step starting from some z; hence it is defective.

A SLCS with $\rho^* = 0$ is nondefective if and only if it is resiliently controllable to the origin in one time step, i.e., for any $z \in \mathbb{R}^n$, there exists $v \in \mathbb{R}^p$ such that $A_i z + B_i v = 0$ for all $i \in \mathcal{M}$. For such a SLCS, any subspace of $\mathbb{R}^n$ will be control σ-invariant; thus the system is reducible if $n \geq 1$.

Assume $\rho^* \neq 0$ and let $\|\cdot\|$ be an arbitrary norm of $\mathbb{R}^n$. Define the extended-value function

$$\zeta(z) := \inf_{u\in\mathcal{U}} \sup_{\sigma\in\mathcal{S}} \sup_{t\in\mathbb{N}} \frac{\|x(t;\sigma,u,z)\|}{(\rho^*)^t} \in \mathbb{R}_+ \cup \{+\infty\}, \quad (5)$$

for $z \in \mathbb{R}^n$. Obviously, $\zeta(z)$ is positively homogeneous of degree one: $\zeta(\alpha z) = \alpha\zeta(z)$, $\forall \alpha \geq 0$. Noting that $\|x(t;\sigma,u,z)\|$ is jointly convex in u and z for fixed t and σ and $\mathcal{U}$ is a vector space hence convex, by applying a result in [14, pp. 87], we deduce that ζ is convex. Thus, the set

$$\mathcal{W} := \{z \,|\, \zeta(z) < \infty\} \quad (6)$$

is a subspace of $\mathbb{R}^n$. We claim that $\mathcal{W}$ is control σ-invariant. Indeed, for any $z \in \mathcal{W}$, $\zeta(z) < \infty$ implies that there exists a policy $u = (u_0, u_1, \dots) \in \mathcal{U}$ and a finite K such that $\|x(t;\sigma,u,z)\| \leq K(\rho^*)^t$ for all t and all $\sigma = (\sigma_0, \sigma_1, \dots) \in \mathcal{S}$. Let $v = u_0(z)$ be the control at time $t = 0$ specified by the policy u and let $\sigma_0 = i$ be arbitrary. Then the solution starting from $x(1) = A_i z + B_i v$ under the control policy $u_+ := (u_1, u_2, \dots)$ satisfies $\|x(t;\sigma_+, u_+, x(1))\| = \|x(t+1;\sigma,u,z)\| \leq K(\rho^*)^{t+1}$ for all $t \in \mathbb{N}$ and all $\sigma_+ := (\sigma_1, \sigma_2, \dots) \in \mathcal{S}$. As a result, $\zeta(x(1)) \leq K\rho^* < \infty$ and thus $x(1) \in \mathcal{W}$, proving that $\mathcal{W}$ is control σ-invariant.

Theorem 2: An irreducible SLCS with $\rho^* \neq 0$ is nondefective.

Proof: By a proper scaling of matrices A_i's if necessary, we can assume without loss of generality that $\rho^* = 1$. Suppose the SLCS is irreducible. Then the subspace $\mathcal{W}$ defined in (6) being control σ-invariant must be either $\{0\}$ or $\mathbb{R}^n$. We will show by contradiction that the former is impossible. Suppose $\mathcal{W} = \{0\}$. Then, for any $z \in \mathbb{S}^{n-1}$ and any $u \in \mathcal{U}$, there exist some $\sigma \in \mathcal{S}$ and $s \in \mathbb{N}$ such that $\|x(s;\sigma,u,z)\| > 2$. We claim that the time s can be chosen to be uniformly bounded w.r.t. z and u. Suppose otherwise. Then there exist a sequence $\{z^{(k)}\}$ in $\mathbb{S}^{n-1}$, a sequence of policies $\{u^{(k)}\}$ in $\mathcal{U}$, and a strictly increasing sequence of times $\{s^{(k)}\}$ such that, for any $\sigma \in \mathcal{S}$, $\|x(t;\sigma,u^{(k)},z^{(k)})\| \leq 2$, $t = 0, \dots, s^{(k)}$, for $k = 1, 2, \dots$. By taking subsequences if necessary, we can assume that $z^{(k)}$ converges to some $z^* \in \mathbb{S}^{n-1}$. We will next construct a control policy u^* under which $\|x(t;\sigma,u^*,z^*)\| \leq 2$ for all t and all $\sigma \in \mathcal{S}$. To this purpose, for each k, we denote by $u_t^{(k)}(\sigma, z^{(k)})$ the actual control at time t produced by the policy $u^{(k)}$ w.r.t. the initial state $x(0) = z^{(k)}$ and $\sigma \in \mathcal{S}$. Note that $u_t^{(k)}(\sigma, z^{(k)})$, $t = 0, \dots, s^{(k)} - 1$, for all k can be chosen to be uniformly bounded as, under them, the state evolves from $x(t)$ with $\|x(t)\| \leq 2$ to $x(t+1)$ with $\|x(t+1)\| \leq 2$. Define the policy u^* to be such that the actual control $u_t^*(\sigma, z^*)$ it produces at time t w.r.t. the initial state $x(0) = z^*$ and $\sigma \in \mathcal{S}$ is the limiting value as $k \to \infty$ of any converging subsequence of $u_t^{(k)}(\sigma, z^{(k)})$. Using the continuity of the state solution w.r.t. initial state and control input, we can easily deduce that $\|x(t;\sigma,u^*,z^*)\| \leq 2$ for all t and all $\sigma \in \mathcal{S}$. This would implies that $\zeta(z^*) \leq 2$ hence $z^* \in \mathcal{W}$, a contradiction with our earlier assumption that $\mathcal{W} = \{0\}$. Hence, we must have $\mathcal{W} = \mathbb{R}^n$, i.e., $\zeta(\cdot)$ is finite on $\mathbb{R}^n$. Thus, $\zeta(\cdot)$ is a norm, and $\zeta(\cdot) \leq K\|\cdot\|$ for some constant $K < \infty$. This is exactly the desired conclusion. ∎

For SLCSs with $\rho^* = 0$, the conclusion in Theorem 2 does not hold. A counter example is given by any defective SLCS with state space dimension one.

Remark 2: The concepts of reducibility and defectiveness were proposed in the study of the joint spectral radius and the stability of autonomous SLSs [15], [16], [17]. Results here are extensions of them to SLCSs. In particular, the proof of Theorem 2 extends the proof of [17, Theorem 2.1].

IV. Bounds of σ-Resilient Stabilizing Rate

In this section, we will derive lower and upper bounds of the σ-resilient stabilizing rate ρ^* via seminorms and norms. We first study a motivating example.

Example 3: Consider the SLCS given by

$$A_1 = \begin{bmatrix} a_1 & 0\\ 0 & f_1\end{bmatrix}, B_1 = \begin{bmatrix} b_1\\ g_1\end{bmatrix}, A_2 = \begin{bmatrix} a_2 & 0\\ 0 & f_2\end{bmatrix}, B_2 = \begin{bmatrix} b_2\\ g_2\end{bmatrix},$$

whose σ-resilient stabilizing rate is ρ^*. This is obtained from two 1D SLCS's, $\{(a_i, b_i)\}_{i=1,2}$ and $\{(f_i, g_i)\}_{i=1,2}$, that share a common control u and switching signal σ.

Assume in the following that $A_1 \neq A_2$ and that B_1 and B_2 are not collinear, i.e., $b_1 g_2 \neq b_2 g_1$. Then, the constants $\alpha := (a_1 - f_1)g_2 - (a_2 - f_2)g_1$ and $\beta := (a_1 - f_1)b_2 - (a_2 - f_2)b_1$ satisfy $\alpha^2 + \beta^2 \neq 0$. Define two nonnegative functions $V, W : \mathbb{R}^2 \to \mathbb{R}$ by, $\forall z = \begin{bmatrix} z_1, z_2\end{bmatrix}^T \in \mathbb{R}^2$,

$$V(z) := |\alpha z_1 - \beta z_2|, \quad W(z) := |\beta z_1 + \alpha z_2|,$$

Their null sets $\mathcal{N}_V := \{z \,|\, V(z) = 0\}$ and $\mathcal{N}_W := \{z \,|\, W(z) = 0\}$ are orthogonal 1D subspaces. For any $x(t) = \begin{bmatrix} z_1 & z_2\end{bmatrix}^T \in \mathbb{R}^2$, it can be verified using (3) that

$$\begin{aligned} &\min_{u(t)} \max_{i=1,2} V(A_i x(t) + B_i u(t)) \\ &= \frac{|a_1 f_2 - a_2 f_1|}{|a_1 - f_1| + |a_2 - f_2|} V(x(t)) := \rho_0 \cdot V(x(t)), \end{aligned} \quad (7)$$

where the $u(t)$ achieving the minimum is given by

$$u^*(t) = -\frac{(\alpha a_1 z_1 - \beta f_1 z_2) \pm (\alpha a_2 z_1 - \beta f_2 z_2)}{(\alpha b_1 - \beta g_1) \pm (\alpha b_2 - \beta g_2)}, \quad (8)$$

with the sign "$\pm$" being "+" if $(a_1 - a_2)(f_1 - f_2) \geq 0$ and "$-$" if otherwise. The result in (7) has several implications. First, $\mathcal{N}_V$ is a control σ-invariant subspace. Second, if at each time t the adversary chooses $\sigma(t) = \arg\max_i V(A_i x(t) + B_i u(t))$, then $V(x(t+1)) \geq \rho_0 V(x(t))$ regardless of the user's choice of $u(t)$. As $V(x(t))$ is positively homogeneous of degree one in $x(t)$, we conclude that $x(t)$ cannot decay at a faster exponential rate than ρ_0, i.e.,

$$\rho^* \geq \rho_0 = \frac{|a_1 f_2 - a_2 f_1|}{|a_1 - f_1| + |a_2 - f_2|}. \tag{9}$$

A. Bounds via Seminorms

We now formalize the bounding technique employed in Example 3. A seminorm of $\mathbb{R}^n$ is a mapping $\xi : \mathbb{R}^n \to \mathbb{R}_+$ with the following properties: it is convex (hence continuous) and positively homogeneous (of degree one): $\xi(\alpha z) = |\alpha| \cdot \xi(z)$ for all $\alpha \in \mathbb{R}$ and $z \in \mathbb{R}^n$. It becomes a norm if it is positive definite: $\xi(z) > 0$ whenever $z \neq 0$.

Lemma 2: For an arbitrary seminorm ξ on $\mathbb{R}^n$, define

$$\xi_\sharp(z) := \inf_{v \in \mathbb{R}^p} \max_{i \in \mathcal{M}} \xi(A_i z + B_i v), \quad \forall z \in \mathbb{R}^n. \tag{10}$$

Then, $\xi_\sharp$ is also a seminorm of $\mathbb{R}^n$. In other words, the mapping $\mathcal{T} : \xi \mapsto \xi_\sharp$ is a self map of seminorms.

Proof: Obviously $\xi_\sharp$ is finite on $\mathbb{R}^n$ as $\mathcal{M}$ is finite. Noting that $\max_i \xi(A_i z + B_i v)$ is convex in (z, v), by [14, pp. 88], $\xi_\sharp$ is convex in z. Homogeneity of $\xi_\sharp$ follows immediately from that of ξ. ■

In particular, if $\xi(\cdot) = \|\cdot\|$ is a norm of $\mathbb{R}^n$, then $\xi_\sharp(\cdot)$, which we denote as $\|\cdot\|_\sharp$, is a seminorm of $\mathbb{R}^n$. Note that $\|\cdot\|_\sharp$ may not be a norm. For instance, if the two 1D subsystem dynamics in Example 1 are scaled version of each other, $a_1/b_1 = a_2/b_2$, then $|\cdot|_\sharp \equiv 0$, which is not a norm of $\mathbb{R}$.

Proposition 1: Let ξ be a non-zero seminorm of $\mathbb{R}^n$ such that $\xi_\sharp(\cdot) \geq \alpha\, \xi(\cdot)$ for some constant $\alpha \geq 0$. Then, $\rho^* \geq \alpha$.

Proof: Assume the adversary adopts the switching strategy $\sigma(t) = \arg\max_i \xi(A_i x(t) + B_i u(t))$, $\forall t \in \mathbb{N}$, and assume $x(0) = z$ is such that $\xi(z) > 0$. Then,

$$\xi(x(t+1)) = \max_i \xi(A_i x(t) + B_i u(t)) \geq \xi_\sharp(x(t)),$$

hence $\xi(x(t+1)) \geq \alpha \cdot \xi(x(t))$, for all $t \in \mathbb{N}$ and all $u \in \mathcal{U}$. This implies that the exponential growth rate of $\xi(x(t))$, hence that of $\|x(t)\|$, is at least α. Therefore, $\rho^* \geq \alpha$. ■

Proposition 1 has been applied in Example 3 with $\xi(\cdot) = V(\cdot)$ and $\alpha = \rho_0$ in equation (7).

Proposition 2: Let $\|\cdot\|$ be a norm of $\mathbb{R}^n$ such that $\|\cdot\|_\sharp \leq \alpha \|\cdot\|$ for some constant $\alpha \geq 0$. Then, $\rho^* \leq \alpha$.

Proof: Suppose the user adopts the input policy $u^*(t) = \arg\min_v \max_{i \in \mathcal{M}} \|A_i x(t) + B_i v\|$ for $t \in \mathbb{N}$, which is admissible. Note that $u^*(t)$ thus defined exists due to the convexity and nonnegativity of $\|\cdot\|$, though it may not be unique (in which case any choice suffices). Then, for any adversary's switching strategy $\sigma \in \mathcal{S}$ and any $t \in \mathbb{N}$,

$$\|x(t+1)\| \leq \max_{i \in \mathcal{M}} \|A_i x(t) + B_i u^*(t)\| = \|x(t)\|_\sharp \leq \alpha \|x(t)\|.$$

This implies that $\|x(t)\| \leq \alpha^t \|x(0)\|$, $\forall t$; hence $\rho^* \leq \alpha$. ■

The following result combines the above two propositions.

Corollary 1: If $\alpha_1 \|\cdot\| \leq \|\cdot\|_\sharp \leq \alpha_2 \|\cdot\|$ for some norm $\|\cdot\|$ on $\mathbb{R}^n$, then $\alpha_1 \leq \rho^* \leq \alpha_2$.

B. Extremal Norms

By Corollary 1, associated with each norm $\|\cdot\|$ are the following lower and upper bounds of ρ^*: $\alpha_\ell := \sup\{\alpha_1 |\, \alpha_1 \|\cdot\| \leq \|\cdot\|_\sharp\}$, $\alpha_u := \inf\{\alpha_2 |\, \|\cdot\|_\sharp \leq \alpha_2 \|\cdot\|\}$. A natural question arises: can such bounds be tight?

Definition 4: A norm $\|\cdot\|$ on $\mathbb{R}^n$ is called an (upper) extremal norm if $\|\cdot\|_\sharp \leq \rho^* \|\cdot\|$.

Suppose an extremal norm $\|\cdot\|$ exists. Then the property $\|\cdot\|_\sharp \leq \rho^* \|\cdot\|$ implies that, for any $z \in \mathbb{R}^n$, the user can find a control $v \in \mathbb{R}^p$ such that $\|A_i z + B_i v\| \leq \rho^* \|z\|$ for all $i \in \mathcal{M}$. This essentially specifies a state feedback control policy $u \in \mathcal{U}$ under which $\|x(t; \sigma, u, z)\| \leq (\rho^*)^t \|z\|$ for arbitrary $\sigma \in \mathcal{S}$. In particular, this implies that the SLCS must be nondefective. The following theorem says that the reverse is also true. The proof can be found in [12].

Theorem 3 ([12]): Extreme norms exist if and only if the SLCS is nondefective.

Definition 5: A nonzero seminorm $\xi(\cdot)$ on $\mathbb{R}^n$ is called a lower extremal seminorm if $\xi_\sharp(\cdot) \geq \rho^* \cdot \xi(\cdot)$.

Theorem 4 ([12]): Lower extremal seminorms exist if the SLCS is nondefective.

The converse of Theorem 4 is not true. For example, we have shown that the SLCS with a single subsystem (A, B) with $A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is defective with $\rho^* = 0$. Then, any seminorm will be a lower extremal seminorm.

A norm that is both upper and lower extremal is called a Barabanov norm.

Definition 6: A norm $\|\cdot\|$ on $\mathbb{R}^n$ is called a *Barabanov norm* if $\|\cdot\|_\sharp = \rho^* \|\cdot\|$.

As a simple example, in Example 1 with $a_1/b_1 = a_2/b_2$, the 1D SLCS has a Barabanov norm $|\cdot|$.

Theorem 5 ([12]): Barabanov norms exist if the SLCS is irreducible.

The proofs of both Theorem 4 and Theorem 5 can be founded in [12]. As a result, when the SLCS is nondefective, suitable norms (resp. seminorms) exist that provide tight upper (resp. lower) bounds for ρ^*. If furthermore the SLCS is irreducible, then it is possible to find a single norm that gives simultaneously tight lower and upper bounds of ρ^*.

Remark 3: The notions of extremal norms and Barabanov norms were originally proposed for the study of joint spectral radius and the stability of autonomous switched linear systems [15], [18], [16], [19]. We extend them to the context of resilient stabilization of SLCSs.

C. Bounds via Ellipsoidal Norms

Denote by $\mathbb{P}_{\succ 0}$ and $\mathbb{P}_{\succeq 0}$ the sets of all positive definite (p.d.) matrices and positive semidefinite (p.s.d.) matrices, respectively. For each $P \succeq 0$, $\|z\|_P := \sqrt{z^T P z}$ defines a seminorm of $\mathbb{R}^n$. If $P \succ 0$, then $\|\cdot\|_P$ is a norm, called an ellipsoidal norm as its unit ball is an ellipsoid. In this section, we will study the bounds on ρ^* derived from $\|\cdot\|_P$.

We first introduce some useful notations. Denote by

$$\Delta := \left\{\theta \in \mathbb{R}^m \,|\, \theta_i \geq 0, \,\forall\, i \in \mathcal{M}, \, \textstyle\sum_{i\in\mathcal{M}} \theta_i = 1\right\}$$

the m-simplex. For each $\theta \in \Delta$ and $P \succeq 0$, define

$$\Gamma_\theta(P) := \sum_{i\in\mathcal{M}} \theta_i A_i^T P A_i - \left(\sum_{i\in\mathcal{M}} \theta_i A_i^T P B_i\right) \times \left(\sum_{i\in\mathcal{M}} \theta_i B_i^T P B_i\right)^\dagger \left(\sum_{i\in\mathcal{M}} \theta_i B_i^T P A_i\right),$$

where $\dagger$ denotes matrix pseudo inverse. Note that $\Gamma_\theta(P)$ is the (generalized) Schur complement [20, pp. 28] of the lower right block of the following p.s.d. matrix:

$$\Upsilon_\theta := \begin{bmatrix} \sum_{i\in\mathcal{M}} \theta_i A_i^T P A_i & \sum_{i\in\mathcal{M}} \theta_i A_i^T P B_i \\ \sum_{i\in\mathcal{M}} \theta_i B_i^T P A_i & \sum_{i\in\mathcal{M}} \theta_i B_i^T P B_i \end{bmatrix}.$$

From this, we conclude that: (i) $\Gamma_\theta(P) \succeq 0$; (ii) for a fixed P (resp. θ), $\Gamma_\theta(P)$ is a concave mapping of θ (resp. P) into $\mathbb{P}_{\succeq 0}$ equiped with the partial order $\preceq$. Define the set

$$\Gamma_\Delta(P) := \{\Gamma_\theta(P) \,|\, \theta \in \Delta\} \subset \mathbb{P}_{\succeq 0}. \tag{11}$$

Lemma 3: For each $P \succeq 0$, denote $\|\cdot\|_{P\sharp} := \mathcal{T}(\|\cdot\|_P)$ where the mapping $\mathcal{T}$ is defined in (10). Then,

$$\|z\|_{P\sharp} = \sup_{\theta\in\Delta} \|z\|_{\Gamma_\theta(P)} = \sup_{Q\in\Gamma_\Delta(P)} \|z\|_Q, \quad \forall z \in \mathbb{R}^n. \tag{12}$$

Proof: By (10), $(\|z\|_{P\sharp})^2$ is the solution of the following optimization problem:

$$\begin{aligned} &\text{minimize} \;\; r \\ &\text{subject to} \;\; (A_i z + B_i v)^T P (A_i z + B_i v) \leq r, \; \forall i \in \mathcal{M}, \end{aligned} \tag{13}$$

with the optimization variables being $v \in \mathbb{R}^p$ and $r \in \mathbb{R}$. By introducing the multipliers (dual variables) $\theta_i \geq 0$ for $i \in \mathcal{M}$, we can define the Lagrangian

$$\begin{aligned} L(v, r, \theta) &:= r + \sum_{i\in\mathcal{M}} \theta_i \cdot \left[(A_i z + B_i v)^T P (A_i z + B_i v) - r\right] \\ &= \left(1 - \textstyle\sum_{i\in\mathcal{M}} \theta_i\right) r + \begin{bmatrix} z^T & v^T \end{bmatrix} \Upsilon_\theta \begin{bmatrix} z \\ v \end{bmatrix}. \end{aligned}$$

The Lagrange dual function is easily verified to be

$$g(\theta) := \inf_{v,\, r} L(v, r, \theta) = \begin{cases} z^T \Gamma_\theta(P) z & \text{if } \sum_{i\in\mathcal{M}} \theta_i = 1 \\ -\infty & \text{if otherwise.} \end{cases}$$

Hence, the dual problem of (13) is

$$\text{maximize} \;\; z^T \Gamma_\theta(P) z \;\; \text{subject to} \;\; \theta \in \Delta, \tag{14}$$

whose solution is exactly the square of the right hand side of (12). Since the original optimization problem (13) is both convex (indeed a second order cone programming) and strongly feasible (r can be arbitrarily large), it has the same solution as that of (14). This completes the proof. ■

To apply Proposition 1 to the ellipsoidal norm associated with $P \succ 0$, we write the condition $\|z\|_{P\sharp} \geq \alpha \|z\|_P$, $\forall z$, as

$$\sup_{\theta\in\Delta} z^T \Gamma_\theta(P) z \geq \alpha z^T P z, \; \forall z.$$

A sufficient condition for the above is $\Gamma_\theta(P) \succeq \alpha P$ for some $\theta \in \Delta$, which by the Schur complement is equivalent to

$$\begin{bmatrix} \sum_{i\in\mathcal{M}} \theta_i A_i^T P A_i - \alpha P & \sum_{i\in\mathcal{M}} \theta_i A_i^T P B_i \\ \sum_{i\in\mathcal{M}} \theta_i B_i^T P A_i & \sum_{i\in\mathcal{M}} \theta_i B_i^T P B_i \end{bmatrix} \succeq 0 \tag{15}$$

for some $\theta \in \Delta$. By Proposition 1, we then have the following result.

Proposition 3: For any $P \in \mathbb{P}_{\succ 0}$, $\rho^* \geq \alpha^*$, where α^* is the solution to the following semidefinite program:

$$\max_{\alpha \geq 0,\, \theta\in\Delta} \;\alpha \quad \text{subject to inequality (15).} \tag{16}$$

Example 4: Consider the following SLCS:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\; B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\; A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\; B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Using Propositin 3 with $P = \begin{bmatrix} 0.7334 & 0.5004 \\ 0.5004 & 0.3414 \end{bmatrix}$, the solution to (16) is $\alpha^* = 0.7588$. Therefore, $\rho^* \geq 0.7588$.

We next apply Propostion 2 to ellipsoidal norms. Suppose $P \succ 0$. The condition $\|\cdot\|_{P\sharp} \leq \alpha \|\cdot\|$ is equivalent to $\sup_{\theta\in\Delta} z^T \Gamma_\theta(P) z \leq \alpha z^T P z$, $\forall z$, or equivalently, $\Gamma_\theta(P) \preceq \alpha P$ for all $\theta \in \Delta$. This leads to the following result.

Proposition 4: Let $P \succ 0$ be given. Then $\rho^* \leq \alpha^*$ where α^* is the solution to the following problem:

$$\min_{\alpha \geq 0} \;\alpha \quad \text{such that } \Gamma_\theta(P) \preceq \alpha P, \; \forall \theta \in \Delta. \tag{17}$$

Problem (17) is difficult to solve due to the infinite number of constraints. The following example shows that it can yield useful bounds of ρ^* in some simple cases.

Example 5: Consider the following controlled double integrator system sampled at integer times:

$$x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ b(t) \end{bmatrix} u(t),$$

where $b(t) \in \{b_1, b_2\}$ has two possible values chosen by the adversary. The underlying SLCS is

$$A_1 = A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ b_1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ b_2 \end{bmatrix}.$$

Consider a generic (homogenized) $P = \begin{bmatrix} 1 & \beta \\ \beta & \gamma \end{bmatrix} \succ 0$. Then we must have $\gamma > 0$ and $\beta^2 < \gamma$. The constraint in (17) is satisfied if and only if for all $\theta_1, \theta_2 \in [0, 1]$ with $\theta_1 + \theta_2 = 1$,

$$\begin{bmatrix} 1 & 1+\beta \\ 1+\beta & 1+2\beta+\gamma \end{bmatrix} - \frac{(\theta_1 b_1 + \theta_2 b_2)^2}{\gamma(\theta_1 b_1^2 + \theta_2 b_2^2)} \times \underbrace{\begin{bmatrix} \beta \\ \beta+\gamma \end{bmatrix} \begin{bmatrix} \beta \\ \beta+\gamma \end{bmatrix}^T}_{Q} \preceq \alpha \begin{bmatrix} 1 & \beta \\ \beta & \gamma \end{bmatrix}. \tag{18}$$

Observe that $Q \succeq 0$ and

$$\min_{\substack{\theta_1,\theta_2 \geq 0 \\ \theta_1+\theta_2=1}} \frac{(\theta_1 b_1 + \theta_2 b_2)^2}{\theta_1 b_1^2 + \theta_2 b_2^2} = \begin{cases} \frac{4 b_1 b_2}{(b_1+b_2)^2} := M_b & \text{if } b_1 b_2 > 0 \\ 0 & \text{if } b_1 b_2 \leq 0. \end{cases}$$

Assume $b_1 b_2 > 0$. Then condition (18) is equivalent to

$$\gamma \begin{bmatrix} 1-\alpha & 1+\beta(1-\alpha) \\ 1+\beta(1-\alpha) & 1+2\beta+\gamma(1-\alpha) \end{bmatrix} \preceq M_b \begin{bmatrix} \beta \\ \beta+\gamma \end{bmatrix} \begin{bmatrix} \beta \\ \beta+\gamma \end{bmatrix}^T.$$

Since β and γ can be freely chosen (as long as $P \succ 0$), we set $\gamma = 2\beta^2$ where $\beta > 0$. Then, the above is reduced to

$$M_b \geq \max\left\{\frac{1-\alpha}{2\beta^2}, \frac{2}{1+2\beta} + \frac{4\beta^2(1-\alpha)}{(1+2\beta)^2}, \frac{2\beta^2(1-\alpha)^2 - 2\alpha^2}{2\beta^2(1-\alpha) - 2\alpha\beta - \alpha}\right\}.$$

Note that the right hand side converges to $\max\{0, 1-\alpha\}$ as $\beta \to +\infty$. Thus, for any $\alpha > 1 - M_b$, we can find $\beta > 0$ large enough such that the above inequality is satisfied. By Proposition 4, this implies that, if $b_1 b_2 > 0$, $\rho^* \leq 1 - M_b = \frac{(b_1-b_2)^2}{(b_1+b_2)^2} < 1$. Hence, the SLCS is σ-resiliently stabilizable.

V. σ-Resilient Generating Function

Define for each $\lambda \geq 0$ and $k \in \mathbb{N}$ the following function:

$$F_\lambda^k(z) := \inf_{u(0)} \sup_{\sigma(0)} \cdots \inf_{u(k-1)} \sup_{\sigma(k-1)} \sum_{t=0}^{k} \lambda^t \|x(t;\sigma,u,z)\|^2, \tag{19}$$

for $z \in \mathbb{R}^n$. Obviously, $F_\lambda^k(\cdot)$ is finite on $\mathbb{R}^n$. Using the map $\mathcal{T}$ in (10), we can obtain $F_\lambda^k(\cdot)$ iteratively as follows:

$$F_\lambda^k(z) = \|z\|^2 + \lambda \cdot \mathcal{T}\left[F_\lambda^{k-1}\right](z),\ \forall z,\ k \geq 1, \tag{20}$$

with $F_\lambda^0(z) = \|z\|^2$. It is easy to see that $F_\lambda^k(\cdot)$ for each k is nonnegative, convex, and homogeneous of degree two on $\mathbb{R}^n$. Moreover, $F_\lambda^k(\cdot)$, $k \in \mathbb{N}$, is nondecreasing as k increases.

Definition 7: The σ-resilient control generating function (σ-CGF) of the SLCS is defined as

$$F_\lambda(z) := \lim_{k\to\infty} F_\lambda^k(z), \quad \forall z \in \mathbb{R}^n,\ \forall \lambda \geq 0. \tag{21}$$

Its radius of convergence, denoted λ_F^*, is defined as

$$\lambda_F^* := \sup\left\{\lambda \geq 0 \,|\, F_\lambda(z) < \infty,\ \forall z \in \mathbb{R}^n\right\}.$$

For $\lambda = 0$, we have $F_\lambda^k(z) = \|z\|^2$, $\forall k$, hence $F_\lambda(z) = \|z\|^2$.

The monotonicity of $F_\lambda^k(\cdot)$ in k implies that $F_\lambda(\cdot)$ is well defined, though possibly of infinite value. From the above discussions, $F_\lambda(\cdot)$ is nonnegative, convex, and homogeneous of degree two on $\mathbb{R}^n$, and satisfies

$$F_\lambda(z) = \|z\|^2 + \lambda \cdot \mathcal{T}\left[F_\lambda\right](z), \quad \forall z \in \mathbb{R}^n. \tag{22}$$

Theorem 6: The SLCS is σ-resiliently exponentially stabilizable if and only if the radius of convergence λ_F^* of its σ-CGF $F_\lambda(\cdot)$ satisfies $\lambda_F^* > 1$. Moreover, $\rho^* = (\lambda_F^*)^{-1/2}$.

Proof: Suppose $\lambda_F^* > 1$. Then $F_1(\cdot)$ is finite on $\mathbb{R}^n$. As $F_1(\cdot)$ is convex hence continuous, $\alpha_2 := \sup_{\|z\|=1} F_1(z) < \infty$. Thus, $\|z\|^2 \leq F_1(z) \leq \alpha_2\|z\|^2$, $\forall z$. Furthermore, by letting $\lambda = 1$ in (22), we have, for all $z \in \mathbb{R}^n$,

$$\inf_{v\in\mathbb{R}^p} \max_{i\in\mathcal{M}} F_1(A_i z + B_i v) = F_1(z) - \|z\|^2 \leq \frac{\alpha_2 - 1}{\alpha_2} F_1(z),$$

Thus, with $F_1(\cdot)$ being a control Lyapunov function, the SLCS is σ-resiliently exponentially stabilizable. Conversely, assume there exists $K < \infty$, $\rho \in [0,1)$, and $u \in \mathcal{U}$ such that $\|x(t;\sigma,u,z)\| \leq K\rho^t\|z\|$, $\forall t$, $\forall z$, $\forall \sigma \in \mathcal{S}$. Adopting this u in (19) and letting $k \to \infty$, we obtain $F_\lambda(z) \leq \frac{K^2}{1-\lambda\rho^2}\|z\|^2 < \infty$ for all $\lambda < 1/\rho^2$. This implies that $\lambda_F^* \geq 1/\rho^2 > 1$.

The second conclusion follows by noting that the scaled SLCS $(\beta A_i, \beta B_0)$ for $\beta > 0$ has generating function $F_{\beta^2\lambda}(z)$ hence radius of convergence λ_F^*/β^2. ■

VI. Conclusions

The concept of switching-resilient stabilization of switched linear control systems is introduced. Bounds on the σ-resilient stabilizing rate are obtained and conditions under which such bounds are tight are characterized through the notions of irreducibility and nondefectiveness.

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