Periodic Stabilization of Discrete-Time Switched Linear Systems

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***Abstract*— In this paper, we investigate the state-feedback switching controller design problem for stabilization of discrete-time linear switched systems. To this end, a periodic state-feedback switching controller is considered along with the generalized periodic Lyapunov inequalities. To compute the control Lyapunov function, a bilinear matrix inequality (BMI) condition is suggested. Then, we focus on developing an iterative algorithm that can efficiently solve the BMI condition. The algorithm is based on the projection of the current Lyapunov matrix onto a matrix polytope. An example is given to illustrate the proposed design method.**

I. INTRODUCTION

This paper aims at investigating a switching controller design method for discrete-time switched linear systems (SLSs). The stability analysis and stabilization of the SLSs have been substantially studied [5]–[15] in recent years mainly based on the Lyapunov theory. For instance, the so-called composite Lyapunov functions have been studied in [6]–[11]. The approximate dynamic programming approach has been applied to design stabilizing controller [12] and optimal controller [13], [14]. More recently, the concept of generating functions has been devised in [15] for the stability analysis of switching systems, and it has been further developed in [16] to analyze the input-to-state $\mathcal{L}_2$ gain of discrete-time switched systems.

In this paper, the problem of designing the state-feedback switching rule $\sigma(k) = u(x(k))$ is addressed in order to asymptotically stabilize the SLSs. To this end, we attempt to apply the periodic control approach investigated in [17]–[19]. As an illustration, let us first consider the simple discrete-time LTI system

$$x(k+1) = Ax(x), \quad x(0) = x_0. \tag{1}$$

To evaluate the asymptotic stability of the system, we can use the Lyapunov theory: discrete-time LTI system (1) is asymptotically stable if and only if there exists a symmetric positive definite matrix P such that $A^T PA - P \prec 0$. Finding P is a convex linear matrix inequality (LMI) problem, for which various numerical solvers exist [2]–[4]. If model uncertainties and nonlinearities need be taken into account, then the problem of finding a constant P becomes more challenging or even impossible. To alleviate these difficulties, various classes of Lyapunov functions, such as parameter-dependent Lyapunov functions [20]–[22] and composite Lyapunov functions [7], can be used. Another approach is to employ a class of periodic Lyapunov functions developed in [17]–[19] for the stabilization of periodic or nonlinear systems. The idea is as follows: for quadratic Lyapunov function $V(x) = x^T Px > 0, \forall x \neq 0$, the conventional Lyapunov inequality is $V(x(k+1)) - V(x(k)) < 0, \forall x \neq 0$. Instead of using the one sample variation of the Lyapunov function, we can consider arbitrary h-sample variation: $V(x(k+h)) - V(x(k)) < 0, \forall x \neq 0$, which corresponds to $(A^T)^h PA^h - P \prec 0$ for LTI systems. If A is Schur, then for any positive definite matrix P, $(A^T)^h PA^h$ will vanish as h goes to infinity, and the Lyapunov inequality becomes $(A^T)^h PA^h - P \cong -P \prec 0$. This approach is useful especially when dealing with uncertain, nonlinear, or periodic systems. In this paper, we will apply this h-sample variation approach to stabilize the discrete-time SLSs. The design procedure is expressed as a non-convex bilinear matrix inequality (BMI) condition. Then, we focus on algorithms to solve the BMI based on the iterative projection of the Lyapunov matrix onto a certain matrix polytope. An example is given to illustrate the proposed methods.

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II. MAIN RESULTS

A. Notation

The adopted notation is as follows: $\mathbb{N}$: set of nonnegative integers; $\mathbb{R}^n$: n-dimensional Euclidean space; $\mathbb{R}^{n\times m}$: set of all $n \times m$ real matrices; A^T: transpose of matrix A; $A \succ 0$ ($A \prec 0$, $A \succeq 0$, and $A \preceq 0$, respectively): symmetric positive definite (negative definite, positive semi-definite, and negative semi-definite, respectively) matrix A; I_n: $n \times n$ identity matrix; $||\cdot||$: Euclidean norm of a vector or spectral norm of a matrix; $\mathbb{S}^n$: symmetric $n \times n$ matrices; $\mathbb{S}^n_+$: cone of symmetric $n \times n$ positive semi-definite matrices; $\mathbb{S}^n_{++}$: symmetric $n \times n$ positive definite matrices; $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$: minimum and maximum eigenvalues of symmetric matrix A, respectively; given two sets $\mathcal{U}$ and $\mathcal{V}$, $\mathcal{U} \oplus \mathcal{V} := \{u + v : u \in \mathcal{U}, v \in \mathcal{V}\}$; $\mathbf{1}_N := [\ 1 \quad 1 \quad \cdots \quad 1\ ]^T \in \mathbb{R}^N$; given $P \in \mathbb{S}^n_{++}$, $||\cdot||_P$ is the ellipsoid norm on $x \in \mathbb{R}^n$ defined by $||x||_P := \sqrt{x^T Px}$.

B. h-samples variation approach

Let us consider the discrete-time (autonomous) SLS

$$x(k+1) = A_{\sigma(k)}x(k), \quad x(0) = z, \tag{2}$$

where $k \in \mathbb{N}$, $x(k) \in \mathbb{R}^n$ is the state, $\sigma(k) \in \mathcal{M} := \{1, 2, \ldots, N\}$ is called the mode, and $A_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{M}$, are the subsystem (dynamics) matrices. We assume that none of the matrices $A_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{M}$, is Schur stable. Starting from $x(0) = z$ and under the switching sequence $\sigma := \{\sigma(0), \sigma(1), \ldots\}$, the trajectory of the SLS is denoted by $x(k; z, \sigma)$.

Definition 1: ([15, Definition 1]) The SLS (2) is called

1) asymptotically switching stabilizable if starting from any initial state z, there exists a switching sequence σ for which the trajectory $x(k;\, z,\, \sigma)$ satisfies $\lim_{k\to\infty} ||x(k;\, z,\, \sigma)|| = 0$.
2) exponentially switching stabilizable (with the parameters κ and r) if there exist $\kappa \geq 1$ and $r \in [0,\, 1)$ such that starting from any initial state z, there exists a switching sequence σ for which the trajectory $x(k;\, z,\, \sigma)$ satisfies $||x(k;\, z,\, \sigma)|| \leq \kappa r^k ||z||$, for all $k \in \mathbb{N}$.

The following result will be used later.

Lemma 1: ([15, Definition 1]) The asymptotic switching stabilizability and the exponential switching stabilizability of the SLS (2) are equivalent.

Remark 1: As a result, we refer to either notions of stabilizability simply as switching stabilizability.

In this paper, we will study the possibility of stabilizing the SLS by using the following periodic state-feedback controller:

$$(\sigma_k(x(k)),\, \ldots,\, \sigma_{k+h-1}(x(k))) = \underset{(i_1,\ldots,\, i_h)\in\mathcal{M}^h}{\arg\min} \; V(A_{i_h}\cdots A_{i_1}x(k)), \tag{3}$$

where $k \in \{0,\, h,\, 2h,\, \ldots\}$, $\sigma_j(x(i))$, $j \geq i$ denotes the mode at time j determined based on state $x(i)$ at time i, and $V(x) = x^T P x > 0, \forall x \neq 0$. The controller is designed in such a way that, every h time steps, the controller generates the current and future sequence of modes of length h selected so that the Lyapunov function's value after h steps is minimized. Then, the switching sequence of length h is applied to the system, and the same process is repeated after h steps. Under (3), the corresponding Lyapunov inequality is

$$\min_{(i_1,\, i_2,\ldots,\, i_h)\in\mathcal{M}^h} V(A_{i_h}\cdots A_{i_1}x(k)) - V(x(k)) < 0, \quad \forall x(k) \neq 0. \tag{4}$$

The Lyapunov inequality (4) implies that the quadratic Lyapunov function does not need to decrease each time step k; it only needs to decrease every h time steps. It should be noted that the suggested periodic state-feedback switching controller is different from the notion of the open-loop periodic stabilization. To explain this, we introduce the following definition.

Definition 2: The SLS (2) is periodic open-loop switching stabilizable if there exists some h and some switching sequence $(i_1, \ldots,\, i_h)$ of length h such that $A_{i_1}A_{i_2}\cdots A_{i_h}$ is Schur.

The periodic state-feedback switching stabilizability does not guarantee the periodic open-loop stabilizability. A counter example of a SLS that is state-feedback switching stabilizable but not periodic open-loop switching stabilizable is given in [15, p. 1068]. To sum up, the proposed method is a periodic state-feedback switching stabilization method, which can stabilize SLSs that are not periodic open-loop switching stabilizable.

Proposition 1: The following two statements are equivalent:

1) The SLS is switching stabilizable.
2) There exist a positive definite matrix P and a positive integer h such that the condition (4) holds.

Proof: 2) $\Rightarrow$ 1): It is straightforwardly proven that the state-feedback switching controller (3) asymptotically stabilizes the SLS (2). 1) $\Rightarrow$ 2): Suppose that for any initial state z, there exists a switching sequence σ_z depending on the initial state such that the SLS (2) is asymptotically stable. Select any $P \succ 0$. From Lemma 1, there exist $\kappa \geq 1$ and $r \in [0,\, 1)$ such that starting from any initial state z, there exists a switching sequence σ_z for which the trajectory $x(k;\, z,\, \sigma_z)$ satisfies $||x(k;\, z,\, \sigma_z)||_P \leq \kappa r^k ||z||_P$, for all $k \in \mathbb{N}$. Therefore, we can select a sufficiently large h such that $||x(h;\, z,\, \sigma_z)||_P < ||z||_P$ for all $z \in \mathbb{R}^n$. Since $\min_{\sigma_h\in\mathcal{M}^h} ||x(h;\, z,\, \sigma_h)||_P \leq ||x(h;\, z,\, \sigma_z)||_P$, where $\sigma_h := (\sigma(0),\, \sigma(1), \ldots,\, \sigma(h-1))$, we have

$$\min_{\sigma_h\in\mathcal{M}^h} ||x(h;\, z,\, \sigma_h)||_P < ||z||_P, \quad \forall z \in \mathbb{R}^n.$$

As a result, (4) holds for $V(x) = x^T P x$ and the chosen h. Therefore, the proof is completed. ∎

Now, for given matrices $P \in \mathbb{S}^n$ and $Q \in \mathbb{S}^n$, let us define the following set:

$$L_h(P,\, Q) := \{(A_{i_h}\cdots A_{i_1})^T P (A_{i_h}\cdots A_{i_1}) + Q \in \mathbb{S}^n : (i_1,\, i_2, \ldots,\, i_h) \in \mathcal{M}^h\} \tag{5}$$

In addition, let $\{\Pi_1(P,\, Q), \ldots,\, \Pi_{N^h}(P,\, Q)\}$ be an enumeration of the elements of the set $L_h(P,\, Q)$. To compute the control Lyapunov function, the following problem is introduced.

Problem 1: For a positive integer h, find scalars $(\alpha_1,\, \alpha_2, \ldots,\, \alpha_{N^h})$ and a matrix $P \in \mathbb{S}^n_{++}$ such that

$$\sum_{i\in\{1,\ldots,N^h\}} \alpha_i \Pi_i(P,\, 0_{n\times n}) - P \prec 0, \quad \mathbf{a} \succeq 0, \quad \mathbf{a}^T\mathbf{1} = 1, \tag{6}$$

where $\mathbf{a} := [\, \alpha_1 \;\cdots\; \alpha_{N^h} \,]^T \in \mathbb{R}^{N^h}$ and $\mathbf{1} := [\, 1 \;\cdots\; 1 \,]^T \in \mathbb{R}^{N^h}$.

Proposition 2: Suppose that Problem 1 admits a solution for a positive integer h. Then, the state-feedback switching controller (3) will stabilize the SLS (2).

Proof: The proof follows the same lines of approaches as [8, Theorem 3]; hence it is omitted here. ∎

The condition (6) is a nonconvex bilinear matrix inequality (BMI) problem with equality constraints, which is not easy to solve. The following result is useful to compute its solution.

Proposition 3: Suppose that there exists a solution $(P^*,\, \mathbf{a}^*)$ to the BMI problem (6) for some h. Then, the following statements are true:

1) For any given $Q \in \mathbb{S}^n_{++}$, define a map $\mathcal{G}_Q$ on $\mathbb{S}^n$ by

$$\mathcal{G}_Q(P) := \sum_{i\in\{1,\ldots,N^h\}} \alpha_i^* \Pi_i(P,\, Q). \tag{7}$$

a) The map has a fixed point $\bar{P}$ defined by (7);
b) The fixed point $\bar{P}$ is unique;
c) $\lim_{k\to\infty} \mathcal{G}_Q^k(P) = \bar{P}$ for all $P \in \mathbb{S}_{++}^n$.

Note that $\bar{P}$ is a solution to the following LMI problem in P:

$$\sum_{i\in\{1,\ldots,N^h\}} \alpha_i^* \Pi_i(P,\, 0_{n\times n}) - P \prec 0.$$

2) For any given $P \in \mathbb{S}_{++}^n$, there exists a sufficiently large $k > h$ such that the following LMI problem in $\mathbf{a} \in \mathbb{R}^{N^k}$ with equality constraints has a solution:

$$\sum_{i\in\{1,\ldots,N^k\}} \alpha_i \Pi_i(P,\, 0_{n\times n}) - P \prec 0,$$
$$\mathbf{a} \succeq 0, \quad \mathbf{a}^T\mathbf{1} = 1. \tag{8}$$

To prove Proposition 3, the following lemma is needed.

Lemma 2: ([23, Theorem 3.1]) Let $Q \in \mathbb{S}_{++}^n$ be given. In addition, let us define the map $\mathcal{R}$ on $\mathbb{S}^n$ by

$$\mathcal{R}(P) := Q + \sum_{i\in\{1,\ldots,m\}} B_i^T P B_i,$$

where $B_i \in \mathbb{R}^{n\times n}$, $\forall i \in \{1, \ldots, m\}$. If there exits $S \in \mathbb{S}_{++}^n$ such that

$$\sum_{i\in\{1,\ldots,m\}} B_i^T S B_i \prec S,$$

then

1) the map $\mathcal{R}$ has a fixed point $\bar{P}$ in $\mathbb{S}^n$;
2) the fixed point $\bar{P}$ is unique;
3) $\lim_{k\to\infty} \mathcal{R}^k(P) = \bar{P}$ for all $P \in \mathbb{S}_{++}^n$.

Proof of Proposition 3:

Part 1): Suppose that there exists a solution $(P^*,\, \mathbf{a}^*)$ to the BMI problem (6) for some h. Let us consider the map $\mathcal{G}_Q(P)$ in (7). Let $(\bar{A}_1, \ldots,\, \bar{A}_{N^h})$ be an enumeration of the elements of the set $\{(A_{i_h} \cdots A_{i_1}) \in \mathbb{R}^{n\times n} : (i_1, \ldots,\, i_h) \in \mathcal{M}^h\}$, and define

$$\tilde{A}_i := \sqrt{\alpha_i^*}\bar{A}_i, \quad \forall i \in \{1, \ldots,\, N^h\}.$$

Then, $\mathcal{G}_Q(P)$ defined in (7) can be rewritten as

$$\mathcal{G}_Q(P) = Q + \sum_{i\in\{1,\ldots,N^h\}} \tilde{A}_i^T P \tilde{A}_i.$$

Since $(P^*,\, \mathbf{a}^*)$ is a solution to the BMI problem (6), we have

$$\sum_{i\in\{1,\ldots,N^h\}} \tilde{A}_i^T P^* \tilde{A}_i \prec P^*.$$

Therefore, we can apply Lemma 2 to prove statement 1).

Part 2): Suppose that there exists a solution $(P^*,\, \mathbf{a}^*)$ to the BMI problem (6) for some h. From the proof of Part 1), we know that for a given $Q \in \mathbb{S}_{++}^n$, the map $\mathcal{G}_Q$ has a unique fixed point $\bar{P}$. Moreover, $\lim_{k\to\infty} \mathcal{G}_Q^k(P) = \bar{P}$ for all $P \in \mathbb{S}_{++}^n$. Then, it can be seen that

$$\begin{aligned}
&\Bigg\| \sum_{i_1\in\{1,\ldots,N^h\}} \cdots \sum_{i_{k+1}\in\{1,\ldots,N^h\}} (\tilde{A}_{i_{k+1}}^T \cdots \tilde{A}_{i_1}^T) P (\tilde{A}_{i_1} \cdots \tilde{A}_{i_{k+1}}) \Bigg\| \\
&= ||\mathcal{G}_Q^{k+1}(P) - \mathcal{G}_Q^k(Q)|| \\
&\le ||\mathcal{G}_Q^{k+1}(P) - \bar{P}|| + ||\bar{P} - \mathcal{G}_Q^k(Q)||.
\end{aligned}$$

Since $\lim_{k\to\infty} \mathcal{G}_Q^k(P) = \bar{P}$ and $\lim_{k\to\infty} \mathcal{G}_Q^k(Q) = \bar{P}$, for any $\varepsilon > 0$, there exists a positive integer K such that

$$||\mathcal{G}_Q^{k+1}(P) - \bar{P}|| + ||\bar{P} - \mathcal{G}_Q^k(Q)|| < \varepsilon, \quad \forall k \ge K.$$

Therefore, one gets

$$\begin{aligned}
&\Bigg\| \sum_{i_1\in\{1,\ldots,N^h\}} \cdots \sum_{i_{k+1}\in\{1,\ldots,N^h\}} (\tilde{A}_{i_{k+1}}^T \cdots \tilde{A}_{i_1}^T) P (\tilde{A}_{i_1} \cdots \tilde{A}_{i_{k+1}}) \Bigg\| \\
&= \Bigg\| \sum_{i\in\{1,\ldots,N^{(k+1)h}\}} \beta_i \hat{A}_i^T P \hat{A}_i \Bigg\| \\
&= \sqrt{\lambda_{\max}\left(\sum_{i\in\{1,\ldots,N^{(k+1)h}\}} \beta_i \hat{A}_i^T P \hat{A}_i \right)} \\
&< \varepsilon, \quad \forall k \ge K,
\end{aligned}$$

where $(\hat{A}_1, \ldots,\, \hat{A}_{N^{(k+1)h}})$ is an enumeration of elements of $\{(A_{i_{(k+1)h}} \cdots A_{i_1}) \in \mathbb{R}^{n\times n} : (i_1, \ldots,\, i_{(k+1)h}) \in \mathcal{M}^{(k+1)h}\}$, and $\mathbf{b} := [\ \beta_1 \ \cdots \ \beta_{N^{(k+1)N}}\ ]$ satisfies $\mathbf{b} \succeq 0$ and $\mathbf{b}^T\mathbf{1} = 1$. If we set $\varepsilon < \sqrt{\lambda_{\min}(P)}$, then $\sum_{i\in\{1,\ldots,N^{(k+1)h}\}} \beta_i \hat{A}_i^T P \hat{A}_i \prec P$ is fulfilled. Therefore, one concludes that there exists a sufficiently large $k > h$ such that the LMI problem in (8) is feasible. ∎

C. Projection algorithm

The BMI problem (6) can be solved locally using various iterative approaches, for instance, the path-following approach [24]. Here, we suggest another simple iterative method to find a solution. To this end, let us define the mapping $M_h(P,\, Q) := \mathrm{co}\{L_h(P,\, Q)\}$, where $Q \in \mathbb{S}_{++}^n$ $L_h\{\cdot,\, \cdot\}$ is defined in (5) and $\mathrm{co}\{\cdot\}$ stands for the convex hull [1]. Furthermore, define $M_h^+(P,\, Q) := M_h(P,\, Q) \oplus \mathbb{S}_+^n$, where $\oplus$ is defined as $\mathcal{U} \oplus \mathcal{V} := \{u + v : u \in \mathcal{U},\, v \in \mathcal{V}\}$ for given two sets $\mathcal{U}$ and $\mathcal{V}$. Then, it is true that P is a solution to the BMI problem (6) if $P \in M_h^+(P,\, Q)$. The concept is shown in Fig. 1. We can see that for any $x \in \mathbb{R}^n$, the minimum of $x^T\Pi x$ over $\Pi \in M_h(P,\, Q)$ is achieved on the vertexes $L_h(P,\, Q)$ of the polytope $M_h(P,\, Q)$. Therefore, $P \in M_h^+(P,\, Q)$ implies

$$\begin{aligned}
\min_{\Pi\in M_h(P,\, Q)} x^T\Pi x &= \min_{i\in\{1,\ldots,\, N^h\}} x^T\Pi_i(P,\, Q)x \\
&\le x^T P x, \quad \forall x \in \mathbb{R}^n.
\end{aligned}$$

This means

$$\begin{aligned}
&\min_{i\in\{1,\ldots,\, N^h\}} x^T\Pi_i(P,\, Q)x \\
&= \min_{(i_1,\ldots,\, i_h)\in\mathcal{M}^h} x^T \left[(A_{i_h} \cdots A_{i_1})^T P (A_{i_h} \cdots A_{i_1}) + Q \right] x
\end{aligned}$$

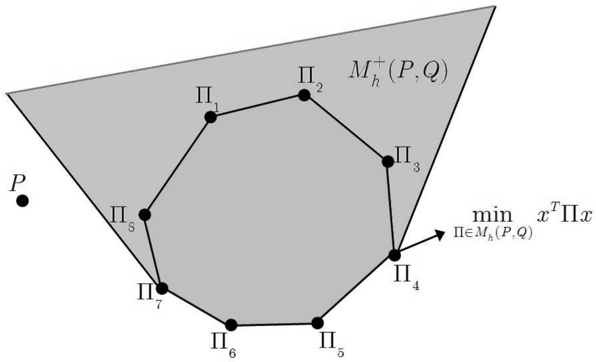

Fig. 1. Visualization of mapping $M_h^+(P,\,Q)$.

$$
\begin{aligned}
&\le x^T P x\\
&\Rightarrow \min_{(i_1,\ldots,i_h)\in\mathcal{M}^h} V(A_{i_h}\cdots A_{i_1}x) < V(x), \quad \forall x(k)\neq 0
\end{aligned}
$$

Therefore, the Lyapunov inequality (4) is satisfied. The task is to find P such that $P \in M_h^+(P,\,Q)$.

Problem 2: Find P such that $P \in M_h^+(P,\,Q)$.

In the sequel, any P^* such that $P^* \in M_h^+(P^*,\,Q)$ will be called a solution to Problem 2. We prove some useful properties of the mapping $M_h^+(\cdot,\,\cdot)$ and the solution P^*.

Proposition 4: The following properties hold:

1) If $P_1 \preceq P_2$, then $M_h^+(P_2,\,Q) \subseteq M_h^+(P_1,\,Q)$;
2) For any $P_1 \succ 0$ and $P_2 \succ 0$, $M_h^+(tP_1+(1-t)P_2,\,Q) = tM_h^+(P_1,\,Q) \oplus (1-t)M_h^+(P_2,\,Q)$ holds for all $t \in [0,\,1]$;
3) For any $P \succ 0$, if $Q_2 \succeq Q_1 \succ 0$, then $M_h^+(P,\,Q_2) \subseteq M_h^+(P,\,Q_1)$;
4) If P^* is a solution, so is λP^* for all $\lambda \ge 1$;
5) If $Q_2 \succeq Q_1 \succ 0$, then the solution set of $P \in M_h^+(P,\,Q_1)$ includes the solution set of $P \in M_h^+(P,\,Q_2)$;

Proof:

1) If $P_1 \preceq P_2$, then there exists $S \succeq 0$ such that $P_2 = P_1 + S$. In addition, suppose $X \in M_h^+(P_2,\,Q)$. Then, there exist $\mathbf{a} := [\,\alpha_1 \ \ \cdots \ \ \alpha_{N^h}\,]^T \in \mathbb{R}^{N^h}$ and $R \succeq 0$ such that $\mathbf{a} \succeq 0$, $\mathbf{a}^T\mathbf{1} = 1$, and
$$X = \sum_{i\in\{1,\ldots,N^h\}} \alpha_i \Pi_i(P_2,\,Q) + R.$$
Then, using relation $P_2 = P_1 + S$, it is easily seen that
$$X = \sum_{i\in\{1,\ldots,N^h\}} \alpha_i \Pi_i(P_1,\,Q) + \bar{R} \in M_h^+(P_1,\,Q),$$
where $\bar{R} := \sum_{i\in\{1,\ldots,N^h\}} \alpha_i \Pi_i(S,\,0_{n\times n}) + R \succeq 0$. Thus, $X \subseteq M_h^+(P_1,\,Q)$. This implies $M_h^+(P_2,\,Q) \subseteq M_h^+(P_1,\,Q)$.
2) For any $P_1 \succ 0$ and $P_2 \succ 0$, assume $X \in M_h^+(tP_1 + (1-t)P_2,\,Q)$. Then, there exist $\mathbf{a} := [\,\alpha_1 \ \ \cdots \ \ \alpha_{N^h}\,]^T \in \mathbb{R}^{N^h}$ and $R \succeq 0$ such that $\mathbf{a} \succeq 0$, $\mathbf{a}^T\mathbf{1} = 1$, and
$$X = \sum_{i\in\{1,\ldots,N^h\}} \alpha_i \Pi_i(tP_1 + (1-t)P_2,\,Q) + R$$
hold. Then, using simple algebraic calculations, it can be seen that
$$
\begin{aligned}
X = &\sum_{i\in\{1,\ldots,N^h\}} \alpha_i \Pi_i(tP_1 + (1-t)P_2,\,Q) + R\\
=&t\sum_{i\in\{1,\ldots,N^h\}} \alpha_i \Pi_i(P_1,\,Q)\\
&+(1-t)\sum_{i\in\{1,\ldots,N^h\}} \alpha_i \Pi_i(P_2,\,Q) + R\\
\in& tM_h^+(P_1,\,Q) + (1-t)M_h^+(P_2,\,Q).
\end{aligned}
$$
The converse is also true. Therefore, the statement is proven.
3) It directly follows from statement 1).
4) Assume that P^* is a solution, i.e., $P^* \in M_h^+(P^*,\,Q)$. Then, there exist $\mathbf{a} := [\,\alpha_1 \ \ \cdots \ \ \alpha_{N^h}\,]^T \in \mathbb{R}^{N^h}$ and $R \succeq 0$ such that $\mathbf{a} \ge 0$, $\mathbf{a}^T\mathbf{1} = 1$, and
$$\sum_{i\in\{1,\ldots,N^h\}} \alpha_i \Pi_i(P^*,\,Q) + R = P^*.$$
Multiplying the above equation by $\lambda \ge 1$, one gets
$$
\begin{aligned}
\lambda P^* =&\lambda\sum_{i\in\{1,\ldots,N^h\}} \alpha_i \Pi_i(P^*,\,Q) + \lambda R\\
=&\sum_{i\in\{1,\ldots,N^h\}} \alpha_i \Pi_i(\lambda P^*,\,\lambda Q) + \lambda R\\
=&\sum_{i\in\{1,\ldots,N^h\}} \alpha_i \Pi_i(\lambda P^*,\,Q) + (\lambda-1)Q + \lambda R\\
\in& M_h^+(\lambda P^*,\,Q).
\end{aligned}
$$
5) It can be easily proven using a similar reasoning of the proof of statement 1).

■

It is possible to devise iterative approaches that generate a sequence of matrices $\{P_0,\,P_1,\ldots,\,P_k\}$ that locally converges to a solution. One of the simple methods is based on the following recursion:
$$P_{k+1} = \mathrm{Proj}_{M_h^+(P_k,\,\gamma I_n)}[P_k] := \mathop{\arg\min}_{P\in M_h^+(P_k,\,\gamma I_n)} ||P - P_k||,$$
where $\mathrm{Proj}_{M_h^+(P_k,\,\gamma I_n)}[P_k]$ denotes the projection of matrix P_k onto the convex set $M_h^+(P_k,\,\gamma I_n)$, where $\gamma > 0$ is a sufficiently small real number. The iteration is visualized in Fig. 2, and the overall algorithm is summarized in Algorithm 1. The projection operator can be implemented using the convex optimization shown in Algorithm 2.

Algorithm 1 Computing P such that $P \in M_h^+(P,\,\gamma I_n)$

$k \leftarrow 0$; $P_0 \leftarrow I_n$; set sufficiently small ε.
repeat
  $P_{k+1} = \mathrm{Proj}_{M_h^+(P_k,\,\gamma I_n)}[P_k]$
  $k \leftarrow k+1$
until $||P_{k+1} - P_k|| \le \varepsilon$
return P_{k+1}

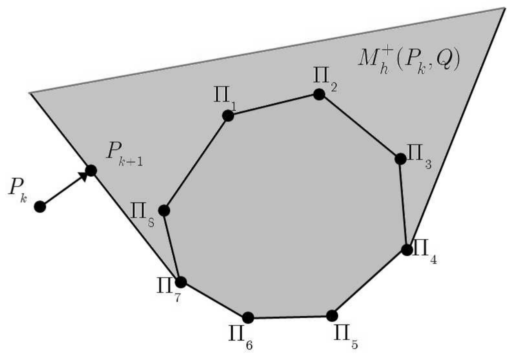

Fig. 2. Visualization of projection $\text{Proj}_{M_h^+(P_k,\,Q)}[P_k]$.

Algorithm 2 Computing $P_{k+1} = \text{Proj}_{M_h^+(P_k,\,\gamma I_n)}[P_k]$

Compute $L_h(P_k, \gamma I_n) = \{\Pi_1(P_k, \gamma I_n), \ldots, \Pi_{N^h}(P_k, \gamma I_n)\}$.

Solve

$$(\mathbf{a}^*, S^*, \varepsilon^*) := \arg\min_{\mathbf{a},\,S,\,\varepsilon} \varepsilon \text{ subject to}$$

$$\begin{bmatrix} -\varepsilon I_n & * \\ \sum_{i\in\{1,\ldots,N^h\}} \alpha_i \Pi_i(P_k, \gamma I_n) + S - P_k & -\varepsilon I_n \end{bmatrix} \preceq 0$$

$$\mathbf{a} \succeq 0, \quad \mathbf{a}^T\mathbf{1} = 1, \quad S \succeq 0$$

return $P_{k+1} = \sum_{i\in\{1,\ldots,N^h\}} \alpha_i^* \Pi_i(P_k, \gamma I_n) + S^*$

Remark 2: Using the fact that $\text{Proj}_{M_h^+(P,\,\gamma I_n)}[P]$ maps any $P \succ 0$ into a set of positive definite matrices, it can be seen that the sequence $\{P_k\}$ generated by using Algorithm 1 satisfies $P_k \succ 0, \forall k \in \{0, 1, \ldots\}$ if $P_0 \succ 0$.

Remark 3: One can observe from Algorithm 1 that the sequence $\{P_k\}$ will, if converges, get closer and closer to the boundary of the polytope $M_h(P^*, \gamma I_n)$, where P^* is a solution such that $P_k \to P^*$ as $k \to \infty$, but not reach the boundary of $M_h(P^*, \gamma I_n)$ in a finite number of iterations. This means that a sufficiently small ε in Algorithm 1 with $||P_{k+1} - P_k|| \leq \varepsilon$ does not guarantee that P_{k+1} is a solution. A possible way to resolve this issue is to use the stopping criterion $\text{Stop}[P_{k+1}, \bar{\gamma} I_n]$ in Algorithm 3 to eventually ensure that P_{k+1} is a solution. Instead of checking $||P_{k+1} - P_k|| \leq \varepsilon$ in Algorithm 1, $\text{Stop}[P_{k+1}, \bar{\gamma} I_n]$ can be evaluated at each iteration.

Remark 4: Let $S(\gamma I_n) := \{P \in \mathbb{S}^n_{++} : P \in M_h^+(P, \gamma I_n)\}$ be the solution space of Problem 1. When applying Algorithm 1 with $\text{Stop}[P_{k+1}, \bar{\gamma} I_n]$ in Algorithm 3 at each iteration, setting $\gamma > \bar{\gamma}$ is recommended. The reason is that if $\gamma > \bar{\gamma}$, then $S(\gamma I_n) \subseteq S(\bar{\gamma} I_n)$. Therefore, with a small margin, approaching a smaller set $S(\gamma I_n)$ using Algorithm 1 enables sequence $\{P_k\}$ to be inside the larger set $S(\bar{\gamma} I_n)$, and eventually, P_k can be identified as a solution.

Algorithm 3 Computing $\text{Stop}[P_{k+1}, \bar{\gamma} I_n]$

Compute $L_h(P_{k+1}, \bar{\gamma} I_n) = \{\Pi_1(P_{k+1}, \bar{\gamma} I_n), \ldots, \Pi_{N^h}(P_{k+1}, \bar{\gamma} I_n)\}$.

Solve the LMI problem for $\mathbf{a}$

$$\sum_{i\in\{1,\ldots,N^h\}} \alpha_i \Pi_i(P_{k+1}, \bar{\gamma} I_n) - P_{k+1} \preceq 0,$$

$$\mathbf{a} \succeq 0, \quad \mathbf{a}^T\mathbf{1} = 1.$$

if the LMI problem is feasible **then** $y = 1$
else $y = 0$
end if
return y.

D. Local convergence

To prove the local convergence of Algorithm 1, the following properties of the projection operator will be used:

Lemma 3 ([25]): Let $\mathcal{C}$ be any nonempty closed convex set in $\mathbb{R}^n$. For any $x, y \in \mathbb{R}^n$, the following property holds:

$$||\text{Proj}_{\mathcal{C}}[x] - \text{Proj}_{\mathcal{C}}[y]||^2 \leq ||x - y||^2 - ||\text{Proj}_{\mathcal{C}}[x] - x + y - \text{Proj}_{\mathcal{C}}[y]||^2.$$

In addition, let us define the set $\Omega(P^*) := \{S \in \mathbb{S}^n_{++} : P^* \in M_h^+(S, \gamma I_n)\}$, where P^* is a solution to Problem 2, i.e., $P^* \in M_h^+(P^*, \gamma I_n)$. In other words, the set $\Omega(P^*)$ can be seen as a set of P whose corresponding polytope $M_h^+(P, \gamma I_n)$ includes solution P^*. The following theorem establishes a local convergence result.

Theorem 1 (Local convergence of Algorithm 1): Assume $\{P_0, P_1, \ldots\} \in \Omega(P^*)$, where P^* is a solution to Problem 2 and $\{P_0, P_1, \ldots\}$ is a sequence of positive definite matrices generated by Algorithm 1. Also, suppose that P^* is a unique solution in $\Omega(P^*)$. Then, $\lim_{k\to\infty} P_k = P^*$.

Proof: Since $P_{k-1} \in \Omega(P^*)$ from the assumption, we have

$$\begin{aligned}
&||\underbrace{\text{Proj}_{M_h^+(P^*,\,\gamma I_n)}(P^*)}_{P^*} - \underbrace{\text{Proj}_{M_h^+(P_{k-1},\,\gamma I_n)}(P_{k-1})}_{P_k}||^2 \\
&= ||\underbrace{\text{Proj}_{M_h^+(P_{k-1},\,\gamma I_n)}(P^*)}_{P^*} - \text{Proj}_{M_h^+(P_{k-1},\,\gamma I_n)}(P_{k-1})||^2 \\
&\leq ||P^* - P_{k-1}||^2 - ||\underbrace{\text{Proj}_{M_h^+(P_{k-1},\,\gamma I_n)}(P^*)}_{P^*} - P^* + P_{k-1} \\
&\quad - \text{Proj}_{M_h^+(P_{k-1},\,\gamma I_n)}(P_{k-1})||^2 \\
&= ||P^* - P_{k-1}||_2^2 - ||P_{k-1} - P_k||^2,
\end{aligned}$$

where Lemma 3 is used in line 3. From the above result, we obtain $||P^* - P_k||^2 \leq ||P^* - P_{k-1}||^2 - ||P_{k-1} - P_k||^2$, which tells us that sequence $\{e_k\}$ with $e_k := ||P^* - P_k||^2$ is (strictly) monotonically decreasing. Since e_k is bounded below by 0, it is true that $\{e_k\}$ converges to a value $e \in \mathbb{R}$, $e \geq 0$. Next, we will show $e = 0$. Suppose that $e_k \to e$ as $k \to \infty$ but $e > 0$. This means that for all $\varepsilon > 0$, there is an integer K such that $|e - e_k| < \varepsilon, \forall k \geq K$. On the other hand, $||P^*-P_k||^2 \leq ||P^*-P_{k-1}||^2 - ||P_{k-1}-P_k||^2$ implies

$$e_k \leq e_{k+1} - ||P_{k-1}-P_k||^2$$
$$\Rightarrow ||P_{k-1}-P_k||^2 \leq e_{k+1}-e_k = |e_{k+1}-e_k|$$

Since $\{e_k\}$ is a convergent sequence, it is also a Cauchy sequence. Therefore, for all $\varepsilon > 0$, there is an integer K such that

$$||P_{k-1}-P_k||^2 \leq |e_{k+1}-e_k| < \varepsilon, \quad \forall k \geq K.$$

As a result, we have $\lim_{k\to\infty} ||P_{k-1}-P_k||^2 = 0$ and $\lim_{k\to\infty} P_k = \tilde{P}$, where $\tilde{P}$ is a point on the surface of sphere $\{S \in \mathbb{R}^{n\times n} : ||P^*-S||^2 = e\}$. In addition, $\tilde{P}$ is a solution to Problem 1 since $\lim_{k\to\infty} P_k = \lim_{k\to\infty} P_{k+1} = \lim_{k\to\infty} \mathrm{Proj}_{M_h^+(P_k,\gamma I_n)}(P_k) = \tilde{P}$. From the assumption $\{P_0, P_1, \ldots\} \in \Omega(P^*)$, it is true that $\tilde{P} \in \Omega(P^*)$. Also, $\tilde{P} \in \{S \in \mathbb{R}^{n\times n} : ||P^*-S||^2 = e\}$ ensures $\tilde{P} \neq P^*$. However, this contradicts the assumption that P^* is a unique solution in $\Omega(P^*)$. Therefore, $e = 0$ and $\lim_{k\to\infty} e_k = 0$, meaning that $\lim_{k\to\infty} P_k = P^*$. This completes the proof. ■

Remark 5: In Theorem 1, the assumption of the uniqueness of the solution can be removed. In this case, the sequence $\{P_0, P_1, \ldots\}$ is still guaranteed to converge to a solution, but not necessarily to P^*.

Corollary 1: Let us assume that P^* is a solution to Problem 1, and $\{P_0, P_1, \ldots\}$ is a sequence of positive definite matrices generated by Algorithm 1. In addition, suppose that

$$P_k \in \Phi(P^*) := \{S \in \mathbb{S}_{++}^{n\times n} : S \preceq P^*\}, \forall k \in \mathbb{N}, \qquad (9)$$

and that P^* is a unique solution in $\Phi(P^*)$. Then, $\lim_{k\to\infty} P_k = P^*$.

Proof: From the statement 1) of Proposition 2, (9) means $P^* \in M_h^+(P^*, \gamma I_n) \subseteq M_h^+(P_k, \gamma I_n), \forall k \in \{0, 1, 2, \ldots\}$. The proof is then completed by applying Theorem 1. ■

Proposition 5: Suppose that the BMI problem of Proposition 1 has a solution. Then, there exists a sufficiently large positive integer h such that Algorithm 1 converges to a solution.

Proof: This can be proven as a direct consequence of Proposition 3. ■

E. Examples

Example 1: Let us consider SLS (2) with

$$A_1 = \begin{bmatrix} 1.0076 & 0.0662 \\ 0.1323 & 0.4122 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.9867 & 0.1527 \\ -0.3054 & 2.2082 \end{bmatrix},$$

which was borrowed from [8]. The eigenvalues are $\lambda = 1.0218, 0.3978$ for A_1 and $\lambda = 1.0262, 2.1688$ for A_2. Algorithm 1 with $\gamma = 0.01$ failed to find a solution for $h = 1, 2, \ldots, 7$. For $h = 8$, a feasible solution was found with $P = \begin{bmatrix} 1.0598 & 0.0571 \\ 0.0571 & 0.9441 \end{bmatrix}$.

Example 2: Let us consider SLS (2) with

$$A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0.5 & 0.5 \\ 1 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.5 & 0.2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0.5 \end{bmatrix}.$$

The eigenvalues are $\lambda = 0.5, 2, 0$ for A_1 and $\lambda = 0.5, 0.5, 1$ for A_2. Algorithm 1 with $\gamma = 0.01$ failed to find a solution for $h = 1, 2, \ldots, 6$. For $h = 7$, a feasible solution was found with $P = I_3$.

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