Semistability of Switched Linear Systems with Applications to Distributed Sensor Networks: A Generating Function Approach

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***Abstract*— This paper investigates semistability of discrete-time, switched linear systems under both deterministic and random switching policies. The notion of semistability pertains to a continuum of initial state dependent equilibria and has found wide applications such as consensus problems in multi-agent systems. The main contributions of the paper are three folds. First, we show that exponential semistability on a common equilibrium space is equivalent to output exponential stability of the switched linear system with a suitably defined output, under both arbitrary and random switchings. Further, their convergence rates are shown to be identical. Second, it is shown that output stability and its convergence rates can be efficiently characterized via the recently developed generating function approach. Third, we consider algorithm development and analysis of resource allocation schemes for topologically changing, distributed sensor networks. We formulate an iteration process of such an algorithm as a switched linear system, and characterize its convergence using the obtained semistability results.**

I. INTRODUCTION

Semistability extends the regular notion of stability pertaining to a single, isolated equilibrium. Roughly speaking, for a dynamical system with a continuum of equilibria, semistability implies that any trajectory converges to a (possibly different) stable equilibrium that is dependent on its initial state. Semistable dynamics have been found in various fields, e.g., mechanical systems [1], [2], network systems [15], [3], biomedical systems [4], [5], chemical kinetics [6], [7], etc, to cite but a few examples.

Motivated by algorithm design and analysis of distributed, multi-agent sensor networks, we perform semistability analysis for a class of switched linear systems under different switching policies. A resource allocation algorithm for a distributed sensor network is an iteration process that can be treated as a discrete-time linear system. A notable feature of this dynamical process is that it may possess a continuum of initial state dependent equilibria. For example, a consensus reached in a distributed, multi-agent network is an equilibrium relying on initial conditions [15]. Further, a sensor network often has a switching topology because of possible communcation link failures/creations. Hence, the iteration process of an algorithm is subject to switching dynamics and thus can be formulated as a discrete-time, switched linear system. To evaluate the performance of an iterative algorithm, e.g., convergence and its convergence rate, it is essential to develop efficient techniques to characterize semistability and (exponential) growth rate of the related switched linear system. This sparks our study in this paper.

Related semistability results in the literature include [16], [17], [18], [19]. In particular, the references [18], [19] study semistability of continuous-time, switched linear systems in the Lyapunov framework. Distinct from this perspective, we investigate semistability and characterize growth rates using the generating function approach recently introduced in [9], [11], [12], [13]. Informally speaking, generating functions are certain power series with coefficients determined from systems trajectories under switching policies. Their convergence radii characterize system growth rates which can be computed via effective algorithms developed in [11]. In this paper, we convert the semistability problem into an equivalent, projection based, output stability problem, under both deterministic and random switchings. The latter problem can be efficiently handled via the generating function approach.

The rest of the paper is organized as follows. In Section II, we introduce semistability and output stability notions. Section III focuses on semistability analysis and generating function characterization under arbitrary switching. Randomly switched linear systems and their generating functions are addressed in Section IV. Finally, the application to distributed sensor networks is treated in Section V.

II. SEMISTABILITY OF SWITCHED LINEAR SYSTEMS

A discrete-time, (autonomous) switched linear system (SLS) on $\mathbb{R}^n$ is:

$$x(t+1) = A_{\sigma(t)}x(t), \quad t = 0, 1, \ldots, \tag{1}$$

where its state $x(t) \in \mathbb{R}^n$ evolves by switching among a finitely family of linear dynamics indexed by the finite index set $\mathcal{M} := \{1, \ldots, M\}$, $\sigma(t) \in \mathcal{M}$ for all t, or simply σ, is the switching sequence, and $A_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{M}$, are the subsystem dynamics matrices. Denoted by $x(t; z, \sigma)$ the state trajectory of the SLS from the initial state $x(0) = z$ under the switching sequence σ. In this paper, unless otherwise stated, the vector norm $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$ and the matrix norm is induced from the Euclidean norm.

Let I_n be the $n \times n$ identity matrix and $\mathcal{N}(\cdot)$ denote the null space of a matrix. Define the (common) equilibrium

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subspace $\mathcal{E}_{\mathrm{e}} := \bigcap_{i\in\mathcal{M}} \mathcal{N}(A_i - I_n)$, and let $\mathcal{E}_{\mathrm{e}}^{\perp}$ be the orthogonal complement of $\mathcal{E}_{\mathrm{e}}$ in $\mathbb{R}^n$. Obviously, $\mathcal{E}_{\mathrm{e}}$ is invariant under $\{A_i\}_{i\in\mathcal{M}}$. Hence each $x_{\mathrm{e}} \in \mathcal{E}_{\mathrm{e}}$ is an equilibrium of the SLS (1) under an arbitrary switching sequence. The following standing assumption asserts that both $\mathcal{E}_{\mathrm{e}}$ and $\mathcal{E}_{\mathrm{e}}^{\perp}$ are nontrivial subspaces of $\mathbb{R}^n$.

Assumption 2.1: The dimension and codimension of $\mathcal{E}_{\mathrm{e}}$ are both at least one.

Hence, there exists at least one $i \in \mathcal{M}$ such that $A_i \neq I_n$.

Definition 2.1: The SLS (1) is *exponentially semistable* under arbitrary switching if there exist constants $\rho \geq 0$ and $r \in [0, 1)$ such that for any $z \in \mathbb{R}^n$ and under any switching sequence σ, there exists a (unique) $x_{\mathrm{e}}(z, \sigma) \in \mathcal{E}_{\mathrm{e}}$ (dependent on z and σ) such that $\|x(t; z, \sigma) - x_{\mathrm{e}}(z, \sigma)\| \leq \rho\, r^t \|z - x_{\mathrm{e}}(z, \sigma)\|$, $\forall\, t \in \mathbb{Z}_+$. Here, r is called the *exponential growth rate of semistability* (under arbitrary switching).

It is known that the asymptotic stability and exponential stability of switched linear systems under arbitrary switching are equivalent [8], [9]. Next, we show that the same holds for semistability and exponential semistability of switched linear systems. Before we state this result, the following definition of semistability for switched linear systems is needed.

Definition 2.2: The SLS (1) is *semistable* under arbitrary switching if there exists a class $\mathcal{KL}$ function $\alpha(\cdot, \cdot)$ such that any $z \in \mathbb{R}^n$ and under any σ, there exists $x_{\mathrm{e}}(z, \sigma) \in \mathcal{E}_{\mathrm{e}}$ such that $\|x(t; z, \sigma) - x_{\mathrm{e}}(z, \sigma)\| \leq \alpha(\|z - x_{\mathrm{e}}(z, \sigma)\|, t), \forall\, t \in \mathbb{Z}_+$.

The following lemma, whose proof is omitted, states the semistability equivalence under arbitrary switching.

Lemma 2.1: The SLS (1) is *exponentially semistable* under arbitrary switching if and only if it is semistable under arbitrary switching.

To characterize the exponential semistability and convergence rate of (1), we project the dynamics onto $\mathcal{E}_{\mathrm{e}}^{\perp}$. Let $O \in \mathbb{R}^{n\times\ell}$ be the matrix whose columns constitute an orthonormal basis of $\mathcal{E}_{\mathrm{e}}^{\perp}$ (this implies that $\mathcal{E}_{\mathrm{e}}^{\perp}$ is of dimension ℓ), and let $P = OO^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ be the matrix representing the orthogonal projection onto $\mathcal{E}_{\mathrm{e}}^{\perp}$. Clearly, P is idempotent, i.e., $P^2 = P$. For a given trajectory $x(t)$, let $x_{\mathcal{E}_{\mathrm{e}}}(t)$ and $x_{\mathcal{E}_{\mathrm{e}}^{\perp}}(t)$ denote the (unique) orthogonal projections of $x(t)$ onto $\mathcal{E}_{\mathrm{e}}$ and $\mathcal{E}_{\mathrm{e}}^{\perp}$, respectively. That is, $x_{\mathcal{E}_{\mathrm{e}}^{\perp}}(t) = Px(t)$ and $x_{\mathcal{E}_{\mathrm{e}}}(t) = [I_n - P]x(t)$. Define the output of the SLS (1) using the projection matrix O:

$$y(t) = O^{\mathrm{T}} x(t), \quad t = 0, 1, 2, \ldots \tag{2}$$

Then, at any t, regardless of the current mode $\sigma(t)$, $\|y(t)\|$ is the Euclidean distance of $x(t)$ to the equilibrium subspace $\mathcal{E}_{\mathrm{e}}$. In the following, denote by $y(t; z, \sigma)$ the output trajectory of the SLS (1) starting from the initial condition z under the switching sequence σ.

Definition 2.3: The SLS (1) with the output (2) is *output exponentially stable* under arbitrary switching if there exist constants $\kappa > 0$ and $\widetilde{r} \in [0, 1)$ such that for any $z \in \mathbb{R}^n$, $\|y(t; z, \sigma)\| \leq \kappa\, \widetilde{r}^t \|z\|$, $\forall\, t \in \mathbb{Z}_+$, under any switching sequence σ. Here, the parameter $\widetilde{r}$ is called the *exponential growth rate of output stability* (under arbitrary switching).

To simplify the subsequent development, we introduce a coordinate transformation as follows. Recall that $\mathcal{E}_{\mathrm{e}}^{\perp}$ is of dimension ℓ. Let $\widehat{O} \in \mathbb{R}^{n\times(n-\ell)}$ be the matrix whose columns constitute an orthonormal basis of $\mathcal{E}_{\mathrm{e}}$, and define the invertible matrix $T := [O\ \widehat{O}]^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ and the state transformation $\widehat{x}(t) = Tx(t)$. In the new coordinates, $\mathcal{E}_{\mathrm{e}}^{\perp} = \mathbb{R}^{\ell} \times \{0\}$, $\mathcal{E}_{\mathrm{e}} = \{0\} \times \mathbb{R}^{(n-\ell)}$, and the relevant matrices can be written as $\widehat{O} = O^{\mathrm{T}}T^{-1} = \begin{bmatrix} I_\ell & 0 \end{bmatrix}$, $\widehat{P} = \widehat{O}\widehat{O}^{\mathrm{T}} = \begin{bmatrix} I_\ell & 0 \\ 0 & 0 \end{bmatrix}$, $\widehat{A}_i = TA_iT^{-1} = \begin{bmatrix} \widehat{A}_{i,11} & 0 \\ \widehat{A}_{i,21} & I_{n-\ell} \end{bmatrix}$, for all $i \in \mathcal{M}$. Furthermore, $\widehat{x}(t) = [y^{\mathrm{T}}(t); \widetilde{x}^{\mathrm{T}}(t)]^{\mathrm{T}}$, $z = [y^{\mathrm{T}}(0); \widetilde{x}^{\mathrm{T}}(0)]^{\mathrm{T}}$, $x^{\mathrm{T}}_{\mathcal{E}_{\mathrm{e}}^{\perp}}(t) = [y^{\mathrm{T}}(t); 0]$, and $x^{\mathrm{T}}_{\mathcal{E}_{\mathrm{e}}}(t) = [0; \widetilde{x}^{\mathrm{T}}(t)]$, where $y(t) \in \mathbb{R}^{\ell}$ and $\widetilde{x}(t) \in \mathbb{R}^{(n-\ell)}$ satisfy

$$y(t+1) = \widehat{A}_{\sigma(t),11}\, y(t), \tag{3}$$

and

$$\widetilde{x}(t) = \widetilde{x}(0) + \sum_{\tau=0}^{t-1} \widehat{A}_{\sigma(\tau),21}\, y(\tau). \tag{4}$$

Note that (3) yields a switched linear system defined by subsystem matrices $\{\widehat{A}_{i,11}\}$ and is decoupled from (4). See Remark 3.1 for the geometry of the above dynamics under exponential stability conditions. Since the state transformation does not affect the semistability and output stability as well as their growth rates, we consider the switching dynamics (3)–(4) throughout the rest of the paper by dropping the notation $\widehat{\ }$ in the equations.

III. Semistability of Switched Linear Systems: Deterministic Case

In this section, we study semistability under deterministic, arbitrary switchings.

A. Semistability and Output Stability

We firstly show the equivalence of exponential semistability and output exponential stability. To this end, we introduce some notions and a technical result that is of its own interest. This result asserts the equivalence of convergence, asymptotic and exponential stability for general switched linear systems under arbitrary switching. Specifically, consider a switched linear system defined by a finite family of matrices. We call the switched linear system *convergent* under arbitrary switching if for any initial state z, $x(t; z, \sigma)$ converges to the origin as $t \to \infty$ under any switching sequence σ.

Theorem 3.1: A switched linear system is convergent under arbitrary switching if and only if it is exponentially stable under arbitrary switching.

Proof: It suffices to prove the "only if". Consider a SLS with the matrices $\{A_i\}_{i=1}^m$. The following claim holds:

Claim: If there exists $T_* \in \mathbb{N}$ (independent of z and σ) such that for any z with $\|z\| = 1$ and under any switching sequence σ, there exists $t_* \in [0, T_*]$ such that $\|x(t_*; z, \sigma)\| \leq 0.5$, then the SLS is exponentially stable under arbitrary switching.

Let $\kappa := \sum_{j=0}^{T_*} (\max_i \|A_i\|)^j$. The claim can be shown via induction that under the given conditions, $\|x(t; z, \sigma)\| \leq \kappa(0.5)^{t/T_* - 1}\|z\|$, $\forall\, t \in \mathbb{Z}_+$ for any z and under any switching sequence σ. This thus yields the exponential stability.

Now suppose that the SLS is convergent but not exponentially stable, under arbitrary switching. It follows from the above claim that there exist an initial state sequence $\{z_k\}$ with $\|z_k\| = 1$, an increasing time sequence $\{T_k\}$ with $\lim_{k\to\infty} T_k = \infty$, and a sequence of switching sequences $\{\sigma_k\}$ such that for each k, $\|x(t; z_k, \sigma_k)\| \geq 0.5$ for all $t = 0, 1, \cdots, T_k$. Using a similar argument as that of [9, Theorem 3], we can construct an initial state z and a switching sequence σ such that $\|x(t; z, \sigma)\| \geq 0.5$ for all t. This is contradictory to the convergence of the SLS. ■

Theorem 3.2: The SLS (1) is exponentially semistable under arbitrary switching if and only if the SLS (1) with the output (2) is output exponentially stable under arbitrary switching. Further, the exponential growth rates of semistability and output stability are equivalent, namely, a constant $r \in [0, 1)$ is the exponential growth rate of semistability if and only if it is that of output stability.

Proof: "If". Suppose $\kappa > 0$ and $\widetilde{r} \in [0, 1)$ exist such that for any $z \in \mathbb{R}^n$ and any switching sequence σ, $\|y(t; z, \sigma)\| \leq \kappa \widetilde{r}^t \|z\|$ for all $t \in \mathbb{Z}_+$. For a initial state $z \in \mathbb{R}^n$ and switching sequence σ, it follows from (4) that, for any $s, w \in \mathbb{Z}_+$ with $s < w$, $\big\|\widetilde{x}(w; z, \sigma) - \widetilde{x}(s; z, \sigma)\big\| = \big\|\sum_{\tau=s}^{w-1} A_{\sigma(\tau),21}\, y(\tau; z, \sigma)\big\| \leq \sum_{\tau=s}^{w-1} \|A_{\sigma(\tau),21}\| \cdot \|y(\tau; z, \sigma)\| \leq \max_i(\|A_{i,21}\|)\Big(\kappa \widetilde{r}^s + \cdots + \kappa \widetilde{r}^{w-1}\Big)\|z\| \leq \max_i(\|A_{i,21}\|)\, \kappa\, \widetilde{r}^s / \big(1 - \widetilde{r}\big)\|z\|$. Hence, for all s, w sufficiently large (no matter how far they are apart), $\|\widetilde{x}(w; z, \sigma) - \widetilde{x}(s; z, \sigma)\|$ is sufficiently small. This shows that $\{\widetilde{x}(t; z, \sigma)\}$ is a Cauchy sequence in $\mathbb{R}^{n-\ell}$ and thus converges in $\mathbb{R}^{n-\ell}$ (as $\mathbb{R}^{n-\ell}$ is complete). Let $\widetilde{x}_{\rm e}(z, \sigma) := \lim_{t\to\infty} \widetilde{x}(t; z, \sigma)$. Since $\mathbb{R}^{n-\ell}$ is closed, $\widetilde{x}_{\rm e}(z, \sigma) \in \mathbb{R}^{n-\ell}$. Further, let $x_{\rm e}(z, \sigma) := [0; \widetilde{x}_{\rm e}^{\rm T}(z, \sigma)]^{\rm T} \in \mathcal{E}_{\rm e}$ and $\widetilde{z} := z - x_{\rm e}(z, \sigma)$. It follows from (3)–(4) that $x(t; \widetilde{z}, \sigma) = \begin{bmatrix} y(t; z, \sigma) \\ \widetilde{x}(t; z, \sigma) \end{bmatrix} - \begin{bmatrix} 0 \\ \widetilde{x}_{\rm e}(z, \sigma) \end{bmatrix}$. This shows that $\|y(t; \widetilde{z}, \sigma)\| = \|y(t; z, \sigma)\| \leq \kappa\, \widetilde{r}^t \|\widetilde{z}\|$. Moreover, $\|\widetilde{x}(t; \widetilde{z}, \sigma)\| = \|\widetilde{x}(t; z, \sigma) - \widetilde{x}_{\rm e}(z, \sigma)\| = \big\|\sum_{\tau=t}^{\infty} A_{\sigma(\tau),21}\, y(\tau; \widetilde{z}, \sigma)\big\| \leq \max_i(\|A_{i,21}\|)\kappa\Big(\widetilde{r}^t + \widetilde{r}^{t+1} + \cdots\Big)\|\widetilde{z}\| \leq \max_i \|A_{i,21}\| \frac{\kappa\, \widetilde{r}^t}{1-\widetilde{r}} \|\widetilde{z}\|, \ \forall\, t$. In view of the above results and $\|x(t; \widetilde{z}, \sigma)\| \leq \|y(t; \widetilde{z}, \sigma)\| + \|\widetilde{x}(t; \widetilde{z}, \sigma)\|$, we obtain a constant $\rho > 0$, independent of z and σ, such that $\|x(t; z, \sigma) - x_{\rm e}(z, \sigma)\| = \|x(t; \widetilde{z}, \sigma)\| \leq \rho\, \widetilde{r}^t \|\widetilde{z}\| = \rho\, \widetilde{r}^t \|z - x_{\rm e}(z, \sigma)\|, \forall\, t \in \mathbb{Z}_+$. This gives rise to the exponential semistability and shows that the exponential growth rate $\widetilde{r}$ of output stability is also that of exponential semistability.

"Only if". We prove this by contradiction. Suppose that the SLS (1) is exponentially semistable but not output exponentially stable, under arbitrary switching. It is seen that the output trajectory $y(t)$ is equivalent to the state trajectory of the SLS (3) defined by the subsystem matrices $\{A_{i,11}\}$. Hence, if the original SLS (1) is not output exponentially stable under arbitrary switching, neither is the SLS (3). We deduce from Theorem 3.1 that the SLS (3) is not convergent under arbitrary switching. Hence, there exist $z \in \mathbb{R}^n$ and a switching sequence σ such that $y(t; z, \sigma)$ does not converge to the origin of $\mathbb{R}^\ell$, contradicting the exponential semistability of the SLS (1) under arbitrary switching.

Finally, we show that a constant $r \in [0, 1)$ is the exponential growth rate of semistability if and only if it is that of output stability. It suffices to consider the "only if" part as the other part has been shown before. Suppose that the SLS (1) is exponentially semistable with the exponential growth rate $r \in [0, 1)$, i.e., for any $z \in \mathbb{R}^n$ and under any σ, there exists $x_{\rm e}(z, \sigma) \in \mathcal{E}_{\rm e}$ such that $\|x(t; z, \sigma) - x_{\rm e}(z, \sigma)\| \leq \rho\, r^t \|z - x_{\rm e}(z, \sigma)\|, \forall\, t$ with $\rho > 0$. This implies, in light of the proof for the "only if" above, that the SLS (3) is exponentially stable with the exponential growth rate $\widetilde{r} \in [0, 1)$ (not necessarily equal to r at this stage) and the parameter $\kappa > 0$, under arbitrary switching. By slightly abusing notation, we use $y(t; y(0), \sigma)$ to denote the trajectory of the SLS (3) starting from $y(0)$ under σ. Hence, we have, for any $y(0) \in \mathbb{R}^\ell$ and under any σ, $\|y(t; y(0), \sigma)\| \leq \kappa \widetilde{r}^t \|y(0)\|, \forall\, t$. It thus follows from the above proof for the "if" part that for any initial state $z = [y^{\rm T}(0); \widetilde{x}^{\rm T}(0)]^{\rm T}$ and any switching sequence σ, $\widetilde{x}_{\rm e}(z, \sigma) = \widetilde{x}(0) + \lim_{t\to\infty} \sum_{\tau=0}^{t-1} A_{\sigma(\tau),21}\, y(\tau; y(0), \sigma)$ such that the latter limit exists. Moreover,

$$
\begin{aligned}
&\|z - x_{\rm e}(z, \sigma)\| \\
&\quad \leq \quad \big\|y(0)\big\| + \left\|\lim_{t\to\infty} \sum_{\tau=0}^{t-1} A_{\sigma(\tau),21}\, y(\tau; y(0), \sigma)\right\| \\
&\quad \leq \quad \big\|y(0)\big\| + \max_i \|A_{i,21}\| \sum_{\tau=0}^{\infty} \big\|y(\tau; y(0), \sigma)\big\| \\
&\quad \leq \quad \big\|y(0)\big\| + \max_i \|A_{i,21}\| \Big(\sum_{\tau=0}^{\infty} \kappa\, \widetilde{r}^\tau \,\big\|y(0)\big\|\Big) \\
&\quad \leq \quad \big\|y(0)\big\| + \max_i \|A_{i,21}\| \frac{\kappa}{1 - \widetilde{r}} \,\big\|y(0)\big\|.
\end{aligned}
$$

Consequently, by the exponential semistability, $\|y(t; z, \sigma)\| \leq \|x(t; z, \sigma) - x_{\rm e}(z, \sigma)\| \leq \rho\, r^t \|z - x_{\rm e}(z, \sigma)\| \leq \widetilde{\rho}\, r^t \|y(0)\| \leq \widetilde{\rho}\, r^t \|z\|, \forall\, t \in \mathbb{Z}_+$ for a constant $\widetilde{\rho} > 0$. Thus, r is the exponential growth rate of output stability. ■

Remark 3.1: The above theorem and the equations (3)–(4) show that under the exponential stability assumption, the SLS (1) can be thought of two dynamical processes: one is the dynamics of $y(t)$ in the fiber direction governed by the exponentially stable SLS (3) defined by $\{A_{i,11}\}$, and the other is the dynamics of $\widetilde{x}(t)$ along the base direction that evolves by integrating $A_{\sigma(t),21}\, y(t)$. The latter dynamics will move at worst in the pace proportional to $\|y(t)\|$ and thus converge at the same exponential rate as that of $y(t)$ to zero.

B. Semistability Analysis via Strong Generating Functions

In view of Theorem 3.2, the maximal exponential growth rate of the semistability of the SLS (1) is completely characterized by that of the SLS (3). This growth rate, denoted by r^*, may serve as a quantitative measure of robustness of the semistability of the SLS (1). Indeed, the SLS (1) is exponentially semistable if and only if $r^* < 1$.

In what follows, let $y(t; v, \sigma)$ denote the trajectory of the SLS (3) starting from the initial state $v \in \mathbb{R}^\ell$ under the switching sequence σ. Certain quantities such as the joint spectral radius and the Lyapunov exponent can determine the maximal exponential growth rate of the SLS (3) and therefore its exponential stability, under arbitrary switching. We next characterize exponential stability and the maximal exponential growth rate of the SLS (3) using the recently proposed generating function approach [9], [12], [11], [13].

The *strong generating function* $G : \mathbb{R}_+ \times \mathbb{R}^\ell \to \mathbb{R}_+ \cup \{+\infty\}$ of the SLS (3) is defined as $G(\lambda, v) := G_\lambda(v) = \sup_\sigma \sum_{t=0}^\infty \lambda^t \|y(t; v, \sigma)\|^2, \quad v \in \mathbb{R}^\ell,\ \lambda \geq 0$, where the supremum is taken over all switching sequences σ. Analytic properties of the generating functions can be found in [11]. The *radius of strong convergence* of G_λ is defined as $\lambda^* := \sup\{\lambda > 0 : G_\lambda(v) < \infty, \forall\, v \in \mathbb{R}^\ell\}$. The following theorem characterizes the exponential stability of the SLS (3) via λ^*:

Theorem 3.3: [11, Theorem 2] The SLS (3) is exponentially stable under arbitrary switching if and only if its radius of strong convergence $\lambda^* > 1$.

Moreover, as shown in [11, Corollary 1], the maximal exponential growth rate of the SLS (3) is given by $r^* = \frac{1}{\sqrt{\lambda^*}}$. Hence, we obtain the following statement without proof:

Theorem 3.4: The SLS (1) is exponentially semistable if and only if the radius of strong convergence λ^* of the SLS (3) satisfies $\lambda^* > 1$. Further, the maximal exponential growth rate of the semistability of the SLS (1) is $(\lambda^*)^{-1/2}$.

To compute the generating function G_λ and the radius of strong convergence λ^* for the SLS (3), we approximate G_λ by a sequence of finite horizon problems. Specifically, define $G_\lambda^k(v) := \max_\sigma \sum_{t=0}^k \lambda^t \|y(t; v, \sigma)\|^2, v \in \mathbb{R}^\ell, k \in \mathbb{Z}_+$. Then the functions $G_\lambda^k(v)$ can be computed recursively by $G_\lambda^0(v) = \|v\|^2$ and $G_\lambda^k(v) = \|v\|^2 + \lambda \max_{i\in\mathcal{M}} G_\lambda^{k-1}(A_{i,11}v),\ k = 1, 2, \ldots$. Based on the Bellman equation and the sub-additivity property, an iterative numerical algorithm can be developed to compute increasingly accurate estimates of G_λ on a grid of the unit sphere. See [12] or [11, Section III] for details.

IV. Semistability of Switched Linear Systems: Random Case

The notion of semistability can be extended to a randomly switched linear system that evolves at each time by a subsystem matrix selected randomly from the set $\{A_i\}_{i\in\mathcal{M}}$ according to a stationary distribution. In this case, the system state is a stochastic process $X(t)$ with the dynamics

$$X(t + 1) = \mathbf{A}(t)X(t), \quad t = 0, 1, 2, \ldots, \tag{5}$$

where at each time t, $\mathbf{A}(t) \in \mathbb{R}^{n\times n}$ is drawn independently randomly from the matrix set $\{A_i\}_{i\in\mathcal{M}}$ with the probability $\mathbb{P}\{\mathbf{A}(t) = A_i\} = p_i, \quad i \in \mathcal{M}$, for some probability distribution $p := \{p_i\}_{i\in\mathcal{M}}$ with $\sum_{i\in\mathcal{M}} p_i = 1$ and $p_i \geq 0$. For a given probability distribution p, denote by $X(t; z, p)$ the stochastic trajectory of the random SLS (5) starting from a deterministic initial state $X(0) = z$, and denote by $\mathbb{E}$ the expectation operator.

Definition 4.1: The random SLS (5) is *mean square exponentially semistable* if there exist constants $\rho \geq 0$ and $r \in [0, 1)$ such that for any $z \in \mathbb{R}^n$, there exists a random vector $X_{\rm e}(z, p) \in \mathcal{E}_{\rm e}$ such that

$$\mathbb{E}\left[\|X(t; z, p) - X_{\rm e}(z, p)\|^2\right] \leq \rho r^t \|z - \mathbb{E}\left[X_{\rm e}(z, p)\right]\|^2, \tag{6}$$

for all $t \in \mathbb{Z}_+$. Here, the parameter r is called the *exponential growth rate of mean square semistability*.

Define the output for the random SLS (5) as

$$Y(t) = O^{\rm T}X(t), \tag{7}$$

where O is the projection matrix defined before. For a given probability distribution p, denote by $Y(t; z, p)$ the stochastic output trajectory of the SLS (5) starting from the deterministic initial state $X(0) = z$.

Definition 4.2: The random SLS (5) with the output (7) is *mean square output exponentially stable* if there exist constants $\kappa \geq 0$ and $\widetilde{r} \in [0, 1)$ such that for any $z \in \mathbb{R}^n$, $\mathbb{E}\left[\|Y(t; z, p)\|^2\right] \leq \kappa \widetilde{r}^t \|z\|^2, \forall\, t \in \mathbb{Z}_+$. Here, $\widetilde{r}$ is called the *exponential growth rate of mean square output stability*.

We adopt the same (deterministic) state transformation introduced in Section II. Therefore, $\mathbf{A}(t)$ can be written as $\mathbf{A}(t) = \begin{bmatrix} \mathbf{A}_{11}(t) & 0 \\ \mathbf{A}_{21}(t) & I_{n-\ell} \end{bmatrix}$, where at each t, $\mathbf{A}_{11}(t) \in \mathbb{R}^{\ell\times\ell}$ and $\mathbf{A}_{21}(t) \in \mathbb{R}^{(n-\ell)\times\ell}$ are drawn independently randomly from $\{A_{i,11}\}$ and $\{A_{i,21}\}$, respectively. Further, let $X^{\rm T}(t) = [Y^{\rm T}(t); \widetilde{X}^{\rm T}(t)]$, and $X_{\mathcal{E}_{\rm e}}^{\rm T}(t) = [0; \widetilde{X}^{\rm T}(t)]$. Hence,

$$Y(t + 1) = \mathbf{A}_{11}(t)Y(t), \tag{8}$$

and

$$\widetilde{X}(t) = \widetilde{X}(0) + \sum_{\tau=0}^{t-1} \mathbf{A}_{21}(\tau)Y(\tau). \tag{9}$$

A. Semistability and Output Stability

The following theorem, as a counterpart of Theorem 3.2, asserts the equivalence of mean square exponential semistability and mean square output exponential stability.

Theorem 4.1: The random SLS (5) is mean square exponentially semistable if and only if the SLS (5) with the output (7) is mean square output exponentially stable. Further, a constant $r \in [0, 1)$ is the exponential growth rate of mean square exponential semistability if and only if it is that of mean square output exponential stability.

Proof: "If". Let $\kappa > 0$ and $\widetilde{r} \in [0, 1)$ be such that for any $z \in \mathbb{R}^n$, $\mathbb{E}\left[\|Y(t; z, p)\|^2\right] \leq \kappa \widetilde{r}^t \|z\|^2$ for all $t \in \mathbb{Z}_+$. Since $\mathbf{A}_{11}(t)$ in (8) is identically distributed, the matrix $V := \mathbb{E}[\mathbf{A}_{11}(t)]$ is independent of t. We claim that V is a stable matrix in the discrete-time sense, i.e., the spectral radius of V is strictly less than 1. To see this, note that the mean square exponential stability assumption implies that $\mathbb{E}[\|Y(t; z, p)\|^2]$, hence $\mathbb{E}[Y(t; z, p)]$, converges to zero as $t \to \infty$, for all z. Since $\mathbb{E}[Y(t+1; z, p)] = V \cdot \mathbb{E}[Y(t; z, p)]$, the spectral radius of V must be less than 1.

Given an initial state z and a probability distribution p, let $\widetilde{X}(t; z, p)$ denote the stochastic state trajectory of $\widetilde{X}(t)$ in (9) and define $L := \max_{i\in\mathcal{M}} \|A_{i,21}\|$. For arbitrary $0 \leq s < w$, $\|\widetilde{X}(w; z, p) - \widetilde{X}(s; z, p)\|^2 \leq L^2\|Y(s; z, p) + \cdots + Y(w - 1; z, p)\|^2 \leq L^2\Big(\sum_{k=s}^{w-1} \|Y(k; z, p)\|^2 +$

$2\sum_{s\le j<k\le w-1} Y^{\mathrm{T}}(j;z,p)\, Y(k;z,p)\Big)$. Since $Y(k;z,p) = \mathbf{A}_{11}(k-1)\cdots\mathbf{A}_{11}(j)Y(j;z,p)$ for any $j<k$, we have

$$\begin{aligned}
&\mathbb{E}\left[Y^{\mathrm{T}}(j;z,p)\, Y(k;z,p)\right] \\
&\quad= \mathrm{trace}\big(\mathbb{E}[\mathbf{A}_{11}(k-1)\cdots\mathbf{A}_{11}(j)] \\
&\qquad\quad \mathbb{E}[Y(j;z,p)Y^{\mathrm{T}}(j;z,p)]\big) \\
&\quad= \mathrm{trace}(V^{k-j}\mathbb{E}[Y(j;z,p)Y^{\mathrm{T}}(j;z,p)]) \\
&\quad\le \alpha\nu^{k-j}\,\mathbb{E}[\|Y(j;z,p)\|^2]
\end{aligned}$$

for some constants $\alpha>0$ and $\nu\in[0,1)$ dependent on V and its order ℓ only. Here the last step follows from the fact that V is stable. In view of the mean square exponential stability of $Y(t;z,p)$, we further have

$$\begin{aligned}
&\mathbb{E}\left[\|\widetilde{X}(w;z,p)-\widetilde{X}(s;z,p)\|^2\right] \\
&\le\; L^2\Big(\sum_{k=s}^{w-1}\kappa\widetilde{r}^k\|z\|^2+2\sum_{s\le j<k\le w-1}\alpha\nu^{k-j}\,\kappa\widetilde{r}^j\|z\|^2\Big) \\
&\le\; \kappa L^2\Big(\frac{\widetilde{r}^s}{1-\widetilde{r}}+2\alpha\sum_{s\le j<k\le w-1}\nu^{k-j}\,\widetilde{r}^j\Big)\|z\|^2.
\end{aligned}$$

Using the pattern of the summation above, we have

$$\begin{aligned}
\sum_{s\le j<k\le w-1}\nu^{k-j}\,\widetilde{r}^j &= \widetilde{r}^{s-1}\sum_{i=1}^{w-s}\widetilde{r}^i\Big(\sum_{k=1}^{w-s+1-i}\nu^k\Big) \\
&\le \widetilde{r}^{s-1}\frac{\nu}{1-\nu}\sum_{i=1}^{w-s}\widetilde{r}^i \le \widetilde{r}^s\frac{\nu}{(1-\nu)(1-\widetilde{r})}.
\end{aligned}$$

This implies that

$$\mathbb{E}\left[\|\widetilde{X}(w;z,p)-\widetilde{X}(s;z,p)\|^2\right] \le \widetilde{L}\,\widetilde{r}^s\,\|z\|^2, \tag{10}$$

where $\widetilde{L}>0$ is a constant independent of w,s and z. By letting $s=0$, we conclude that $X(t;z,p)$ has bounded second moment. Hence, $\{\widetilde{X}(t;z,p)\}_{t=0,1,\ldots}$ is a Cauchy sequence of random vectors in the L^2-space with respect to the underlying probability measure. Since the L^2-space is complete, $\widetilde{X}(t;z,p)$ converges in mean square to some random vector $\widetilde{X}_{\mathrm{e}}(z,p)$ (with the finite second moment) as $t\to\infty$. Let $\widetilde{z} := z-\big[0;\mathbb{E}[\widetilde{X}_{\mathrm{e}}^{\mathrm{T}}(z,p)]\big]^{\mathrm{T}}$. Hence, $Y(t;\widetilde{z},p)=Y(t;z,p)$ (in distribution), and this, together with (9), shows that $\widetilde{X}(t;\widetilde{z},p)=\widetilde{X}(t;z,p)-\mathbb{E}\left[\widetilde{X}_{\mathrm{e}}(z,p)\right]$. Further, as $t\to\infty$, $\widetilde{X}(t;\widetilde{z},p)$ converges to the random vector $\widetilde{X}_{\mathrm{e}}(\widetilde{z},p)=\widetilde{X}_{\mathrm{e}}(z,p)-\mathbb{E}[\widetilde{X}_{\mathrm{e}}(z,p)]$ (with zero mean) in mean square. Let $X_{\mathrm{e}}^{\mathrm{T}}(z)=[0;\widetilde{X}_{\mathrm{e}}^{\mathrm{T}}(z)]$ and $X_{\mathrm{e}}^{\mathrm{T}}(\widetilde{z},p)=[0;\widetilde{X}_{\mathrm{e}}^{\mathrm{T}}(\widetilde{z},p)]$. Then $\mathbb{E}\left[\|X(t;z,p)-X_{\mathrm{e}}(z,p)\|^2\right] = \mathbb{E}\left[\|X(t;\widetilde{z},p)-X_{\mathrm{e}}(\widetilde{z},p)\|^2\right] = \mathbb{E}\left[\|Y(t;\widetilde{z},p)\|^2\right] + \mathbb{E}\big[\|\widetilde{X}(t;\widetilde{z},p)-\widetilde{X}_{\mathrm{e}}(\widetilde{z},p)\|^2\big]$. By letting $s=t$, $w=\infty$, and replacing z by $\widetilde{z}$ in (10), we see $\mathbb{E}\left[\|\widetilde{X}(t;\widetilde{z},p)-\widetilde{X}_{\mathrm{e}}(\widetilde{z},p)\|^2\right] \le \widetilde{L}\widetilde{r}^t\|\widetilde{z}\|^2$. Along with the mean square output exponential stability, we obtain $\mathbb{E}\left[\|X(t;z,p)-X_{\mathrm{e}}(z,p)\|^2\right] \le (\kappa+\widetilde{L})\widetilde{r}^t\|\widetilde{z}\|^2 \le (\kappa+\widetilde{L})\widetilde{r}^t\big\|z-\mathbb{E}[X_{\mathrm{e}}(z,p)]\big\|^2$ for all t and z. This yields the mean square exponential semistability and shows that the exponential growth rate $\widetilde{r}$ of the mean square output stability is also that of the mean square semistability.

"Only if". Suppose that the random SLS is mean square exponentially semistable but not mean square output exponentially stable. In light of (8)–(9), we deduce, via the equivalence of mean square asymptotic stability and mean square exponential stability of random jumped linear systems [14, Theorem 4.1.1], that the random SLS (8) is not mean square asymptotically stable, leading to a contradiction via a similar argument of Theorem 3.2.

To complete the proof, we only need to show that the exponential growth rate $r\in[0,1)$ of mean square exponential semistability is that of mean square output exponential stability. By observing the above "only if" part, we deduce that $\mathbb{E}[\|y(t;z,p)\|^2] \le \widetilde{\kappa}\widetilde{r}^t\|z\|^2$ for all t and z, for some constants $\widetilde{\kappa}>0$ and $\widetilde{r}\in[0,1)$ (not necessarily equal to r at this stage). It follows from the above "if" part that $\widetilde{X}(t;z,p)$ converges to $\widetilde{X}_{\mathrm{e}}(z,p)$ in mean square as $t\to\infty$. By (9), this further implies that $\sum_{\tau=0}^{t}\mathbf{A}_{21}(\tau)Y(\tau;z,p)$ also converges in the L^2-space as $t\to\infty$. Consequently, $\Big\|\widetilde{X}(0)-\mathbb{E}\Big[\widetilde{X}_{\mathrm{e}}(z,p)\Big]\Big\|^2 = \Big\|\mathbb{E}\Big[\sum_{\tau=0}^{\infty}\mathbf{A}_{21}(\tau)Y(\tau;z,p)\Big]\Big\|^2 \le \mathbb{E}\Big[\Big\|\sum_{\tau=0}^{\infty}\mathbf{A}_{21}(\tau)Y(\tau;z,p)\Big\|^2\Big] \le \mathbb{E}\Big[\|\widetilde{X}_{\mathrm{e}}(z,p)-\widetilde{X}(0)\|^2\Big] \le \widetilde{L}\|z\|^2$, where the last inequality follows from (10) by letting $s=0$ and $w=\infty$. Finally, recalling that $X_{\mathrm{e}}(z,p)=[0;\widetilde{X}_{\mathrm{e}}^{\mathrm{T}}(z,p)]^{\mathrm{T}}$, we have $\mathbb{E}\left[\|Y(t;z,p)\|^2\right] \le \mathbb{E}\left[\|X(t;z,p)-X_{\mathrm{e}}(z,p)\|^2\right] \le \rho\, r^t\|z-\mathbb{E}\left[X_{\mathrm{e}}(z,p)\right]\|^2 \le \rho\, r^t\Big(\|Y(0)\|^2+\widetilde{L}\|z\|^2\Big) \le \widetilde{\rho}\, r^t\|z\|^2$, $\forall\ t\in\mathbb{Z}_+$ for some constant $\widetilde{\rho}>0$. This leads to the desired exponential growth rate for the mean square output exponential stability. ■

B. Semistability Analysis via Mean Generating Functions

Theorem 4.1 allows us to determine the mean square exponential semistability and its maximal exponential growth rate via the mean generating function of the random SLS (8).

Let $Y(t;v,p)$ denote the stochastic state trajectory of the random SLS (8) starting from the deterministic initial state $v\in\mathbb{R}^\ell$ under the switching probability distribution p. The *mean generating function* $F:\mathbb{R}_+\times\mathbb{R}^\ell\to\mathbb{R}_+\cup\{+\infty\}$ of the random SLS (8) is defined as $F(\lambda,z)=F_\lambda(z) := \mathbb{E}\left[\sum_{t=0}^{\infty}\lambda^t\|Y(t;v,p)\|^2\right]=\sum_{t=0}^{\infty}\lambda^t\mathbb{E}\left[\|Y(t;v,p)\|^2\right]$. The mean generating function F_λ shares the similar properties of the strong generating function G_λ, and a collection of its properties can be found in [11, Proposition 13]. The radius of convergence of F_λ is defined as $\lambda_p^* := \sup\{\lambda\ge 0: F_\lambda(v)<\infty, \forall\ v\in\mathbb{R}^\ell\}$. This quantity can be used to determine the mean square exponential stability as shown below.

Lemma 4.1: [11, Theorem 4] The random SLS (8) is mean square exponentially stable if and only if $\lambda_p^*>1$.

It also follows from a similar argument of [11, Corollary 1] that $(\lambda_p^*)^{-\frac{1}{2}}$ is the maximal exponential growth rate of the random SLS (8). In view of this and Theorem 4.1, we have:

Theorem 4.2: The random SLS (5) is mean square exponentially semistable if and only if the radius of convergence of the random SLS (8) satisfies $\lambda_p^*>1$. Further, the maximal exponential growth rate of the mean square semistability of the SLS (5) is $(\lambda_p^*)^{-1/2}$.

V. Application to Sensor Network Allocation Algorithms

We apply the stability results developed in the preceding sections to the consensus problem of distributed sensor networks under possible topology switching. Specifically, consider a network characterized by a strongly connected directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consisting of a set of nodes $\mathcal{V} = \{1, \ldots, q\}$ and a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ representing the communication links between two nodes, where each edge $(i, j) \in \mathcal{E}$ is an ordered pair of distinct nodes. The set of neighbors of node i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. Let $x_i(t) \in \mathbb{R}$ denote the number of mobile sensors that node i has at time t. Those mobile sensors are used to collect information at node i and travel along the graph network based on some algorithms.

At the initial time, the number of mobile sensors at node i is given by $x_i(0)$. During the t-th time window, suppose node i contacts a neighboring node j to see how many mobile sensors both nodes have. Then at this time, both nodes will relocate their mobile sensors in such a way that the number of mobile sensors at each node is proportional to a value defined by a certain merit function. Obviously node i may have more than just one neighboring node. When the execution command is sent out for action, these mobile sensors start moving together so that more mobile sensors will aggregate at some location of interest. This movement is a coordinated motion in the sense that these mobile sensors will self-organize their collective moves by updating their information based on neighbor-to-neighbor interaction.

Due to possible link failures and link creations of communications between sensors, the sensors may connect or disconnect with each other via wireless communications during the movement. Hence, the communication topology for this mobile sensor network is not fixed, leading to a switching topology. This calls for a distributed, iterative allocation algorithm to efficiently redistribute mobile sensors in a topologically changing graph. By viewing the iteration process of such an algorithm as a discrete-time, switched linear system, the stability results developed before can be used to address the semistability and convergence rate of an algorithm under arbitrary switching and random switching, so that the growth rate of the proposed algorithm can be established on a solid theoretical foundation.

Let $x(0) = [x_1(0), \ldots, x_q(0)]^{\mathrm{T}}$ be the initial vector and $\mathbf{1}^{\mathrm{T}} x(0)$ be the total number of mobile sensors for the network, where $\mathbf{1} := [1, \ldots, 1]^{\mathrm{T}} \in \mathbb{R}^q$. Then an iterative allocation algorithm for updating $x_i(t)$ is given by the form

$$x_i(t+1) = W_{\sigma(t),i,i} x_i(t) + \sum_{j \in \mathcal{N}_i} W_{\sigma(t),i,j} x_j(t), \qquad (11)$$

or, equivalently, in the vector form $x(t+1) = W_{\sigma(t)} x(t), \quad t = 0, 1, 2, \ldots,$ where $\sigma(t) \in \Sigma := \{1, \ldots, m\}$ represents the switching sequence and $W_k \in \mathbb{R}^{q \times q}$, $k \in \Sigma$, are the subsystem dynamics matrices. Here we set $W_{\sigma(t),i,j} = 0$ if $j \notin \mathcal{N}_i$. We assume there is no sensor dropping or adding to the network. Thus, the design aim here is to identify $W_{\sigma(t)}$ and its convergence rate so that (11) exponentially converges to x_{e} as $t \to \infty$, where x_{e} denotes the final distribution pattern of mobile sensors among the sites, which is a function of $x(0)$ ($\mathbf{1}^{\mathrm{T}} x_{\mathrm{e}} = \mathbf{1}^{\mathrm{T}} x(0)$).

A. Applications to Gossip Algorithms

We analyze mean square exponential semistability of random gossip algorithms proposed in [20] using the semistability techniques. Consider an asynchronous randomized gossip algorithm described as follows. Each node has a clock that ticks according to a rate 1 Poisson process. Thus, the random inter-tick times at each node are exponentially distributed, and independent across nodes and over time. We discretize time according to clock ticks since these are the only times at which the value of $X(t)$ changes. In the t-th time slot, let node i's clock tick and let it contact some neighboring node j with probability p_{ij}. At this time, both nodes set their values equal to the average of their current values. Formally, let $X(t)$ denote the vector of values at the end of the time slot t. Then $X(t)$ is updated by the algorithm

$$X(t+1) = \mathbf{W}(t) X(t), \quad t = 0, 1, 2, \ldots, \qquad (12)$$

where $\mathbf{W}(t)$ is a random matrix drawn independently from the set $\{W_{ij}\}$ with $W_{ij} = I_q - (e_i - e_j)(e_i - e_j)^{\mathrm{T}}/2$ and with probability p_{ij}/q (the probability that the i-th node's clock ticks is $1/q$, and the probability that it contacts node j is p_{ij}). Here $e_i \in \mathbb{R}^q$ is a vector with the i-th component equal to 1 and the rest equal to 0. In other words, $\mathbb{P}\{\mathbf{W}(t) = W_{ij}\} = \frac{p_{ij}}{q}, \quad i, j = 1, \ldots, q, \quad i \neq j$.

It is easy to verify that $\mathbf{1}^{\mathrm{T}} W_{ij} = \mathbf{1}^{\mathrm{T}}$ and $W_{ij} \mathbf{1} = \mathbf{1}$ for all i, j such that $\mathbf{1}^{\mathrm{T}} \mathbf{W}(t) = \mathbf{1}^{\mathrm{T}}$, $\forall t$. Further, $\mathcal{E}_{\mathrm{e}} = \mathrm{span}\{\mathbf{1}\}$ and $P = OO^{\mathrm{T}} = I_q - \frac{1}{q} \mathbf{1}\mathbf{1}^{\mathrm{T}}$ for a suitable matrix O. Therefore, for any initial state z, it can be shown via direct calculation that the equilibrium $X_{\mathrm{e}}(z)$ is deterministic and is given by $\mathbf{1}\mathbf{1}^{\mathrm{T}} z/q$ and that $X_{\mathcal{E}_{\mathrm{e}}^{\perp}}(t; z, p) = X(t; z, p) - X_{\mathrm{e}}(z)$. Consequently, the following results hold.

Corollary 5.1: The gossip algorithm (12) is mean square exponentially semistable if and only if the auxiliary system (12) and (7) is mean square output exponentially stable.

Corollary 5.2: The gossip algorithm (12) is mean square exponentially semistable if and only if the radius of convergence for the auxiliary system (12) and (7) satisfies $\lambda_p^* > 1$.

Proof: This follows directly from Theorem 4.2. ■

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