A Study of the Generalized Input-to-State $\mathcal{L}_2$-Gain of Discrete-Time Switched Linear Systems

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***Abstract*— A generalized notion of the input-to-state $\mathcal{L}_2$-gains of discrete-time switched linear systems is proposed in this paper. Such gains are then characterized using the radii of convergence of a family of suitably defined functions called the generating functions. Properties of the generating functions are studied and their numerical computation algorithms are developed. Some numerical examples are presented.**

I. Introduction

A switched linear system is a dynamical system consisting of a number of linear subsystems along with a switching rule that determines the switchings among subsystems. Switched linear systems are an important class of hybrid systems, and their stability has been studied extensively; see the recent surveys [1], [2] and the references therein. Approaches to analyzing stability include multiple Lyapunov functions, LMI methods and Lie algebraic methods.

An important concept in the study of robust control of linear and nonlinear systems is the $\mathcal{L}_2$-gain [3], namely, the maximum output energy that can be excited using a given input/perturbation energy. This concept has a natural extension to switched linear systems, whose (hybrid) control includes both continuous inputs and the switching signal [4]. A common storage function approach was used to bound the $\mathcal{L}_2$-gain of switched systems in [5], [6], while [7] applied the variational approaches. Conditions for characterizing the $\mathcal{L}_2$-gain of switched linear systems under average dwell time constraints were given in [8]. The design of switching signal to achieve a certain $\mathcal{L}_2$-gain and the subsequent stability analysis were presented in [9].

In this paper, we propose a generalized notion of the input-to-state $\mathcal{L}_2$-gain for discrete-time switched linear systems. Compared with the classical definition, the input energy and state energy are weighted by an exponential discount factor. By studying how the induced $\mathcal{L}_2$-gain evolves with the discount factor, not just for the single non-discounted case as in the classical studies, we may gain new insights into the robustness of the switched linear systems.

In our previous work [10], a method based on the notion of *generating functions* is proposed to study the stability of autonomous switched linear systems. These functions are power series with coefficients dependent on the system trajectories, and their radii of convergence characterize the maximum exponential growth rate of the SLS trajectories. In this paper, we will extend this method to controlled switched linear systems by defining the corresponding generating functions and showing that their radii of convergence characterize precisely the generalized $\mathcal{L}_2$-gain under study. The notion of generating functions lends itself particularly well to $\mathcal{L}_2$-gain analysis of discrete-time switched linear systems, a topic which has received less attention compared to its continuous-time counterpart.

The paper is organized as follows. In Section II, controlled switched linear systems are briefly reviewed. The generalized $\mathcal{L}_2$-gain is defined in Section III. In Section IV, the strong generating functions of controlled switched linear systems are defined and some of their properties are derived. We show that their radii of convergence characterize the generalized $\mathcal{L}_2$-gain. An algorithm for computing the strong generating functions is presented in Section V, and a numerical example is presented in Section VI. Finally, some concluding remarks are given in Section VII.

II. Controlled and Autonomous Switched Linear Systems

A discrete-time controlled switched linear system (SLS) has the dynamics

$$x(t+1) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad t = 0, 1, \ldots, \tag{1}$$

that switches among M linear control subsystems ($A_i \in \mathbb{R}^{n\times n}, B_i \in \mathbb{R}^{n\times m}$) indexed by $i \in \mathcal{M} := \{1, \ldots, M\}$. We assume that at least one B_i is nonzero. Here, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, and $\sigma(t) \in \mathcal{M}$ is the mode, all at time t. For simplicity, we often use u to denote the control input sequence $\{u(t)\}_{t=0,1,\ldots}$, and σ the switching sequence $\{\sigma(t)\}_{t=0,1,\ldots}$.

Denote by $x(t;\sigma,z,u)$ the state trajectory of the controlled SLS (1) starting from the initial state $x(0) = z$ under the switching sequence σ and the control input u. For a fixed σ, system (1) becomes a linear time-varying system, whose solution $x(t;\sigma,z,u)$ is jointly linear in z and u.

By setting $u \equiv 0$, the dynamics (1) reduces to that of an autonomous SLS:

$$x(t+1) = A_{\sigma(t)}x(t), \quad t = 0, 1, \ldots. \tag{2}$$

Denote by $x(t;\sigma,z)$ the solution to (2) starting from $x(0) = z$ under the switching sequence σ. Then $x(t;\sigma,z)$ is exactly the solution $x(t;\sigma,z,u)$ to the controlled SLS (1) with $u = 0$.

The reachable set of the SLS (1) is defined as the set of all states that the state trajectory can reach within a finite

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time starting from zero initial state, under arbitrary switching sequences and control inputs:

$$\mathcal{R} := \{x(t;\sigma,0,u) \,|\, t = 0,1,\ldots,\forall\sigma,\ \forall u\}.$$

The reachable set $\mathcal{R}$ for the SLS (1) is in general a countable union of subspaces and not necessarily a subspace in itself [11]. However, in the rest of the paper we assume that $\mathcal{R}$ is a subspace in $\mathbb{R}^n$ and will refer to it as the reachable subspace. Indeed, a randomly generated SLS is completely reachable over $\mathbb{R}^n$ with probability one.

III. Generalized $\mathcal{L}_2$-Gain of Controlled SLSs

A. Definition

The definition of the input-to-state $\mathcal{L}_2$-gain $\kappa \in \mathbb{R}_+ \cup \{\infty\}$ of the discrete-time SLS (1), adapted from the corresponding definition of continuous-time SLSs in [4], is defined by

$$\begin{aligned}\kappa^2 &:= \sup_{\sigma} \sup_{0\neq u\in\mathcal{L}_2} \frac{\sum_{t=0}^{\infty}\|x(t;\sigma,0,u)\|^2}{\sum_{t=0}^{\infty}\|u(t)\|^2} \\ &= \sup_{\sigma} \sup_{0\neq u\in\mathcal{L}_2} \frac{\sum_{t=0}^{\infty}\|x(t+1;\sigma,0,u)\|^2}{\sum_{t=0}^{\infty}\|u(t)\|^2}, \end{aligned} \tag{3}$$

where $\mathcal{L}_2$ is the space of all u with finite $\mathcal{L}_2$-norm. In this paper, we study a generalized version of the $\mathcal{L}_2$-gain, denoted by $\kappa(\lambda) \in \mathbb{R}_+ \cup \{\infty\}$, whose square is given by

$$[\kappa(\lambda)]^2 := \sup_{\sigma} \sup_{0\neq u\in\mathcal{L}_{2,\lambda}} \frac{\sum_{t=0}^{\infty}\lambda^t\|x(t+1;\sigma,0,u)\|^2}{\sum_{t=0}^{\infty}\lambda^t\|u(t)\|^2}, \tag{4}$$

where $\lambda \in \mathbb{R}_+ := [0,\infty)$ is a discount factor, and $\mathcal{L}_{2,\lambda}$ is the space of all u with finite λ-discounted $\mathcal{L}_2$-norm: $\|u\|_{2,\lambda} := \left(\sum_{t=0}^{\infty}\lambda^t\|u(t)\|^2\right)^{1/2}$. The $\mathcal{L}_2$-gain κ defined in (3) is a special case of (4) by setting $\lambda = 1$.

B. Approximations by Truncation

For later analysis and computation, we define finite-horizon versions of the $\mathcal{L}_2$-gain as follows. For each $k \in \mathbb{N}$ where $\mathbb{N}$ is the set of positive integers, denote by $\mathcal{U}_k$ the set of controls u that are identically zero after time $k-1$. Then the k-horizon $\mathcal{L}_2$-gain $\kappa_k(\lambda)$ of system (1) is defined by:

$$[\kappa_k(\lambda)]^2 := \sup_{\sigma} \sup_{0\neq u\in\mathcal{U}_k} \frac{\sum_{t=0}^{k-1}\lambda^t\|x(t+1;\sigma,0,u)\|^2}{\sum_{t=0}^{k-1}\lambda^t\|u(t)\|^2}. \tag{5}$$

Lemma 1: For each $k \in \mathbb{N}$, $\kappa_k(\lambda)$ is a lower semi-continuous function of $\lambda \in \mathbb{R}_+$.

Proof: By (5), $\kappa_k(\lambda)$ is the supremum of a family of continuous functions in $\lambda \in \mathbb{R}_+$ over all σ and u. Thus, $\kappa_k(\lambda)$ itself must be lower semi-continuous in $\lambda \in \mathbb{R}_+$. ■

Proposition 1: For any $\lambda \geq 0$, $\kappa_k(\lambda) \uparrow \kappa(\lambda)$ as $k \to \infty$.

Proof: That $\kappa_k(\lambda) \leq \kappa(\lambda)$ follows as the supremum in (5) is taken over a smaller space $\mathcal{U}_k$ than $\mathcal{L}_{2,\lambda}$ in (4). A similar argument shows that $\kappa_k(\lambda)$ is non-decreasing in k.

As a result, we have $\lim_{k\to\infty}\kappa_k(\lambda) \leq \kappa(\lambda)$. To show the other direction, assume first $\kappa(\lambda)$ is finite. Then for any small $\varepsilon > 0$, a control u and a switching sequence σ exist such that $\sum_{t=0}^{\infty}\lambda^t\|u(t)\|^2 = 1$ and $\sum_{t=0}^{\infty}\lambda^t\|x(t+1;\sigma,0,u)\|^2 \geq [\kappa(\lambda)]^2 - \varepsilon$. By choosing k large enough, we have $\sum_{t=0}^{k-1}\lambda^t\|x(t+1;\sigma,0,u)\|^2 \geq [\kappa(\lambda)]^2 - 2\varepsilon$ while $\sum_{t=0}^{k}\lambda^t\|u(t)\|^2 \leq 1$; thus $[\kappa_k(\lambda)]^2 \geq [\kappa(\lambda)]^2 - 2\varepsilon$. As $\varepsilon > 0$ is arbitrary, this implies that $\lim_{k\to\infty}\kappa_k(\lambda) \geq \kappa(\lambda)$, hence $\lim_{k\to\infty}\kappa_k(\lambda) = \kappa(\lambda)$. The case when $\kappa(\lambda) = \infty$ can be similarly proved. ■

Let $\mathcal{U}_c := \cup_{k=0}^{\infty}\mathcal{U}_k$ be the set of all controls u with finite duration. Note that $\mathcal{U}_c$ is dense in $\mathcal{L}_{2,\lambda}$. Then,

$$[\kappa(\lambda)]^2 = \sup_{\sigma} \sup_{0\neq u\in\mathcal{U}_c} \frac{\sum_{t=0}^{\infty}\lambda^t\|x(t+1;\sigma,0,u)\|^2}{\sum_{t=0}^{\infty}\lambda^t\|u(t)\|^2}, \tag{6}$$

which follows directly from Proposition 1. This will inspire the definition of the generating functions in the next section.

C. Properties

We first derive some basic properties of the generalized $\mathcal{L}_2$-gain. More properties will be derived in the next section.

Proposition 2: The $\mathcal{L}_2$-gain $\kappa(\lambda)$ as a function of $\lambda \in \mathbb{R}_+$ has the following properties:

1) At $\lambda = 0$, $\kappa(0) = \max_{i\in\mathcal{M}}\sigma_{\max}(B_i)$, where $\sigma_{\max}(B_i)$ denotes the largest singular value of B_i;
2) $\kappa(\lambda)$ is a lower semi-continuous function in $\lambda \in \mathbb{R}_+$.

Proof: 1). At $\lambda = 0$, we have

$$\begin{aligned}[\kappa(0)]^2 &= \sup_{\sigma} \sup_{u(0)\neq 0} \frac{\|x(1;\sigma,0,u)\|^2}{\|u(0)\|^2} \\ &= \sup_{i\in\mathcal{M}} \sup_{u(0)\neq 0} \frac{\|B_i u(0)\|^2}{\|u(0)\|^2} = \max_{i\in\mathcal{M}}[\sigma_{\max}(B_i)]^2.\end{aligned}$$

2). This follows from Proposition 1 as $\kappa(\lambda) = \sup_k \kappa_k(\lambda)$ and each $\kappa_k(\lambda)$ is lower semi-continuous in λ. ■

Although not clear yet at this point, we will prove later on that $\kappa(\lambda)$ is a non-decreasing function of λ.

IV. Generating Functions of Controlled Switched Linear Systems

A. Autonomous Generating Function

For each $\lambda \in \mathbb{R}_+$, the strong generating function of the autonomous SLS (2) is the function $G_\lambda : \mathbb{R}^n \to \mathbb{R}_+ \cup \{\infty\}$ defined as [10]:

$$G_\lambda(z) := \sup_{\sigma}\sum_{t=0}^{\infty}\lambda^t\|x(t;\sigma,z)\|^2, \quad \forall z \in \mathbb{R}^n. \tag{7}$$

The radius of convergence of the strong generating function $G_\lambda(z)$ is defined as:

$$\lambda^* := \sup\{\lambda \in \mathbb{R}_+ \,|\, G_\lambda(z) < \infty,\ \forall z \in \mathbb{R}^n\}. \tag{8}$$

In [10], it is shown that the autonomous SLS (2) is exponentially stable under arbitrary switching if and only if its radius of convergence $\lambda^* > 1$.

Let $\mathcal{V}$ be a subspace of $\mathbb{R}^n$ invariant under $\{A_i\}_{i\in\mathcal{M}}$, i.e., $A_i\mathcal{V} \subset \mathcal{V}$ for all $i \in \mathcal{M}$. The radius of convergence of the strong generating function $G_\lambda(z)$ on $\mathcal{V}$ is defined as:

$$\lambda^*_{\mathcal{V}} := \sup\{\lambda \in \mathbb{R}_+ \,|\, G_\lambda(z) < \infty,\ \forall z \in \mathcal{V}\}. \tag{9}$$

Proposition 3: At $\lambda = \lambda^*_{\mathcal{R}}$, where $\lambda^*_{\mathcal{R}}$ is defined in (9) with $\mathcal{V}$ being the reachable subspace $\mathcal{R}$, the $\mathcal{L}_2$-gain $\kappa(\lambda)$ satisfies $\kappa(\lambda^*_{\mathcal{R}}) = \infty$.

Proof: At $\lambda = \lambda^*_{\mathcal{R}}$, by definition (9), there exists $x_* \in \mathcal{R}$ such that $G_{\lambda^*_{\mathcal{R}}}(x_*) = \infty$. As $x_* \in \mathcal{R}$, we can find a control sequence u_* and a switching sequence σ_* that together steer the SLS from $x(0) = 0$ to $x(k) = x_*$ at a finite time k. By setting $u = (u_*, 0, 0, \ldots)$ and $\sigma = (\sigma_*, \sigma')$ for any remaining switching sequence σ' in (6), we obtain

$$[\kappa(\lambda^*_{\mathcal{R}})]^2 \geq (\lambda^*_{\mathcal{R}})^k \frac{\sum_{t=0}^{\infty}(\lambda^*_{\mathcal{R}})^t \|x(t+1;\sigma',x_*,0)\|^2}{\sum_{t=0}^{k-1}(\lambda^*_{\mathcal{R}})^t \|u_*(t)\|^2}.$$

Since the summation on top can be arbitrarily large for properly chosen σ' due to our assumption that $G_{\lambda^*_{\mathcal{R}}}(x_*) = \infty$, we must have $\kappa(\lambda^*_{\mathcal{R}}) = \infty$. ■

B. Controlled Generating Function

For each $\lambda, \gamma \in \mathbb{R}_+$, the strong generating function $G_{\lambda,\gamma} : \mathbb{R}^n \to \mathbb{R}_+ \cup \{\infty\}$ of the SLS (1) is defined as

$$G_{\lambda,\gamma}(z) := \sup_{\sigma, u \in \mathcal{U}_c} \left[\sum_{t=0}^{\infty} \lambda^t \|x(t;\sigma,z,u)\|^2 - \gamma^2 \lambda \sum_{t=0}^{\infty} \lambda^t \|u(t)\|^2 \right] \quad (10)$$

$$= \|z\|^2 + \lambda \cdot \sup_{\sigma, u \in \mathcal{U}_c} \sum_{t=0}^{\infty} \lambda^t \left[\|x(t+1;\sigma,z,u)\|^2 - \gamma^2 \|u(t)\|^2 \right] \quad (11)$$

for $\lambda, \gamma \in \mathbb{R}_+$ and $z \in \mathbb{R}^n$. The signs of the two summations in (10) are chosen so that the u and σ achieving the supremum tend to excite the largest state energy $\sum_{t=0}^{\infty} \lambda^t \|x(t;\sigma,z,u)\|^2$ using the least control energy $\sum_{t=0}^{\infty} \lambda^t \|u(t)\|^2$. Due to the restriction $u \in \mathcal{U}_c$, the power series in (11) is always well defined (with possible value ∞) as its coefficients will be non-negative after a finite time. By choosing $u = 0$ in the supremum, it is easily seen that the function $G_{\lambda,\gamma}(z)$ is non-negative. Moreover, at $\lambda = 0$, $G_{0,\gamma}(z) = \|z\|^2$, $\forall z$.

Inspired by (5), for each $k \in \mathbb{N}$, we define the k-horizon strong generating function as:

$$G_{\lambda,\gamma,k}(z) := \sup_{\sigma, u \in \mathcal{U}_k} \left[\sum_{t=0}^{k} \lambda^t \|x(t;\sigma,z,u)\|^2 - \gamma^2 \lambda \sum_{t=0}^{k-1} \lambda^t \|u(t)\|^2 \right] \quad (12)$$

$$= \|z\|^2 + \lambda \cdot \sup_{\sigma, u \in \mathcal{U}_k} \sum_{t=0}^{k-1} \lambda^t \left[\|x(t+1;\sigma,z,u)\|^2 - \gamma^2 \|u(t)\|^2 \right]. \quad (13)$$

Proposition 4: For each fixed $\lambda, \gamma \in \mathbb{R}_+$ and $z \in \mathbb{R}^n$, $G_{\lambda,\gamma,k}(z) \uparrow G_{\lambda,\gamma}(z)$ as $k \to \infty$.

Proof: For fixed $\lambda, \gamma \in \mathbb{R}_+$ and $z \in \mathbb{R}^n$, it follows directly from the definitions (13) and (11) that $G_{\lambda,\gamma,k}(z)$ is non-decreasing in k and that $G_{\lambda,\gamma,k}(z) \leq G_{\lambda,\gamma}(z)$. Assume that $G_{\lambda,\gamma}(z)$ is finite. For any $\varepsilon > 0$, we can find $u \in \mathcal{U}_k$ for some k large enough and σ such that

$$\sum_{t=0}^{\infty} \lambda^t \|x(t;\sigma,z,u)\|^2 - \gamma^2 \lambda \sum_{t=0}^{k-1} \lambda^t \|u(t)\|^2 > G_{\lambda,\gamma}(z) - \varepsilon.$$

Since the summation $\sum_{t=0}^{\infty} \lambda^t \|x(t;\sigma,z,u)\|^2$ converges, by choosing $k' \geq k$ large enough, we have

$$\sum_{t=0}^{k'} \lambda^t \|x(t;\sigma,z,u)\|^2 - \gamma^2 \lambda \sum_{t=0}^{k-1} \lambda^t \|u(t)\|^2 > G_{\lambda,\gamma}(z) - 2\varepsilon.$$

This implies that $G_{\lambda,\gamma,k'}(z) \geq G_{\lambda,\gamma}(z) - 2\varepsilon$, and thus $G_{\lambda,\gamma,k}(z) \uparrow G_{\lambda,\gamma}(z)$ as $k \to \infty$ as ε is arbitrary. The case when $G_{\lambda,\gamma}(z) = \infty$ can be proved in a similar way. ■

Using Proposition 4, we can prove many properties of the strong generating function $G_{\lambda,\gamma}(z)$ by first establishing them for $G_{\lambda,\gamma,k}(z)$ whose definitions involve only finite summations and then taking the limit $k \to \infty$.

C. Properties of Generating Functions

Proposition 5: For any $\lambda, \gamma \in \mathbb{R}_+$, the strong generating function $G_{\lambda,\gamma}(\cdot)$ and its k-horizon version $G_{\lambda,\gamma,k}(\cdot)$ for any $k \in \mathbb{N}$ have the following properties.

1) (**Homogeneity**): $G_{\lambda,\gamma}(\cdot)$ and $G_{\lambda,\gamma,k}(\cdot)$ are both homogeneous of degree two, i.e., for any nonzero $\alpha \in \mathbb{R}$, $G_{\lambda,\gamma}(\alpha z) = \alpha^2 G_{\lambda,\gamma}(z)$ and $G_{\lambda,\gamma,k}(\alpha z) = \alpha^2 G_{\lambda,\gamma,k}(z)$, $\forall\, z \in \mathbb{R}^n$. Thus, $G_{\lambda,\gamma}(0) \in \{0, \infty\}$.
2) (**Bellman Equation**): For all $z \in \mathbb{R}^n$,

$$G_{\lambda,\gamma,k+1}(z) = \|z\|^2 + \lambda \cdot \sup_{i \in \mathcal{M}, v \in \mathbb{R}^m} \left[-\gamma^2 \|v\|^2 + G_{\lambda,\gamma,k}(A_i z + B_i v) \right],$$
$$G_{\lambda,\gamma}(z) = \|z\|^2 + \lambda \cdot \sup_{i \in \mathcal{M}, v \in \mathbb{R}^m} \left[-\gamma^2 \|v\|^2 + G_{\lambda,\gamma}(A_i z + B_i v) \right]. \quad (14)$$

3) (**Sub-Additivity**): For any $z_1, z_2 \in \mathbb{R}^n$, we have

$$\sqrt{G_{\lambda,\gamma,k}(z_1 + z_2)} \leq \sqrt{G_{\lambda,\gamma,k}(z_1)} + \sqrt{G_{\lambda,\gamma,k}(z_2)},$$
$$\sqrt{G_{\lambda,\gamma}(z_1 + z_2)} \leq \sqrt{G_{\lambda,\gamma}(z_1)} + \sqrt{G_{\lambda,\gamma}(z_2)}.$$

4) (**Convexity**): $\sqrt{G_{\lambda,\gamma}(\cdot)}$ and $\sqrt{G_{\lambda,\gamma,k}(\cdot)}$ are both convex functions on $\mathbb{R}^n$.
5) (**Monotonicity**): For any $z \in \mathbb{R}^n$, $G_{\lambda,\gamma}(z)$ and $G_{\lambda,\gamma,k}(z)$ are non-increasing in $\gamma \in \mathbb{R}_+$ (for fixed λ); and non-decreasing in $\lambda \in \mathbb{R}_+$ (for fixed γ).
6) (**Lower Bound**): $G_{\lambda,\gamma}(z) \geq G_{\lambda}(z)$, $\forall z \in \mathbb{R}^n$, where $G_{\lambda}(z)$ is defined in (7) for the autonomous SLS (2).
7) (**Invariant Subspace**): The subset $\mathcal{G}_{\lambda,\gamma} := \{z \in \mathbb{R}^n \,|\, G_{\lambda,\gamma}(z) < \infty\}$ is a subspace of $\mathbb{R}^n$ invariant under subsystem dynamics $\{(A_i, B_i)\}_{i \in \mathcal{M}}$.
8) (**Quadratic Bound**): If $\lambda, \gamma \in \mathbb{R}_+$ are such that $G_{\lambda,\gamma}(z) < \infty$ for all $z \in \mathbb{R}^n$, then $G_{\lambda,\gamma}(z) \leq g\|z\|^2$, $\forall z \in \mathbb{R}^n$, for some finite constant $g > 0$.

Proof: Let $\lambda, \gamma \in \mathbb{R}_+$ and $k \in \mathbb{N}$ be arbitrary.

1). The homogeneity property follows directly from the observation that $x(t;\sigma,\alpha z,\alpha u) = \alpha \cdot x(t;\sigma,z,u)$, $\forall\, t$.

2). Partition $u \in \mathcal{U}_{k+1}$ as $u = (v, u')$ where $v \in \mathbb{R}^m$ and $u' \in \mathcal{U}_k$, and σ as $\sigma = (i, \sigma')$ where $i \in \mathcal{M}$. Then by (12),

$$\begin{aligned} G_{\lambda,\gamma,k+1}(z) &= \sup_{i,v} \Bigg\{ \|z\|^2 - \gamma^2\lambda\|v\|^2 + \lambda \cdot \sup_{\sigma',u'} \\ &\quad \Bigg[\sum_{t=0}^{k} \lambda^t \|x(t;\sigma', A_i z + B_i v, u')\|^2 - \gamma^2\lambda \sum_{t=0}^{k-1} \lambda^t \|u'(t)\|^2 \Bigg] \Bigg\} \\ &= \|z\|^2 + \sup_{i,v} \left[-\gamma^2\lambda\|v\|^2 + \lambda\, G_{\lambda,\gamma,k}(A_i z + B_i v) \right]. \end{aligned}$$

The Bellman equation for $G_{\lambda,\gamma}(z)$ can be proved similarly.

3). For each fixed σ, we write the term inside the bracket of the definition (12) as an explicit quadratic function as

$$\begin{bmatrix} z \\ \mathbf{u} \end{bmatrix}^T \begin{bmatrix} Q_{zz} & Q_{zu} \\ Q_{uz} & Q_{uu} \end{bmatrix} \begin{bmatrix} z \\ \mathbf{u} \end{bmatrix},$$

where $\mathbf{u} := \begin{bmatrix} u(0)^T & \cdots & u(k-1)^T \end{bmatrix}^T \in \mathbb{R}^{nk}$, for some matrices $Q_{zz} = Q_{zz}^T \in \mathbb{R}^{n\times n}$, $Q_{zu} = Q_{uz}^T \in \mathbb{R}^{n\times nk}$, and $Q_{uu} = Q_{uu}^T \in \mathbb{R}^{nk\times nk}$. Note that $Q_{zz} \succeq I$ is necessarily positive definite; while Q_{uu} may be indefinite. Define

$$f_\sigma(z) := \sup_{\mathbf{u}} \begin{bmatrix} z \\ \mathbf{u} \end{bmatrix}^T \begin{bmatrix} Q_{zz} & Q_{zu} \\ Q_{uz} & Q_{uu} \end{bmatrix} \begin{bmatrix} z \\ \mathbf{u} \end{bmatrix}.$$

Then $G_{\lambda,\gamma,k}(z) = \sup_\sigma f_\sigma(z)$. To prove the sub-additivity of $\sqrt{G_{\lambda,\gamma,k}(\cdot)}$, it suffices to show that, for each σ,

$$\sqrt{f_\sigma(z_1 + z_2)} \le \sqrt{f_\sigma(z_1)} + \sqrt{f_\sigma(z_2)}, \ \forall z_1, z_2. \tag{15}$$

We differentiate three cases.

Case 1: If $Q_{uu} \prec 0$ is negative definite, then the $\mathbf{u}$ achieving the supremum in the definition of $f_\sigma(z)$ above is given by $\mathbf{u} = -Q_{uu}^{-1}Q_{uz}z$; hence $f_\sigma(z) = z^T(Q_{zz} - Q_{zu}Q_{uu}^{-1}Q_{uz})z$, where $Q_{zz} - Q_{zu}Q_{uu}^{-1}Q_{uz} \succ 0$ as $Q_{zz} \succ 0$ and $Q_{uu} \prec 0$. Thus, $\sqrt{f_\sigma(z)}$ defines a norm on $\mathbb{R}^n$, and (15) holds.

Case 2: If Q_{uu} has at least one positive eigenvalue, then $f_\sigma(z) = \infty$, $\forall z \in \mathbb{R}^n$, and (15) is trivially true.

Case 3: If $Q_{uu} \preceq 0$ has its largest eigenvalue at exactly 0, then we let $\mathcal{N}(Q_{uu}) \neq 0$ be its null space and $\mathcal{R}(Q_{uu})$ be its range space. If z belongs to the subspace $Q_{uz}^{-1}[\mathcal{R}(Q_{uu})]$, then $Q_{uz}z = Q_{uu}\mathbf{u}_0$ for some $\mathbf{u}_0 \in \mathbb{R}^{nk}$. Therefore,

$$\begin{aligned} f_\sigma(z) &= \sup_{\mathbf{u}\in\mathbb{R}^{nk}} \left(z^T Q_{zz} z + 2\mathbf{u}^T Q_{uz} z + \mathbf{u}^T Q_{uu}\mathbf{u} \right) \\ &= \sup_{\mathbf{u}\in\mathbb{R}^{nk}} \left(z^T Q_{zz} z + 2\mathbf{u}^T Q_{uu}\mathbf{u_0} + \mathbf{u}^T Q_{uu}\mathbf{u} \right) \\ &= \sup_{\mathbf{u}\in\mathbb{R}^{nk}} \Big[z^T Q_{zz} z - \mathbf{u}_0^T Q_{uu}\mathbf{u}_0 \\ &\qquad + (\mathbf{u} + \mathbf{u}_0)^T Q_{uu}(\mathbf{u} + \mathbf{u}_0) \Big] \\ &= z^T Q_{zz} z - \mathbf{u}_0^T Q_{uu}\mathbf{u}_0 \\ &= z^T Q_{zz} z - \mathbf{u}_0^T Q_{uu}^T Q_{uu}^\dagger Q_{uu}\mathbf{u}_0 \\ &= z^T (Q_{zz} - Q_{zu}Q_{uu}^\dagger Q_{uz}) z < \infty, \end{aligned}$$

where $Q_{uu}^\dagger$ denotes the Moore-Penrose pseudo inverse of Q_{uu}. Note that $Q_{zz} - Q_{zu}Q_{uu}^\dagger Q_{uz} \succ 0$ as $Q_{zz} \succ 0$ and $Q_{uu}^\dagger \preceq 0$. On the other hand, if $z \notin Q_{uz}^{-1}[\mathcal{R}(Q_{uu})]$, then $Q_{uz}z \notin \mathcal{R}(Q_{uu}) = \mathcal{N}(Q_{uu})^\perp$; hence we can find $\mathbf{u}_1 \in \mathcal{N}(Q_{uu})$ such that $\mathbf{u}_1^T Q_{uz} z > 0$. By choosing $\mathbf{u} = \alpha\mathbf{u}_1$ for arbitrarily large $\alpha > 0$, we obtain that $f_\sigma(z) = \infty$. To sum up, $\sqrt{f_\sigma(z)}$ defines a norm on the subspace $Q_{uz}^{-1}[\mathcal{R}(Q_{uu})]$, and is infinite everywhere else. As a result, (15) holds on this subspace, and is trivially true if one or both of z_1 and z_2 is outside.

The above proves the sub-additivity of $\sqrt{G_{\lambda,\gamma,k}(\cdot)}$. Sub-additivity of $\sqrt{G_{\lambda,\gamma}(\cdot)}$ is proved by letting $k \to \infty$.

4). For any $z_1, z_2 \in \mathbb{R}^n$ and $\alpha_1, \alpha_2 \ge 0$ with $\alpha_1 + \alpha_2 = 1$, by the sub-additivity and homogeneity properties,

$$\begin{aligned} \sqrt{G_{\lambda,\gamma}(\alpha_1 z_1 + \alpha_2 z_2)} &\le \sqrt{G_{\lambda,\gamma}(\alpha_1 z_1)} + \sqrt{G_{\lambda,\gamma}(\alpha_2 z_2)} \\ &= \alpha_1\sqrt{G_{\lambda,\gamma}(z_1)} + \alpha_2\sqrt{G_{\lambda,\gamma}(z_2)}. \end{aligned}$$

This shows that $\sqrt{G_{\lambda,\gamma}(\cdot)}$ is convex on $\mathbb{R}^n$. The convexity of $\sqrt{G_{\lambda,\gamma,k}(\cdot)}$ can be proved in the same way.

5). By Proposition 4, we need only to prove the monotonicity property for $G_{\lambda,\gamma,k}(z)$. That $G_{\lambda,\gamma,k}(z)$ is non-increasing in γ for fixed z and λ is obvious from its definition (12). We next prove by induction that $G_{\lambda,\gamma,k}(z)$ is non-decreasing in λ for fixed z and γ. At $k = 1$, $G_{\lambda,\gamma,1}(z) = \|z\|^2 + \lambda\cdot\sup_{i\in\mathcal{M},v\in\mathbb{R}^n}(\|A_i z + B_i v\|^2 - \gamma^2\|v\|^2)$ is obviously non-decreasing in λ as the supremum term is non-negative. Suppose for some $k \ge 1$, $G_{\lambda,\gamma,k}(z)$ is non-decreasing in λ for any fixed z and γ. Then by the Bellman equation, for any $z \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}_+$ and any $\lambda > \lambda' \ge 0$,

$$\begin{aligned} &G_{\lambda,\gamma,k+1}(z) \\ &= \|z\|^2 + \lambda \sup_{i\in\mathcal{M},v\in\mathbb{R}^m} \left[-\gamma^2\|v\|^2 + G_{\lambda,\gamma,k}(A_i z + B_i v) \right] \\ &\ge \|z\|^2 + \lambda' \sup_{i\in\mathcal{M},v\in\mathbb{R}^m} \left[-\gamma^2\|v\|^2 + G_{\lambda,\gamma,k}(A_i z + B_i v) \right] \\ &\ge \|z\|^2 + \lambda' \sup_{i\in\mathcal{M},v\in\mathbb{R}^m} \left[-\gamma^2\|v\|^2 + G_{\lambda',\gamma,k}(A_i z + B_i v) \right] \\ &= G_{\lambda',\gamma,k+1}(z). \end{aligned}$$

Here the first inequality follows as the supremum term is non-negative; while the second inequality is due to the induction hypothesis. This proves the desired conclusion.

6). By setting $u = 0$ in (10) we arrive at the definition (7).

7). That $\mathcal{G}_{\lambda,\gamma}$ is a subspace follows from the sub-additivity property of $\sqrt{G_{\lambda,\gamma}(\cdot)}$. Its invariance to subsystem dynamics follows from the Bellman equation of $G_{\lambda,\gamma}(\cdot)$.

8). Write $z = \sum_{i=1}^n \alpha_i e_i$ in an orthonormal basis $\{e_i\}$ of $\mathbb{R}^n$. Then by sub-additivity, $G_{\lambda,\gamma}(z) \le \left[\sum_{i=1}^n |\alpha_i|\sqrt{G_{\lambda,\gamma}(e_i)}\right]^2 \le g \cdot \sum_{i=1}^n \alpha_i^2 = g\|z\|^2$, where $g := n \cdot \max\{G_{\lambda,\gamma}(e_i) \,|\, i = 1, \ldots, n\}$. ■

Remark 1: The set $\{z \in \mathbb{R}^n | G_{\lambda,\gamma,k}(z) < \infty\}$ is still a subspace of $\mathbb{R}^n$ for $k \in \mathbb{N}$. Unlike $\mathcal{G}_{\lambda,\gamma}$ in Proposition 5, it is not necessarily invariant under subsystem dynamics.

D. Radius of Convergence

By Proposition 5, the generating function $G_{\lambda,\gamma}(z)$ is non-decreasing in λ. For $\gamma \in \mathbb{R}_+$, define the *radius of convergence* of the generating function $G_{\lambda,\gamma}(z)$ (on $\mathbb{R}^n$) as

$$\lambda^*(\gamma) := \sup\{\lambda \,|\, G_{\lambda,\gamma}(z) < \infty, \ \forall z \in \mathbb{R}^n\}.$$

More generally, let $\mathcal{V}$ be a subspace of $\mathbb{R}^n$ invariant under subsystem dynamics $\{(A_i, B_i)\}_{i\in\mathcal{M}}$. Then the radius of convergence of $G_{\lambda,\gamma}(z)$ on $\mathcal{V}$ is defined as

$$\lambda^*_{\mathcal{V}}(\gamma) := \sup\{\lambda \,|\, G_{\lambda,\gamma}(z) < \infty,\ \forall z \in \mathcal{V}\},$$

which can also be thought of as the radius of convergence of the generating function of the restricted SLS on $\mathcal{V}$. Note that $\lambda^*_{\mathcal{V}}(\gamma)$ depends on γ and $\mathcal{V}$.

Lemma 2: For any $\lambda, \gamma \in \mathbb{R}_+$, $G_{\lambda,\gamma}(0) < \infty$ if and only if $G_{\lambda,\gamma}(z) < \infty$ for all $z \in \mathcal{R}$. As a result,

$$\lambda^*_{\mathcal{R}}(\gamma) = \sup\{\lambda \,|\, G_{\lambda,\gamma}(0) < \infty\}, \quad \forall \gamma \in \mathbb{R}_+.$$

Proof: The sufficient part is trivial. For the necessary part, assume $G_{\lambda,\gamma}(0) < \infty$. By applying the Bellman equation (14) for $z = 0$, we have $G_{\lambda,\gamma}(B_i v) < \infty$ for all $i \in \mathcal{M}$ and $v \in \mathbb{R}^m$, i.e., $G_{\lambda,\gamma}(z) < \infty$ for all z reachable by the SLS (1) in one time step starting from 0. Similarly, by applying the Bellman equation for all such $z = B_i v$, we conclude that $G_{\lambda,\gamma}(z) < \infty$ for all z reachable by the SLS (1) in two time steps. By induction, we have $G_{\lambda,\gamma}(z) < \infty$ for all $z \in \mathcal{R}$. ■

Proposition 6: The radius of convergence $\lambda^*_{\mathcal{R}}(\gamma)$ of the generating function on the reachable subspace $\mathcal{R}$ as a function of $\gamma \in \mathbb{R}_+$ has the following properties.

1) $\lambda^*_{\mathcal{R}}(\gamma) \equiv 0$ for $0 \le \gamma < \max_{i\in\mathcal{M}} \sigma_{\max}(B_i)$;
2) $\lambda^*_{\mathcal{R}}(\gamma)$ is a non-decreasing function of γ for $\gamma \ge \max_{i\in\mathcal{M}} \sigma_{\max}(B_i)$;

Proof: 1. Fix an arbitrary $\lambda > 0$. By Proposition 4 and (13), $G_{\lambda,\gamma}(z)$ at $z = 0$ has the lower bound:

$$G_{\lambda,\gamma}(0) \ge G_{\lambda,\gamma,1}(0) = \lambda \cdot \sup_{i\in\mathcal{M}, v\in\mathbb{R}^n} \left(\|B_i v\|^2 - \gamma^2 \|v\|^2\right).$$

If $0 \le \gamma < \max_{i\in\mathcal{M}} \sigma_{\max}(B_i)$, then $\|B_i v\|^2 - \gamma^2\|v\|^2$ can be made arbitrarily large by letting v move towards infinity along a singular vector direction of the B_i with the largest singular value. This implies that $G_{\lambda,\gamma}(0) = \infty$, hence $\lambda^*_{\mathcal{R}}(\gamma) \le \lambda$. Since $\lambda > 0$ is arbitrary, we have $\lambda^*_{\mathcal{R}}(\gamma) = 0$.

2. The property follows directly from the non-increasing nature of $G_{\lambda,\gamma}(z)$ in γ. ■

We next show how the $\mathcal{L}_2$-gain can be characterized by the radius of convergence of $G_{\lambda,\gamma}(z)$ on $\mathcal{R}$.

Theorem 1: (**$\mathcal{L}_2$-gain characterization**) For $\lambda > 0$ and $\gamma \in \mathbb{R}_+$, the following statements are equivalent:

1) $\kappa(\lambda) \le \gamma$, where $\kappa(\lambda)$ is the generalized $\mathcal{L}_2$-gain defined in (4);
2) $G_{\lambda,\gamma}(0) = 0$, where $G_{\lambda,\gamma}(\cdot)$ is the generating function of the SLS (1);
3) $\lambda \le \lambda^*_{\mathcal{R}}(\gamma)$, where $\mathcal{R}$ is the reachable subspace of the SLS (1) from the origin.

Proof: $1 \Leftrightarrow 2$: By definition (10), the condition that $G_{\lambda,\gamma}(0) = 0$ is equivalent to

$$\sum_{t=0}^{\infty} \lambda^t \|x(t+1;\sigma,0,u)\|^2 \le \gamma^2 \sum_{t=0}^{\infty} \lambda^t \|u(t)\|^2,$$

for any $u \in \mathcal{U}_c$ and any σ. By (6), this in turn is equivalent to $[\kappa(\lambda)]^2 \le \gamma^2$, i.e., $\kappa(\lambda) \le \gamma$.

$2 \Leftrightarrow 3$: This follows directly from Lemma 2. ■

In the above theorem, the equivalence of 1) and 3) implies the following.

Theorem 2: The two functions $\kappa(\lambda)$ for $\lambda \in (0, \lambda^*_{\mathcal{R}})$ and $\lambda^*_{\mathcal{R}}(\gamma)$ for $\gamma \in \mathbb{R}_+$ are generalized inverse functions of each other:

$$\kappa(\lambda) = \inf\{\gamma \in \mathbb{R}_+ : \lambda^*_{\mathcal{R}}(\gamma) \ge \lambda\},$$
$$\lambda^*_{\mathcal{R}}(\gamma) = \sup\{\lambda > 0 : \kappa(\lambda) \le \gamma\}.$$

Proof: By Theorem 1, we have, for any $\lambda > 0$,

$$\inf\{\gamma \in \mathbb{R}_+ : \lambda^*_{\mathcal{R}}(\gamma) \ge \lambda\} = \inf\{\gamma \in \mathbb{R}_+ : \kappa(\lambda) \le \gamma\},$$

which is exactly $\kappa(\lambda)$. The second equality can be proved similarly. Note that when $0 \le \gamma < \max_{i\in\mathcal{M}} \sigma_{\max}(B_i)$, the set $\{\lambda > 0 : \kappa(\lambda) \le \gamma\}$ is empty; thus the second equality also holds as long as the supremum of an empty subset of $\mathbb{R}_+$ is understood to be lower boundary 0. ■

As a result of Theorem 2 and the fact that $\lambda^*_{\mathcal{R}}(\gamma)$ is non-decreasing in γ, the following result follows immediately.

Corollary 1: The generalized $\mathcal{L}_2$-gain $\kappa(\lambda)$ is a non-decreasing function of $\lambda \in \mathbb{R}_+$.

V. Algorithm for Computing $G_{\lambda,\gamma}(z)$

We next present an algorithm for computing the finite strong generating functions. By Proposition 4, the value of $G_{\lambda,\gamma}(z)$ can be estimated by computing $G_{\lambda,\gamma,k}(z)$ for increasingly larger k. The Bellman's equation in Proposition 5 can be used to compute $G_{\lambda,\gamma,k}(z)$ recursively, provided we have a suitable representation of the values of $G_{\lambda,\gamma,k}(z)$ in $\mathbb{R}^n$. One possible representation is by its values on a fine grid of the unit sphere. By homogeneity, it is then possible to approximate the values of the generating function at other points in $\mathbb{R}^n$ through scaling and interpolation. Specifically, in each iteration of the Bellman equation, a line search needs to be performed to optimize $G_{\lambda,\gamma,k}(A_i z + B_i v) - \gamma^2\|v\|^2$ with respect to v. The unknown value of $G_{\lambda,\gamma,k}(A_i z + B_i v)$ is obtained through extrapolation.

The above idea is summarized in Algorithm 1. The computational complexity of the algorithm increases linearly with the number of subsystems present but exponentially with the state-space dimension. The algorithm is initialized with $G_{\lambda,\gamma,0} = \|z\|^2$. The output of the algorithm is a sequence of functions $\hat{G}_{\lambda,\gamma,k}(z)$ which represent approximations of $G_{\lambda,\gamma,k}(z)$, for z on a grid of the unit sphere.

This algorithm performs well for small values of γ. For large γ, however, the error in finding the optimal v is magnified leading to inaccurate approximations. Thus, the algorithm encounters more difficulty in computing the strong generating function when λ is close to the radius of convergence as this entails a large value of γ for characterizing the finiteness of the generating function.

VI. Numerical Example

We consider the SLS with the following subsystems.

$$A_1 = \begin{bmatrix} \frac{1}{2} & \frac{2}{5} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}; \qquad A_2 = \begin{bmatrix} \frac{3}{5} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix};$$
$$A_3 = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}, B_3 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}; \qquad A_4 = \begin{bmatrix} \frac{1}{6} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}, B_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Algorithm 1 Computing $G_{\lambda,\gamma,k}(z)$ on a grid on $\mathbb{S}$.

Let $\mathcal{S} = \{z_j\}_{j=1}^N$ be a set of grid points of $\mathbb{S}$;
Initialize $k := 0$, $\hat{G}_{\lambda,\gamma,0}(z_j) = 1, \forall z_j \in \mathcal{S}$;
repeat
  $k := k + 1$;
  for each $z_j \in \mathcal{S}$ **do**
    for each $i \in \mathcal{M}$ **do**
      Find v_l, K_l such that $A_i z_j + B_i v_l = K_l z_l$ for all $\{z_l\} \in \mathcal{S}$;
      Find $g_i = \max_{v_l} K_l^2 \hat{G}_{\lambda,\gamma,k-1}(z_l) - \gamma^2 \|v_l\|^2$;
    end for
    Set $\hat{G}_{\lambda,\gamma,k}(z) = 1 + \lambda \max_{i\in\mathcal{M}} g_i$;
  end for
until $\hat{G}_{\lambda,\gamma,k}(z_j)$ converges for all z_j within tolerance (or appears to diverge);
return $\hat{G}_{\lambda,\gamma,k}$.

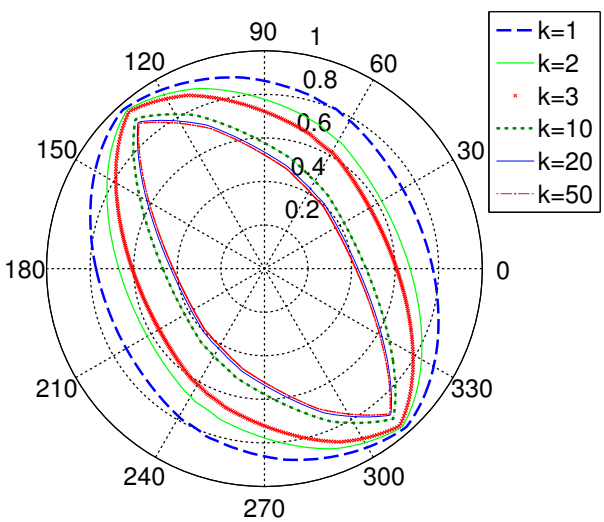

Fig. 1: Level curves $G_{\lambda,\gamma,k}(\cdot) = 1$ on the unit circle for $\lambda = 1.1$, $\gamma = 8$ with k varying .

Algorithm 1 is used to compute the generating function $G_{\lambda,\gamma,k}(\cdot)$ for $\lambda = 1.1$ and $\gamma = 8$ on 500 evenly distributed grid points on the unit circle. By homogeneity, this will yield estimates of $G_{\lambda,\gamma,k}(z)$ for arbitrary $z \in \mathbb{R}^2$. Fig. 1 depicts the level curves of $G_{\lambda,\gamma,k}(\cdot) = 1$ at various k. Convergence of $G_{\lambda,\gamma,k}(\cdot)$ as $k \to \infty$ is observed. By Proposition 4, we conclude that the strong generating function $G_{\lambda,\gamma}(\cdot)$ is finite everywhere for $\lambda = 1.1$ and $\gamma = 8$.

By repeating the above process for different values of λ and γ, we can estimate the region on the (λ, γ)-plane where the strong generating function is finite everywhere. See the shaded region in Fig. 2 for such a plot. According to Theorem 2, the boundary curve of the shaded region is exactly the graph of the radius of convergence $\lambda^*(\gamma)$ as a function of γ, or after a reflection, the graph of the generalized $\mathcal{L}_2$-gain $\kappa(\lambda)$ as a function of $\lambda \in \mathbb{R}_+$.

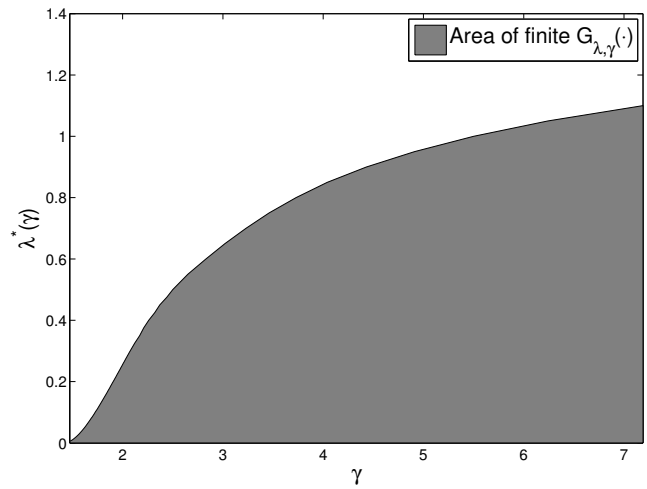

Fig. 2: Plot of $\lambda^*(\gamma)$ vs γ.

VII. Conclusion

We show that the a family of properly defined functions, the strong generating functions, provide an alternative and effective way to study the generalized $\mathcal{L}_2$-gain of discrete-time switched linear systems. Numerical algorithms can be developed to compute the generating functions, hence the generalized $\mathcal{L}_2$-gain as well.

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