Stability of Switched Linear Systems on Cones: A Generating Function Approach

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***Abstract*— This paper extends the recent study of the generating function approach to stability analysis of switched linear systems from the Euclidean space to a closed convex cone. Examples of the latter class of switched systems include switched positive systems that model various biologic and economic systems with positive states. Strong and weak stability notions are considered in this paper. In particular, it is shown that asymptotic and exponential stability are equivalent for both notions. Strong and weak generating functions on cones are introduced and their properties are established. Necessary and sufficient conditions for strong/weak exponential stability of switched linear systems on cones are obtained in terms of the radii of convergence of strong/weak generating functions.**

I. Introduction

There has been a surging interest in switched and hybrid systems and their applications across a number of fields, such as engineering, robotics, and systems biology. A fundamental issue in the analysis and design of switched dynamical systems are their stability [11], [14], [15]. Numerous techniques have been proposed for the stability analysis, e.g., the Lie-algebraic approach [10] and the Lyapunov framework [3], [7]. In the vast literature on switched systems, switched linear systems have received particular attention due to their relatively simple structure and yet rich dynamical behaviors.

Recently introduced in [8], the generating functions have been proven to be an efficient and unified tool for studying the exponential stability of discrete-time switched linear systems. Roughly speaking, generating functions are suitably defined power series with coefficients determined from the systems trajectories under certain switching policies. Their radii of convergence characterize the exponential growth rates of the system trajectories. Therefore, the exponential stability of a switched linear system can be completely described in terms of the radii of convergence of its generating functions. Furthermore, generating functions are closely related to the value functions of properly defined optimal control problems and admit efficient numerical computation. This allows one to develop effective algorithms to determine the exponential growth rates, and in turn the exponential stability, under different switching policies, e.g., arbitrary switching, optimal switching, or random switching. The generating function approach can also be extended to handle state-dependent switchings [13].

Most literature on switched systems concentrates on those on the Euclidean space. However, a variety of applied systems have their states confined within certain regions. A prominent example is positive systems [5] that model a wide range of industrial, biological, economic, and social systems. Stability of switched positive systems and their extension, i.e., switched systems over cones, has also received increasing attention due to applications in such areas as communication and multi-agent systems; certain stability tools have been studied, e.g. the common Lyapunov function approach [2], [4], [6], [12]. In this paper, we carry out the stability analysis for switched linear systems on closed convex cones by using the generating function approach. Specifically, we consider the strong and weak stability notions on a cone, and show that for both notions, the asymptotic and exponential stability are equivalent. Analytic properties of strong and weak generating functions on a cone, as well as their stability implications and numerical approximations, are established.

The paper is organized as follows. In Section II, switched linear systems on cones are introduced and their stability notions are defined. In Section III, the equivalence of the asymptotic and exponential stability is proven. Sections IV and V treat the strong and the weak generating functions of switched linear systems on closed convex cones, respectively.

II. Stability of Switched Linear Systems on Cones

The dynamics of a discrete-time autonomous switched linear system (SLS) is given by

$$x(t+1) = A_{\sigma(t)} x(t), \quad t = 0, 1, \dots. \tag{1}$$

Here $x(t) \in \mathbb{R}^n$ is the state; $\{A_1, \dots, A_m\}$ is a set of subsystem dynamics matrices; and $\sigma(t) \in \mathcal{M} := \{1, \dots, m\}$ for all t, or simply σ, is the switching sequence. Given the initial state $x(0) = z$, the trajectory of the SLS under the switching sequence σ is denoted by $x(t; z, \sigma)$.

Let $\mathcal{C}$ be a closed convex cone. Throughout this paper, we assume that the SLS (1) is positively invariant with respect to $\mathcal{C}$, namely, each subsystem defined by A_i satisfies $A_i z \in \mathcal{C}$ whenever $z \in \mathcal{C}$. This assumption ensures that a trajectory $x(t; z, \sigma)$ starting from $z \in \mathcal{C}$ will remain in $\mathcal{C}$ at all subsequent times regardless of the switching sequence σ. Because of the positive invariance assumption, the restriction of the SLS (1) on the cone $\mathcal{C}$ is a valid dynamical system, which we refer to as the SLS (1) on $\mathcal{C}$. A particular example of SLSs on cones is *switched positive systems*, where $\mathcal{C} =$

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$\mathbb{R}^n_+$ is the nonnegative orthant of $\mathbb{R}^n$, and A_i, $i \in \mathcal{M}$, are all positive matrices.

A. Stability Notions of SLSs on Cones

The stability of the SLS (1) on the cone $\mathcal{C}$ can be defined as follows.

Definition 1: The SLS (1) on the cone $\mathcal{C}$ is called

- *exponentially stable under arbitrary switching* (with the parameters κ and r) if there exist $\kappa \geq 0$ and $r \in [0, 1)$ such that starting from any initial state $z \in \mathcal{C}$ and under any switching sequence σ, the trajectory $x(t; z, \sigma)$ satisfies $\|x(t; z, \sigma)\| \leq \kappa r^t \|z\|$ for all $t \in \mathbb{Z}_+$;
- *exponentially stable under optimal switching* (with the parameters κ and r) if there exist $\kappa \geq 1$ and $r \in [0, 1)$ such that starting from any initial state $z \in \mathcal{C}$, there exists a switching sequence σ for which $x(t; z, \sigma)$ satisfies $\|x(t; z, \sigma)\| \leq \kappa r^t \|z\|$, for all $t \in \mathbb{Z}_+$.

Similar to linear systems, we can define the notions of stability (in the sense of Lyapunov) and asymptotic stability for the SLS on $\mathcal{C}$ under both arbitrary and optimal switchings. Due to the homogeneity of the SLS, the local and global stability notions are equivalent. For simplicity, we also refer to the stability under arbitrary switching as *strong* stability and stability under optimal switching as *weak* stability.

In Definition 1, by replacing the cone $\mathcal{C}$ with $\mathbb{R}^n$, we obtain the corresponding notions of strong and weak exponential stability for the original SLS (1) (on $\mathbb{R}^n$). It is easily seen that stability of the SLS on $\mathbb{R}^n$ in any particular sense (such as strongly or weakly exponentially stable) implies that on $\mathcal{C}$ but not vice versa. As a result, the stability study for SLSs on cones poses new challenges beyond that for SLSs on $\mathbb{R}^n$.

Before ending this section, we briefly review some basic notions of cones [1]. A cone $\mathcal{C}$ is called *pointed* if the condition that $x_1 + \cdots + x_k = 0$ with $x_i \in \mathcal{C}$, $i = 1, \ldots, k$, implies that $x_i = 0$ for all i. A convex cone $\mathcal{C}$ is pointed if and only if $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$, or equivalently, if $\mathcal{C}$ does not contain a nontrivial subspace. For example, $\mathbb{R}^n_+$ is pointed but the half space $\{x \in \mathbb{R}^n \mid x_1 \geq 0\}$ is not. A convex cone $\mathcal{C}$ can always be decomposed as $\mathcal{C} = \mathcal{K} + \mathcal{V}$, where $\mathcal{K}$ is a pointed cone and $\mathcal{V} = \mathcal{C} \cap (-\mathcal{C})$ is a subspace (called the linearity space of $\mathcal{C}$) orthogonal to $\mathcal{K}$: $\mathcal{K} \perp \mathcal{V}$. A cone $\mathcal{C}$ is *solid* if it has nonempty interior. For example, $\mathbb{R}^n_+$ is solid and hence *proper* (i.e. closed, convex, solid, and pointed.)

III. Asymptotic and Exponential Stability of SLSs on Cones

It is well known that the notions of asymptotic stability and exponential stability are equivalent for linear systems. In this section, we will extend this result to SLSs on cones, first for strong stability and then for weak stability. The previous result in [13] on the strong stability of SLSs on $\mathbb{R}^n$ then becomes a special case of our proof.

A. Equivalence of Strong Asymptotic and Exponential Stability for SLSs on Cones

To show the equivalence of asymptotic and exponential stability for the SLS (1) on the cone $\mathcal{C}$ under arbitrary switching, we consider a more general setting: conewise linear inclusions (CLIs) on the cone $\mathcal{C}$. SLSs on $\mathcal{C}$ are special instances of CLIs on $\mathcal{C}$.

Let $\Xi := \{\mathcal{X}_i\}_{i=1}^{\ell}$ be a finite family of nonempty closed cones whose union is $\mathcal{C}$, namely, $\cup_{i=1}^{\ell} \mathcal{X}_i = \mathcal{C}$. Each $\mathcal{X}_i$ is neither necessarily polyhedral nor convex, and two cones in Ξ may overlap. Associated with each cone $\mathcal{X}_i$ is a linear dynamics $x \mapsto A_i x$ if $x \in \mathcal{X}_i$, for some matrix $A_i \in \mathbb{R}^{n\times n}$ positively invariant on $\mathcal{C}$. The conewise linear inclusion on $\mathcal{C}$ is the dynamical system defined by:

$$x(t+1) \in f(x(t)), \quad t \in \mathbb{Z}_+. \tag{2}$$

Here, $f : \mathcal{C} \rightrightarrows \mathcal{C}$ is the set-valued map defined by $f(x) := \{A_i x \mid \text{ for all } i \text{ such that } x \in \mathcal{X}_i\}$. Thus, at any time t, each $\mathcal{X}_i$ which the current state $x(t) = x$ belongs to offers a possible location $A_i x$ to which the state may evolve at the next step. Obviously, by setting $\ell = m$ and $\mathcal{X}_i = \mathcal{C}$ for all $i = 1, \ldots, m$, the CLI (2) on $\mathcal{C}$ reduces to the SLS (1) on $\mathcal{C}$.

Starting from an initial state $z \in \mathcal{C}$, denote by $x(t, z)$ a solution trajectory of the CLI (2). Due to the set-valued nature of the dynamics, there are in general (infinitely) many choices of $x(t, z)$. The (local) stability notions of the CLI (2) at the equilibrium point $x_e = 0$ are defined as follows.

Definition 2: At $x_e = 0$, the CLI (2) on $\mathcal{C}$ is called

- *strongly stable* if, for each $\varepsilon > 0$, there is a $\delta_\varepsilon > 0$ such that $\|x(t, z)\| < \varepsilon, \forall\, t \in \mathbb{Z}_+$, for any trajectory $x(t, z)$ starting from $z \in \mathcal{C}$ with $\|z\| \leq \delta_\varepsilon$;
- *strongly asymptotically stable* if it is strongly stable and there is a $\delta > 0$ such that $x(t, z) \to 0$ as $t \to \infty$ for any trajectory $x(t, z)$ starting from $z \in \mathcal{C}$ with $\|z\| < \delta$;
- *strongly exponentially stable* if there exist $\delta > 0$, $\kappa \geq 1$, and $r \in [0, 1)$ such that $\|x(t, z)\| \leq \kappa r^t \|z\|$, $\forall t \in \mathbb{Z}_+$, for any $x(t, z)$ starting from $z \in \mathcal{C}$ with $\|z\| < \delta$.

Due to homogeneity of the dynamics (2), the local and global stability notions of the CLI (2) are equivalent. In other words, in the above definitions we can equivalently set $\delta = \infty$.

In the Appendix, we shall prove the following result.

Theorem 1: The CLI (2) on $\mathcal{C}$ is strongly asymptotically stable if and only if it is strongly exponentially stable.

Since the SLS (1) on $\mathcal{C}$ is a special instance of CLIs on $\mathcal{C}$, Theorem 1 implies the following.

Corollary 1: The asymptotic and exponential stability of the SLS (1) on $\mathcal{C}$ under arbitrary switching are equivalent.

B. Equivalence of Weak Asymptotic and Exponential Stability for SLSs on Cones

It has been shown through a counter example in [13, Example 5] that weak asymptotic and weak exponential stability are not equivalent for general CLIs on $\mathcal{C}$ (or even on $\mathbb{R}^n$). In this subsection, however, we establish the equivalence of these two weak stability notions for SLSs on cones. The underlying reason for the difference in these two cases, as evidenced by the following proof, is that solutions to SLSs on $\mathcal{C}$ under a fixed switching sequence depend continuously on initial states, while this is not the case for CLIs on $\mathcal{C}$.

Theorem 2: The SLS (1) on $\mathcal{C}$ is weakly asymptotically stable if and only if it is weakly exponentially stable.

Proof: It suffices to show that weak asymptotic stability implies weak exponential stability as the other direction is trivial. Assume that the SLS (1) on $\mathcal{C}$ is asymptotically stable under optimal switching, namely, for any initial state $z \in \mathcal{C}$, the state trajectory $x(t; z, \sigma) \to 0$ as $t \to \infty$ for at least one switching sequence σ. For each $z \in \mathcal{C} \cap \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1} := \{z \in \mathbb{R}^n \,|\, \|z\| = 1\}$, there exist a switching sequence σ_z and a time $T_z \in \mathbb{Z}_+$ such that $\|x(T_z; z, \sigma_z)\| \leq \frac{1}{4}$. Since under the fixed switching sequence σ_z, the solution $x(t; z, \sigma_z)$ at time T_z depends continuously on the initial state z, we can find a neighborhood U_z of z in $\mathcal{C} \cap \mathbb{S}^{n-1}$ such that $\|x(T_z; y, \sigma_z)\| \leq \frac{1}{2}$ for all $y \in U_z$. The union of all such neighborhoods, $\{U_z \,|\, z \in \mathcal{C} \cap \mathbb{S}^{n-1}\}$, is an open covering of the compact set $\mathcal{C} \cap \mathbb{S}^{n-1}$; hence there must exist a finite sub-covering: $\mathcal{C} \cap \mathbb{S}^{n-1} \subseteq \cup_{i=1}^{\ell} U_{z_i^*}$ for some $\ell < \infty$ and $z_1^*, \ldots, z_\ell^* \in \mathcal{C} \cap \mathbb{S}^{n-1}$.

The above obtained finite covering enables us to construct a state-feedback switching policy that leads to an exponentially converging state trajectory. To see this, define $T_* := \max_i T_{z_i^*}$. For any initial state $z \in \mathcal{C} \cap \mathbb{S}^{n-1}$, the above argument implies that $z \in U_{z_i^*}$ for some $1 \leq i \leq \ell$. By our construction, $x(T_1) := x(T_{z_i^*}; z, \sigma_{z_i^*})$ satisfies $\|x(T_1)\| \leq \frac{1}{2}$. Assume without loss of generality that $x(T_1) \neq 0$. Then $x(T_1)/\|x(T_1)\| \in U_{z_j^*}$ for some $1 \leq j \leq \ell$, and as a result, $x(T_2) := x(T_{z_j^*}; x(T_1), \sigma_{z_j^*})$ satisfies $\|x(T_2)\| \leq \frac{1}{2}\|x(T_1)\|$. Repeating this process inductively, we obtain a switching sequence σ_z concatenated by $\sigma_{z_i^*}, \sigma_{z_j^*}, \ldots$ and a sequence of times $0 = T_0 \leq T_1 \leq T_2 \leq \cdots$ with at most T_* between successive ones such that the resulting trajectory $x(t; z, \sigma_z)$ satisfies $\|x(T_{k+1}; z, \sigma_z)\| \leq \frac{1}{2}\|x(T_k; z, \sigma_z)\|$ for all k. Let $\kappa := \sum_{j=0}^{T_*} (\max_{i \in \mathcal{M}} \|A_i\|)^j$. Then it is easily seen that $\|x(t; z, \sigma_z)\| \leq \kappa (0.5)^{t/T_* - 1} \|z\|$ for all $t \in \mathbb{Z}_+$. Since neither κ nor T_* depends on z, the SLS (1) on $\mathcal{C}$ is exponentially stable under optimal switching. ■

Remark 1: We call the SLS (1) on $\mathcal{C}$ *weakly convergent* if for any $z \in \mathcal{C}$, a switching sequence σ_z exists such that $x(t; z, \sigma_z) \to 0$ as $t \to \infty$. This condition seems weaker than weak asymptotic stability as weak Lyapunov stability is not required. However, the proof of Theorem 2 essentially shows that weak convergence is equivalent to weak exponential (thus asymptotic) stability. This observation will be exploited in Theorems 5 and 6 in Section V.

IV. Strong Generating Functions of SLSs on Cones

A. Strong Generating Functions of SLSs on $\mathbb{R}^n$

In [8], the notion of strong generating functions is proposed to study the exponential stability under arbitrary switching of SLSs. The strong generating function of the SLS (1) on $\mathbb{R}^n$ is the map $G : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+ \cup \{\infty\}$ defined as follows: for each $z \in \mathbb{R}^n$ and $\lambda \geq 0$,

$$G_\lambda(z) := G(\lambda, z) := \sup_\sigma \sum_{t=0}^{\infty} \lambda^t \|x(t; z, \sigma)\|^q, \quad (3)$$

where the supremum is taken over all the possible switching sequences, q is a positive integer, and $\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^n$.

The *radius of convergence* of the strong generating function on $\mathbb{R}^n$ is defined as

$$\lambda^*_{\mathbb{R}^n} := \sup\{\lambda \geq 0 \,|\, G_\lambda(z) < \infty, \ \forall\, z \in \mathbb{R}^n\}. \quad (4)$$

The following result is proved in [8].

Theorem 3: The SLS (1) on $\mathbb{R}^n$ is exponentially stable under arbitrary switching if and only if $\lambda^*_{\mathbb{R}^n} > 1$.

Thus the radius of convergence of the strong generating function on $\mathbb{R}^n$ fully characterizes the strong exponential stability of the SLS on $\mathbb{R}^n$.

B. Strong Generating Functions of SLSs on Cones

For the closed convex cone $\mathcal{C}$, define $\mathcal{W}$ to be the smallest subspace of $\mathbb{R}^n$ invariant with respect to $\{A_i\}_{i \in \mathcal{M}}$ containing $\mathcal{C}$, or equivalently, the set of all vectors generated from elements of $\mathcal{C}$ through repeated operations of multiplication by matrices in $\{A_1, \ldots, A_m\}$ and linear combinations:

$$\mathcal{W} := \text{span}\Big\{ \mathcal{C}, \cup_{i \in \mathcal{M}} A_i \mathcal{C}, \cup_{i,j \in \mathcal{M}} A_i A_j \mathcal{C}, \ldots \Big\}.$$

In particular, if $\mathcal{C}$ is solid, then $\mathcal{W} = \mathbb{R}^n$. If $\mathcal{C}$ is polyhedral, i.e. it is finitely (and positively) generated such that $\mathcal{C} = \{\sum_{k=1}^{\ell} \alpha_k v^k \,|\, \alpha_i \geq 0\}$ for some vectors $v^k \in \mathbb{R}^n, k = 1, \cdots, \ell$, then $\mathcal{W} = \text{span}\{\{v^k\}_{k=1}^{\ell}, \cup_{i \in \mathcal{M}} A_i \{v^k\}_{k=1}^{\ell}, \cup_{i,j \in \mathcal{M}} A_i A_j \{v^k\}_{k=1}^{\ell}, \ldots\}$.

Note that $\mathcal{C} \subseteq \mathcal{W} \subseteq \mathbb{R}^n$ form a cascade of sets invariant with respect to $\{A_i\}_{i \in \mathcal{M}}$. Hence, the SLS (1) restricted to each set is well defined and the definition of the generating function in (3) can be extended to $\mathcal{C}$ and $\mathcal{W}$ as well. In particular, the strong generating function of the SLS (1) on the cone $\mathcal{C}$ is defined as, for $\lambda \geq 0$,

$$G_\lambda(z) := \sup_\sigma \sum_{t=0}^{\infty} \lambda^t \|x(t; z, \sigma)\|^q, \quad \forall z \in \mathcal{C}. \quad (5)$$

Here, the same notation $G_\lambda(\cdot)$ is used as in (3) as (5) is exactly the restriction of (3) on $\mathcal{C}$. For this reason, we simply refer to (5) as the strong generating function on $\mathcal{C}$. Similarly, we can define $G_\lambda(\cdot)$ on $\mathcal{W}$ as the restriction of (3) on $\mathcal{W}$.

Define the radii of convergence of the strong generating functions on $\mathcal{C}$ and $\mathcal{W}$ respectively as

$$\lambda^*_{\mathcal{C}} := \sup\{\lambda \geq 0 \,|\, G_\lambda(z) < \infty, \ \forall z \in \mathcal{C}\},$$
$$\lambda^*_{\mathcal{W}} := \sup\{\lambda \geq 0 \,|\, G_\lambda(z) < \infty, \ \forall z \in \mathcal{W}\}.$$

For each $\lambda \geq 0$, define the three subsets

$$\mathcal{G}_\lambda(\mathcal{C}) := \{z \in \mathcal{C} \,|\, G_\lambda(z) < \infty\} \subseteq \mathcal{C},$$
$$\mathcal{G}_\lambda(\mathcal{W}) := \{z \in \mathcal{W} \,|\, G_\lambda(z) < \infty\} \subseteq \mathcal{W},$$
$$\mathcal{G}_\lambda(\mathbb{R}^n) := \{z \in \mathbb{R}^n \,|\, G_\lambda(z) < \infty\} \subseteq \mathbb{R}^n,$$

which satisfy $\mathcal{G}_\lambda(\mathcal{C}) \subseteq \mathcal{G}_\lambda(\mathcal{W}) \subseteq \mathcal{G}_\lambda(\mathbb{R}^n)$, and

$$\mathcal{G}_\lambda(\mathcal{C}) = \mathcal{G}_\lambda(\mathbb{R}^n) \cap \mathcal{C}, \quad \mathcal{G}_\lambda(\mathcal{W}) = \mathcal{G}_\lambda(\mathbb{R}^n) \cap \mathcal{W}. \quad (6)$$

These sets will be useful in the next subsection.

C. Properties of Strong Generating Functions

Obtained through restriction, the strong generating functions on $\mathcal{C}$ and $\mathcal{W}$ inherit many of the properties of their counterpart on $\mathbb{R}^n$ established in [8], as listed below.

Proposition 1: For any $q \in \mathbb{N}$ and any vector norm $\|\cdot\|$, the strong generating functions $G_\lambda(z)$ of the SLS (1) on $\mathcal{C}$ and $\mathcal{W}$ have the following properties.

1. (Bellman Equation): For all $\lambda \geq 0$ and $z \in \mathcal{C}$ (or $\mathcal{W}$),
$$G_\lambda(z) = \|z\|^q + \lambda \cdot \max_{i\in\mathcal{M}} G_\lambda(A_i z).$$
2. (Sub-additivity): For each $\lambda \geq 0$, we have
$$\big(G_\lambda(z_1+z_2)\big)^{1/q} \leq \big(G_\lambda(z_1)\big)^{1/q} + \big(G_\lambda(z_2)\big)^{1/q},$$
for all $z_1, z_2 \in \mathcal{C}$ (or $\mathcal{W}$).
3. (Convexity): For each $\lambda \geq 0$, the function $\big(G_\lambda(z)\big)^{1/q}$ is convex on $\mathcal{C}$ (or $\mathcal{W}$).
4. (Invariant Cone): Let $\lambda \geq 0$ be arbitrary. The set $\mathcal{G}_\lambda(\mathcal{C})$ is a closed convex cone in $\mathcal{C}$ invariant with respect to $\{A_i\}_{i\in\mathcal{M}}$. Particularly, if $\mathcal{C}$ is polyhedral, so is $\mathcal{G}_\lambda(\mathcal{C})$.
5. (Invariant Subspace): Let $\lambda \geq 0$ be arbitrary. The set $\mathcal{G}_\lambda(\mathcal{W})$ is a subspace of $\mathcal{W}$ invariant with respect to $\{A_i\}_{i\in\mathcal{M}}$.
6. For $0 \leq \lambda < (\max_{i\in\mathcal{M}} \|A_i\|^q)^{-1}$, where the matrix norm is induced from the vector norm $\|\cdot\|$, $G_\lambda(z)$ is finite everywhere on $\mathcal{C}$.

Proof: 1. This follows directly from the dynamic programming principle.

2. For any $z_1, z_2 \in \mathcal{C}$, $z_1 + z_2 \in \mathcal{C}$ as $\mathcal{C}$ is a convex cone. Then, by definition,

$$\begin{aligned}
G_\lambda(z_1+z_2) &= \sup_\sigma \sum_{t=0}^{\infty} \lambda^t \|x(t;z_1,\sigma) + x(t;z_2,\sigma)\|^q \\
&\leq \sup_\sigma \sum_{t=0}^{\infty} \lambda^t \Big(\|x(t;z_1,\sigma)\| + \|x(t;z_2,\sigma)\|\Big)^q \\
&\leq \sup_\sigma \Bigg[\bigg(\sum_{t=0}^{\infty} \lambda^t \|x(t;z_1,\sigma)\|^q\bigg)^{1/q} \\
&\qquad + \bigg(\sum_{t=0}^{\infty} \lambda^t \|x(t;z_2,\sigma)\|^q\bigg)^{1/q}\Bigg]^q \\
&\leq \Big[\big(G_\lambda(z_1)\big)^{1/q} + \big(G_\lambda(z_2)\big)^{1/q}\Big]^q,
\end{aligned}$$

where the second inequality is due to the Minkowski inequality. The case for $\mathcal{W}$ is entirely similar.

3. This is due to the subadditivity and the positive homogeneity of $\big(G_\lambda(z)\big)^{1/q}$.

4. The conic property and the convexity of $\mathcal{G}_\lambda(\mathcal{C})$ follow from the positive homogeneity and the convexity of $\big(G_\lambda(z)\big)^{1/q}$, respectively. The invariance with respect to $\{A_i\}_{i\in\mathcal{M}}$ is a consequence of the Bellman equation. To show that $\mathcal{G}_\lambda(\mathcal{C})$ is a closed convex cone, we note that $\mathcal{G}_\lambda(\mathcal{C}) = \mathcal{G}_\lambda(\mathbb{R}^n) \cap \mathcal{C}$, and $\mathcal{G}_\lambda(\mathbb{R}^n)$ is easily shown to be a subspace [8]. Thus, $\mathcal{G}_\lambda(\mathcal{C})$ as the intersection of the convex cone $\mathcal{C}$ and a subspace is convex and closed, and is polyhedral whenever $\mathcal{C}$ is polyhedral.

5. $\mathcal{G}_\lambda(\mathcal{W}) = \mathcal{G}_\lambda(\mathbb{R}^n) \cap \mathcal{W}$ is evidently a subspace.

6. The proof is similar to that in [8], hence omitted. ■

Besides the above inherited properties, the strong generating functions on $\mathcal{C}$ and $\mathcal{W}$ also have some other shared properties. Obviously, the former is the restriction of the latter on the cone $\mathcal{C}$. Less obviously, we have the following.

Proposition 2: For any $\lambda \geq 0$, the strong generating functions $G_\lambda(z)$ of the SLS (1) on $\mathcal{C}$ and $\mathcal{W}$ satisfy:

$$G_\lambda(z) < \infty,\ \forall\, z \in \mathcal{C} \iff G_\lambda(z) < \infty,\ \forall\, z \in \mathcal{W}.$$

Proof: It suffices to show "⇒" direction. Suppose $G_\lambda(\cdot)$ is finite on $\mathcal{C}$, i.e., $\mathcal{G}_\lambda(\mathcal{C}) = \mathcal{C}$. Since $\mathcal{G}_\lambda(\mathbb{R}^n)$ is a subspace of $\mathbb{R}^n$ invariant with respect to $\{A_i\}_{i\in\mathcal{M}}$ and contains $\mathcal{G}_\lambda(\mathcal{C})$, hence $\mathcal{C}$, it must also contain $\mathcal{W}$, as $\mathcal{W}$ is the smallest invariant subspace containing $\mathcal{C}$. In other words, $\mathcal{W} \subseteq \mathcal{G}_\lambda(\mathbb{R}^n)$. Hence $G_\lambda(z)$ is finite for all $z \in \mathcal{W}$. ■

As a result, the radii of convergence of the strong generating functions on $\mathcal{C}$, $\mathcal{W}$, and $\mathbb{R}^n$ have the following relation.

Corollary 2: $\lambda^*_{\mathcal{C}} = \lambda^*_{\mathcal{W}} \geq \lambda^*_{\mathbb{R}^n}$. In particular, if $\mathcal{C}$ is solid, then $\lambda^*_{\mathcal{C}} = \lambda^*_{\mathcal{W}} = \lambda^*_{\mathbb{R}^n}$.

Proposition 3: The strong generating functions $G_\lambda(z)$ of the SLS (1) on $\mathcal{C}$ and $\mathcal{W}$ have the following properties.

1. If $\lambda \in [0, \lambda^*_{\mathcal{C}})$ (hence $G_\lambda(z) < \infty$ for all $z \in \mathcal{C}$), then there exists a constant $c \in [1,\infty)$ such that
$$\|z\| \leq \big(G_\lambda(z)\big)^{1/q} \leq c\|z\|, \quad \forall\ z \in \mathcal{W}.$$
2. (Relative Lipschitz Property) Let $\lambda \in [0, \lambda^*_{\mathcal{C}})$. Then $\big(G_\lambda(z)\big)^{1/q}$ is relatively Lipschitz on $\mathcal{W}$ (thus on $\mathcal{C}$), i.e., there exists $L > 0$ such that for any $x, y \in \mathcal{W}$,
$$\big|\big(G_\lambda(x)\big)^{1/q} - \big(G_\lambda(y)\big)^{1/q}\big| \leq L\|x-y\|.$$

Proof: 1. The first inequality is obvious as $G_\lambda(z) \geq \|z\|^q$ follows directly from the definition. To show the second inequality, by homogeneity, it suffices to show that $\big(G_\lambda(z)\big)^{1/p} \leq c$, $\forall z \in \mathcal{W} \cap \mathbb{S}^{n-1}$, for some constant $c \geq 1$. Let $\{u^i\}_{i=1}^{\ell}$ be a basis of $\mathcal{W}$. Since $\mathcal{W} \cap \mathbb{S}^{n-1}$ is bounded, we can find $\gamma > 0$ such that for each $z \in \mathcal{W} \cap \mathbb{S}^{n-1}$, there exists a unique real tuple $\{\alpha_1, \cdots, \alpha_\ell\}$ satisfying $z = \sum_{j=1}^{\ell} \alpha_j u^j$ and $\sum_{j=1}^{\ell} |\alpha_j| < \gamma$. Therefore, by virtue of the subadditivity and positive homogeneity of $\big(G_\lambda(z)\big)^{1/q}$, we conclude that $\big(G_\lambda(z)\big)^{1/q} \leq c := \gamma \sum_{i=1}^{\ell} \big(G_\lambda(u^i)\big)^{1/q}$ for all $z \in \mathcal{W} \cap \mathbb{S}^{n-1}$. It is easy to verify that $c \geq 1$.

2. It follows from the subadditivity of $\big(G_\lambda(z)\big)^{1/q}$ on $\mathcal{W}$ that for any $x, y \in \mathcal{W}$,

$$\big(G_\lambda(x)\big)^{1/q} - \big(G_\lambda(y)\big)^{1/q} \leq \big(G_\lambda(x-y)\big)^{1/q}.$$

Switching x and y, we have

$$\big(G_\lambda(y)\big)^{1/q} - \big(G_\lambda(x)\big)^{1/q} \leq \big(G_\lambda(y-x)\big)^{1/q}.$$

Combining the above two inequalities, we obtain

$$\big|\big(G_\lambda(x)\big)^{1/q} - \big(G_\lambda(y)\big)^{1/q}\big| \leq \big(G_\lambda(x-y)\big)^{1/q} \leq c\|x-y\|,$$

where the last step is due to $(x-y) \in \mathcal{W}$ and the first property. ■

Remark 2: By the results of this subsection, $\mathcal{G}_\lambda(\mathcal{C})$ is a closed convex sub-cone of $\mathcal{C}$: suppose $\mathcal{C}$ admits the decomposition $\mathcal{C} = \mathcal{K} + \mathcal{V}$ where $\mathcal{K}$ is a pointed cone and $\mathcal{V}$ is a subspace, then $\mathcal{G}_\lambda(\mathcal{C}) = \mathcal{K}_\lambda + \mathcal{V}_\lambda$ with $\mathcal{K}_\lambda \subset \mathcal{K}$ a pointed cone and $\mathcal{V}_\lambda \subset \mathcal{V}$ a subspace. As λ increases, G_λ will increase, hence the invariant subsets $\mathcal{G}_\lambda(\mathcal{C})$, $\mathcal{G}_\lambda(\mathcal{W})$, and $\mathcal{G}_\lambda(\mathbb{R}^n)$ will shrink. In particular, if $\mathcal{C}$ is not pointed (i.e., $\mathcal{V} \neq \{0\}$ is nontrivial), then as λ increases, $\mathcal{G}_\lambda(\mathcal{C})$ will change from non-pointed to pointed, or equivalently, $\mathcal{V}_\lambda$ will shrink to $\{0\}$, at exactly $\lambda^*_{\mathcal{V}} := \inf\{\lambda \geq 0 \,|\, G_\lambda(z) = \infty,\ \forall z \in \mathcal{V}\}$.

D. Strong Exponential Stability Characterization

The radii of convergence of the strong generating functions characterize the strong exponential stability of the SLS on $\mathcal{C}$, as stated by the following theorem.

Theorem 4: The following are equivalent:

1. the SLS (1) on the cone $\mathcal{C}$ (or on the subspace $\mathcal{W}$) is exponentially stable under arbitrary switching;
2. $\lambda^*_{\mathcal{C}} = \lambda^*_{\mathcal{W}} > 1$;
3. $G_1(z)$ is finite for all $z \in \mathcal{C}$ (or $\mathcal{W}$).

Proof: In view of Corollary 1 and Proposition 3, the proof is essentially the same as that of [8, Theorem 1]. ■

E. Numerical Computation of Strong Generating Functions

The algorithm developed in [8] for computing strong generating functions on $\mathbb{R}^n$ can be extended to compute those on closed convex cones. We briefly discuss this in this section. For any $\lambda \in [0, \lambda^*_{\mathcal{C}})$, define $g_\lambda := \sup_{z\in\mathcal{C},\,\|z\|=1} G_\lambda(z)$. Moreover, define the following functions that approximate the strong generating function $G_\lambda(z)$ on $\mathcal{C}$: $G^k_\lambda(z) := \max_\sigma \sum_{t=0}^{k} \lambda^t \|x(t;z,\sigma)\|^q,\ \forall\ z \in \mathcal{C}$. It is easy to see that $G^k_\lambda(z)$ satisfies $G^k_\lambda(z) = \|z\|^q + \lambda \max_{i\in\mathcal{M}} G^{k-1}_\lambda(A_i z),\ \forall\ z \in \mathcal{C}$, with $G^0_\lambda(z) = \|z\|^q$. This yields a recursive procedure to compute these functions. Applying Propositions 1 and 3 as well as the similar argument as in [8, Proposition 6], we have the following.

Proposition 4: The functions $G^k_\lambda(z)$ satisfy

(1) $G^0_\lambda(z) \leq G^1_\lambda(z) \leq \cdots \leq G_\lambda(z)$, $\forall \lambda \geq 0$, $\forall z \in \mathcal{C}$.

(2) $|G^k_\lambda(z) - G_\lambda(z)| \leq g_\lambda(1 - 1/g_\lambda)^{k+1}\|z\|^q$, $\forall k \in \mathbb{Z}_+$, $\forall z \in \mathcal{C}$, for any $\lambda \in [0, \lambda^*_{\mathcal{C}})$.

These results show that the sequence of functions $\{G^k_\lambda\}$ converges uniformly and exponentially fast to G_λ and as such can provide numerical approximations of the latter. To efficiently implement this numerical procedure, an over-approximation can be developed using the convex, conic structure of $\mathcal{C}$; we refer the interested reader to Algorithm 1 in [8] for further details.

V. Weak Generating Functions of SLSs on Cones

Similar to the strong generating functions, weak generating functions can be defined to address weak asymptotic/exponential stability of SLSs on cones, i.e. the stability of the SLSs on cones under optimal switching. Specifically, for the SLS (1) on the closed convex cone $\mathcal{C}$, define its weak generating function $H : \mathbb{R}_+ \times \mathcal{C} \to \mathbb{R}_+ \cup \{\infty\}$ as

$$H_\lambda(z) := H(\lambda, z) := \inf_\sigma \sum_{t=0}^{\infty} \lambda^t \|x(t;z,\sigma)\|^q, \qquad (7)$$

where $\lambda \geq 0$, $z \in \mathcal{C}$, and the infimum is over all switching sequences σ of the SLS on $\mathcal{C}$. In addition, $q \in \mathbb{N}$, and $\|\cdot\|$ is an arbitrary norm in $\mathbb{R}^n$. The radius of convergence for the weak generating function of the SLS on $\mathcal{C}$ is defined as

$$\lambda^{\mathcal{C}}_* := \sup\{\lambda \geq 0 \,|\, H_\lambda(z) < \infty,\ \forall\, z \in \mathcal{C}\}.$$

Proposition 5: For any $q \in \mathbb{N}$ and any vector norm $\|\cdot\|$, the weak generating function $H_\lambda(z)$ of the SLS (1) on $\mathcal{C}$ has the following properties.

1. (Bellman Equation): For any $\lambda \geq 0$ and $z \in \mathcal{C}$,
$$H_\lambda(z) = \|z\|^q + \lambda \cdot \min_{i\in\mathcal{M}} H_\lambda(A_i z).$$
2. (Invariant Cone): For any $\lambda \geq 0$, the set
$$\mathcal{H}_\lambda(\mathcal{C}) := \{z \in \mathcal{C} \,|\, H_\lambda(z) = \infty\}$$
is a cone in $\mathcal{C}$ not containing 0. Further, $\mathcal{H}_\lambda(\mathcal{C})$ is invariant with respect to $\{A_i\}_{i\in\mathcal{M}}$, i.e. $A_i\mathcal{H}_\lambda(\mathcal{C}) \subseteq \mathcal{H}_\lambda(\mathcal{C})$, $\forall i \in \mathcal{M}$.
3. For $0 \leq \lambda < (\min_{i\in\mathcal{M}} \|A_i\|^q)^{-1}$, where the matrix norm is induced from the vector norm $\|\cdot\|$, $H_\lambda(z)$ is finite everywhere on $\mathcal{C}$.

Proof: The proofs of these properties are similar to those in [8] for the corresponding properties of the weak generating function on $\mathbb{R}^n$; hence they are omitted. ■

The subsequent theorem shows two important results: (i) As λ increases, $\lambda^{\mathcal{C}}_*$ is the exact value at which $H_\lambda(z)$ starts to have the infinite value; (ii) if for some $\lambda \geq 0$, the weak generating function $H_\lambda(\cdot)$ is finite everywhere on $\mathcal{C}$, then it must be bounded by a homogeneous function $c\|z\|^q$ uniformly on $\mathcal{C}$.

Theorem 5: For each $\lambda \geq 0$, the following are equivalent:

(a) $H_\lambda(z) \leq c\|z\|^q$, $\forall\, z \in \mathcal{C}$, for some constant $c > 0$ (generally dependent on λ);

(b) $H_\lambda(z) < \infty$ for all $z \in \mathcal{C}$;

(c) $\lambda \in [0, \lambda^{\mathcal{C}}_*)$.

Proof: It is obvious that $(a) \Rightarrow (b)$ and $(c) \Rightarrow (b)$. We shall show $(b) \Rightarrow (a)$ and $(b) \Rightarrow (c)$ as follows, which leads to the equivalence of the three statements.

To prove $(b) \Rightarrow (a)$, consider $\lambda \geq 0$ such that (b) holds. Then for any $z \in \mathcal{C}$, there exists a switching sequence σ_z such that $\sum_{t=0}^{\infty} \lambda^t \|x(t;z,\sigma_z)\|^q < \infty$. Define $\widetilde{A}_i := \lambda^{1/q} A_i, \forall i \in \mathcal{M}$. For the initial state $z \in \mathcal{C}$ and the switching sequence σ_z, let $\widetilde{x}(t;z,\sigma_z)$ denote the trajectory from z under σ_z with the dynamics matrices A_i in the corresponding $x(t;z,\sigma_z)$ replaced by $\widetilde{A}_i$. Then, $\|\widetilde{x}(t;z,\sigma_z)\|^q = \lambda^t \|x(t;z,\sigma_z)\|^q$ for all $t \in \mathbb{Z}_+$. Since $\sum_{t=0}^{\infty} \|\widetilde{x}(t;z,\sigma_z)\|^q < \infty$ for each $z \in \mathcal{C}$, $\widetilde{x}(t;z,\sigma_z) \to 0$ as $t \to \infty$. In other words, the SLS defined by subsystem dynamics matrices $\{\widetilde{A}_i\}_{i\in\mathcal{M}}$ is weakly convergent on $\mathcal{C}$. We thus deduce from Remark 1 that it is weakly exponentially stable on $\mathcal{C}$. Thus there exist $\kappa > 0$ and $\rho \in (0,1)$ such that $\|\widetilde{x}(t;z,\sigma_z)\|^q \leq \kappa\rho^t\|z\|^q$, $\forall\, t \in \mathbb{Z}_+$ for each $z \in \mathcal{C}$. Hence

$$H_\lambda(z) \leq \sum_{t=0}^{\infty} \lambda^t \|x(t;z,\sigma_z)\|^q = \sum_{t=0}^{\infty} \|\widetilde{x}(t;z,\sigma_z)\|^q \leq \frac{\kappa}{1-\rho}\|z\|^q, \quad \forall\ z \in \mathcal{C}.$$

Letting $c := \kappa/(1-\rho)$ yields (a). Further, $(c) \Rightarrow (a)$ holds.

To show $(b) \Rightarrow (c)$, it suffices to show that $H_{\lambda_*^{\mathcal{C}}}(z) = \infty$ for some $z \in \mathcal{C}$. To this end, define $h_\lambda := \sup_{z\in\mathcal{C},\,\|z\|=1} H_\lambda(z), \ \forall\, \lambda \in [0, \lambda_*^{\mathcal{C}})$. It follows from the implication $(c) \Rightarrow (a)$ and a similar argument as in [8, Proposition 9] that for any $\lambda \in (0, \lambda_*^{\mathcal{C}})$, $\lambda/(1-1/h_\lambda) \le \lambda_*^{\mathcal{C}}$, or equivalently, $0 \le 1/h_\lambda \le 1 - \lambda/\lambda_*^{\mathcal{C}}$. Therefore, h_λ tends to infinity as $\lambda \uparrow \lambda_*^{\mathcal{C}}$. This in turn implies that $H_{\lambda_*^{\mathcal{C}}}(z) = \infty$ for some $z \in \mathcal{C}$, since otherwise, by using $(b) \Rightarrow (a)$, we would have $h_{\lambda_*^{\mathcal{C}}} \le c$ for some $c > 0$, a contradiction to the fact that $h_\lambda \to \infty$ as $\lambda \uparrow \lambda_*^{\mathcal{C}}$. ∎

The next result shows that the radius of convergence $\lambda_*^{\mathcal{C}}$ characterizes the weak exponential stability of the SLSs on $\mathcal{C}$.

Theorem 6: The SLS (1) on $\mathcal{C}$ is exponentially stable under optimal switching if and only if $\lambda_*^{\mathcal{C}} > 1$.

Proof: The proof for necessity is straightforward. To show sufficiency, suppose $\lambda_*^{\mathcal{C}} > 1$. Thus for $\lambda = 1$, $H_1(z)$ is finite for any $z \in \mathcal{C}$. This implies that the SLS on $\mathcal{C}$ is weakly convergent and thus weakly exponentially stable, in view of Remark 1 and Theorem 2. ∎

Using Theorem 6, it can be shown as in [8] that

Corollary 3: For any $r > (\lambda_*^{\mathcal{C}})^{-1/q}$, there exists $\kappa_r > 0$ such that for any $z \in \mathcal{C}$, there exists a switching sequence σ_z such that $\|x(t; z, \sigma_z)\| \le \kappa_r r^t \|z\|, \forall\, t \in \mathbb{Z}_+$.

Similar numerical approximations for the weak generating function on $\mathcal{C}$ can be obtained; see [8] for more details.

VI. Appendix: Proof of Theorem 1

The following technical lemma is easy to show and its proof is omitted.

Lemma 1: Let $\{\mathcal{S}_i\}_{i=1}^{\ell}$ be a finite family of closed sets in $\mathbb{R}^n$ and let $\mathcal{S} := \cup_{i=1}^{\ell} \mathcal{S}_i$. For any $x^* \in \mathcal{S}$, there exists a neighborhood $\mathcal{U}$ of x^* such that $(\mathcal{U} \cap \mathcal{S}) \subseteq \cup_{i\in\mathcal{I}(x^*)} \mathcal{S}_i$, where the index set $\mathcal{I}(x^*) := \{i \,|\, x^* \in \mathcal{S}_i\}$.

Assume that the CLI (2) is strongly stable on $\mathcal{C}$. Then for any given $r > 0$, there exists $\delta > 0$ such that $\|x(t, x^0)\| < r$, $\forall t \in \mathbb{Z}_+$, for any trajectory $x(t, x^0)$ starting from $x^0 \in \mathcal{C}$ with $\|x^0\| \le \delta$.

Proposition 6: If the CLI (2) is asymptotically stable on $\mathcal{C}$, then for the $\delta > 0$ obtained above and a given $c \in (0, 1)$, there is $T_{\delta,c} \in \mathbb{Z}_+$ (depending on δ and c only) such that $[x^0 \in \mathcal{C} \text{ with } \|x^0\| \le \delta] \ \Rightarrow \ \|x(t, x^0)\| \le c\,\delta, \ \forall\, t \ge T_{\delta,c}$ for any $x(t, x^0)$ starting from x^0.

Proof: For the given $\delta > 0$ and a given $c \in (0, 1)$, suppose the proposition fails. Then there exist an initial state sequence $\{x_k^0\} \subseteq \mathcal{B}_\delta \cap \mathcal{C}$, the corresponding trajectories $\{x(t, x_k^0)\}$, and a strictly increasing time sequence $\{t_k\} \subseteq \mathbb{Z}_+$ with $\lim_{k\to\infty} t_k = \infty$ such that $\|x(t_k, x_k^0)\| > c\,\delta$. Furthermore, it follows from the stability of $x_e = 0$ that a positive scalar μ exists (with $\mu < \delta$), along with the positive scalar $r > \delta$, such that (i) $\|x(t, x_k^0)\| \le r, \forall\, t \in \mathbb{Z}_+$, for all k; (ii) $x^0 \in (\mathcal{B}_\mu \cap \mathcal{C}) \Rightarrow \|x(t, x^0)\| \le c\,\delta, \ \forall\, t \in \mathbb{Z}_+$. By (ii) and the semi-group property, we have $\|x(t, x_k^0)\| \ge \mu$ for all $t \in \{0, 1, \cdots, t_k\}$. Since $\mu \le \|x_k^0\| \le \delta$ for all k and $\mathcal{C}$ is closed, there exists a subsequence of $\{x_k^0\}$ converging to $x_*^0 \in \mathcal{C}$ with $\mu \le \|x_*^0\| \le \delta$. Without loss of generality, let $\{x_k^0\}$ be that subsequence converging to x_*^0. In view of (i)-(ii) and the construction of $\{t_k\}$, we see that the sequence $\{x(1, x_k^0)\}_{k\ge t_1} \subseteq \mathcal{C}$ satisfies $\mu \le \|x(1, x_k^0)\| \le r$ for all $k \ge t_1$. Thus it has a subsequence converging to $x_*^1 \in \mathcal{C}$ with $\mu \le \|x_*^1\| \le r$. By Lemma 1, a neighborhood $\mathcal{N}$ of x_*^0 can be found such that $(\mathcal{N} \cap \mathcal{C}) \subseteq \cup_{i\in\mathcal{I}(x_*^0)} \mathcal{X}_i$. Note that $x(1, x_k^0) = A_j\, x_k^0$ for some j and $x_k^0 \in \mathcal{N}$ for all large k. Furthermore, since the index set $\mathcal{I}(x_*^0)$ is finite, we deduce that there exist a subsequence $\{x(1, x_{k'}^0)\}$ of $\{x(1, x_k^0)\}_{k\ge t_1}$ and an index $j_1 \in \mathcal{I}(x_*^0)$ such that $x(1, x_{k'}^0) = A_{j_1} x_{k'}^0$ for all k' with $x(1, x_{k'}^0) \to x_*^1$ and $x_{k'}^0 \to x_*^0$. This shows that $x_*^1 = A_{j_1} x_*^0$. Recalling $j_1 \in \mathcal{I}(x_*^0)$, we have $x_*^1 \in f(x_*^0)$.

Repeating this argument and using induction, we obtain $\{x_*^t\}_{t\in\mathbb{Z}_+} \subseteq \mathcal{C}$ such that (i) $\mu \le \|x_*^t\| \le r$ for all $t \in \mathbb{Z}_+$; (ii) for each $t \in \mathbb{Z}_+$, $x_*^{t+1} \in f(x_*^t)$. This shows that the trajectory $x(t, x_*^0) = \{x_*^t\}_{t\in\mathbb{Z}_+}$ is such that $\|x(t, x_*^0)\| \ge \mu, \forall\, t \in \mathbb{Z}_+$. This contradicts the assumption of asymptotic stability of the CLI on $\mathcal{C}$. ∎

With Proposition 6 in hand, the remaining proof of Theorem 1 essentially follows from the similar argument in the proof of [13, Theorem 3].

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