A Case Study of Formation Constrained Optimal Multi-Agent Coordination

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***Abstract*— The centralized optimal multi-agent coordination problem is studied for a particular formation constraint. The problem is formulated as an optimal control problem, and the first-order necessary condition is derived analytically in a suitably chosen coordinate system. A candidate solution satisfying the necessary condition is proposed. The Jacobi equation characterizing the conjugate points along the candidate solution is derived and solved analytically by using the theory of matrix differential equations. The optimal centralized coordinated motions derived in this paper will yield a performance lower bound for those generated by decentralized algorithms.**

I. INTRODUCTION

Multi-agent coordination arises in various context, such as air traffic management (ATM [1], [2]), robotics [3], Unmanned Aerial Vehicle (UAVs [4]), and spacecraft [5], etc. In these applications, the system under study consists of a group of agents that can coordinate their motions to achieve a common goal or complete a common task. Often times, the coordinated motions are subject to some formation constraints, namely, distances between certain pairs of agents need to be kept constant throughout the process. For example, a group of UAVs may need to fly in a certain formation to reduce their fuel expenditure and keep active communication links among them. As another example, a team of mobile robots may coordinate their motions to carry a common object from one end of the room to the other end without dropping it or running into the obstacles.

In this paper, we study the optimal multi-agent coordination problem for a particular formation constraint, namely, during the motion a number of agents must keep a constant distance from the central agent. With this formation constraint, we aim to find the coordinated motions with the minimum energy cost that can move the group of agents from given initial positions to given destination positions within a certain time horizon. While many other studies on formation-constrained multi-agent coordination focus on aspects such as stability [6], feasibility [7], and consensus forming [8], this paper is among the few [9] dealing with the optimality of the coordinated motions. Although a similar problem is studied in [10] for a snake formation constraint, the formation considered in this paper is much more complicated than [10], which posts new challenges on many aspects, especially on characterizing the conjugate points.

The contribution of this paper consists of the following. Firstly, the optimal multi-agent coordination problem is formulated as an optimal control problem, and various necessary conditions for the optimal solutions are derived. Secondly, a natural candidate solution satisfying the necessary conditions is proposed. The Jacobi equation characterizing the conjugate points along this solution is derived. Finally, the Jacobi equation is treated as a second-order matrix differential equation (MDE), and the general MDE theory is introduced to solve the Jacobi equation and derive the conjugate points analytically. The Jacobi equation and conjugate point arise in various optimal control problems. Using the MDE theory to solve the Jacobi equation is relatively new in the literature. Therefore, besides presenting a solution to the multi-agent coordination problem, this paper also introduces the method of using the MDE theory to find the conjugate points for optimal control problems.

The rest of this paper is organized as follows. In Section II, the general formation-constrained optimal coordination problem is formulated. In Section III, we focus on a particular formation constraint, for which we propose an optimal solution and derive the Jacobi equation characterizing the conjugate points along this solution. In Section IV, the theory of matrix differential equation is introduced to solve the Jacobi equation analytically. Conclusion remarks are given in Section V

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II. OPTIMAL FORMATION CONSTRAINED MULTI-AGENT COORDINATION

In this section, the general problem of *optimal formation constrained multi-agent coordination (OFC)* is formulated. We first introduce some notations.

Consider $n+1$ agents moving on a plane $\mathbb{R}^2$. Their positions are denoted by the ordered $(n+1)$-tuple $\langle q_i\rangle_{i=0}^n = (q_0,\dots,q_n)$, where $q_i \in \mathbb{R}^2$ is the position of agent i, $i=0,\dots,n$. A formation constraint on the locations of the $n+1$ agents can be described in terms of an undirected graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$, whose set of vertices $\mathcal{V}=\{0,\dots,n\}$ consists of $n+1$ nodes that correspond to the $n+1$ agents, and whose set of edges $\mathcal{E}$ is a subset of $\mathcal{V}\times\mathcal{V}$. An $(n+1)$-tuple $\langle q_i\rangle_{i=0}^n$ is said to satisfy the *$\mathcal{G}$-formation constraint* if and only if for each edge $(i,j)\in\mathcal{E}$, $0\le i,j\le n$, the distance of agent i and agent j is at a prescribed value (say, unity):

$$\|q_i - q_j\| = 1 \text{ for each } (i,j)\in\mathcal{E}.$$

Note that in the above definition, if (i,j) is not an edge in $\mathcal{E}$, there is no constraint on the distance between agents i and j: $\|q_i-q_j\|$ can be either greater or smaller than 1.

Problem 1 (OFC): Given a formation graph $\mathcal{G}$, and the starting position $\langle a_i\rangle_{i=0}^n$ and the destination position $\langle b_i\rangle_{i=0}^n$ of the $n+1$ agents, find the motions $\langle q_i(t)\rangle_{i=0}^n$ of the agents over a time interval $[0,t_f]$ so that

1) for each agent i, it starts from a_i at time 0 and ends at b_i at time t_f, i.e., $q_i(0) = a_i$, $q_i(t_f) = b_i$;
2) the locations $\langle q_i(t)\rangle_{i=0}^n$ of the $n+1$ agents satisfy the $\mathcal{G}$-formation constraint at all times t in $[0, t_f]$;
3) the total energy expenditure J is minimized, where J is defined by

$$J = \sum_{i=0}^{n} \int_0^{t_f} \|\dot{q}_i\|^2 \, dt. \qquad (1)$$

Remark 1: J is the sum of $n+1$ terms; each term $\int_0^{t_f} \|\dot{q}_i\|^2 \, dt$ is the standard energy of $q_i(t)$ as a curve in $\mathbb{R}^2$ ([11]). Intuitively, minimizing $\int_0^{t_f} \|\dot{q}_i\|^2 \, dt$ will tend to make the motion $q_i(t)$ for agent i follow a straighter path with less speed variations. In comparison, if term $\int_0^{t_f} \|\dot{q}_i\| \, dt$ is used in the definition of J instead of $\int_0^{t_f} \|\dot{q}_i\|^2 \, dt$, then, as $\int_0^{t_f} \|\dot{q}_i\| \, dt$ is the length of the curve $q_i(t)$, $0 \le t \le t_f$, motions that follows the same path will have the same cost, even though some may be smoother than others. Obviously, in practice, smoother motions should be favored.

The OFC problem tries to find the coordinated motions of the $n+1$ agents that can move them from $\langle a_i\rangle_{i=0}^n$ at time 0 to $\langle b_i\rangle_{i=0}^n$ at time t_f with the minimal energy expenditure, while at the same time maintaining the $\mathcal{G}$-formation constraint, namely, the distance between agents i and j is kept at the constant 1 at all times for $(i,j) \in \mathcal{E}$.

The starting position $\langle a_i\rangle_{i=0}^n$ and the destination position $\langle b_i\rangle_{i=0}^n$ are called *aligned* if they have the same centroid: $\frac{1}{n+1}\sum_{i=0}^n a_i = \frac{1}{n+1}\sum_{i=0}^n b_i = c$. In [12], it is proved that the general OFC problems can be reduced to the OFC problem where the initial and the destination positions are aligned. Hence without loss of generality we assume in the rest of this paper that $\langle a_i\rangle_{i=0}^n$ and $\langle b_i\rangle_{i=0}^n$ have a common centroid c, say, at the origin.

Assumption 1: Assume that the initial and the destination positions are aligned at the same centroid $c = 0$:

$$\frac{1}{n+1}\sum_{i=0}^{n} a_i = \frac{1}{n+1}\sum_{i=0}^{n} b_i = 0.$$

Under this assumption, the following result can be applied to reduce the complexity of solving the OFC problem.

Lemma 1 ([12] Align Condition): Suppose that in the OFC problem the starting position $\langle a_i\rangle_{i=0}^n$ and the destination position $\langle b_i\rangle_{i=0}^n$ are aligned at the common centroid $c = 0$. Then the optimal solutions $\langle q_i\rangle_{i=0}^n$ to the OFC problem under any formation constraint $\mathcal{G}$ satisfy

$$\frac{1}{n+1}\sum_{i=0}^{n} q_i(t) = 0, \quad \forall t \in [0, t_f].$$

In other words, the positions of the $n+1$ agents during the optimal coordinated motions are also centered at the origin under arbitrary formation constraints. This in effect reduces the dimension of the problem by two: the optimal solution $(q_0, \ldots, q_n)$ as a curve in $\mathbb{R}^{2(n+1)}$ lies in a subspace of codimension two.

III. An Example of OFC Problem

In this paper, we focus on the OFC problem with a particular tree formation constraint $\mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0)$, where the root node 0 has n ($n \ge 2$) immediate successors, i.e., $\mathcal{V}_0 = \{0, 1, \ldots, n\}$ and $\mathcal{E}_0 = \{(0,i) : i = 1, \ldots, n\}$. Suppose that the starting positions $a = \langle a_i\rangle_{i=0}^n$ of the $n+1$ agents are given by:

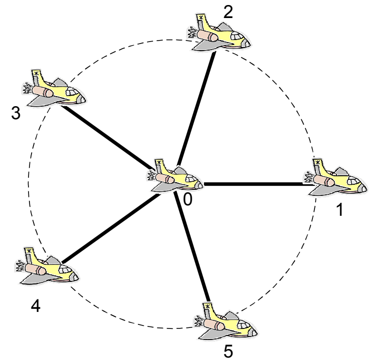

Fig. 1. Formation pattern when $n = 5$.

$$a_0 = (0,0),\ a_i = (\cos((i-1)\phi_n), \sin((i-1)\phi_n)), \qquad (2)$$

$$\text{where} \qquad \phi_n = \frac{2\pi}{n}, \qquad (3)$$

for $i = 1, \ldots, n$. Suppose that the destination positions are $b = \langle b_i\rangle_{i=0}^n = \langle R_{t_f}(a_i)\rangle_{i=0}^n$, where for each $\alpha \in R$, R_α denotes the rotation operation around the origin by the angle α counterclockwise. In other words, at both the initial and destination positions, agent 0 is located at the origin while all the other agents are evenly distributed on the unit circle. See Fig. 1 for an example when $n = 5$. Note that $\langle a_i\rangle_{i=0}^n$ and $\langle b_i\rangle_{i=0}^n$ are not only aligned at the common centroid 0, but also can be obtained from each other by a rotation around the origin.

A. A New Coordinate System

Let $q(t) = \langle q_i(t)\rangle_{i=0}^n$ be a trajectory satisfying the $\mathcal{G}_0$ constraints. Since $(0,i) \in \mathcal{E}_0$ for $i = 1, \ldots, n$, we must have $\|q_i(t) - q_0(t)\| = 1$ for any $t \in [0, t_f]$. It turns out that in studying the above OFC problem, it is more convenient to work in the polar coordinate centered at $q_0(t)$ than the canonical one, as the formation constraint is intrinsically encoded in the former system. Define θ_k as the phase angle of the vector $q_k - q_0$, $k = 1, \ldots, n$. Thus if we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, we have

$$q_i = q_0 + e^{j\theta_i}, \quad k = 1, \ldots, n, \qquad (4)$$

where $j = \sqrt{-1}$. Since the initial and destination positions are aligned at common centroid $q_0 = 0$, by Lemma 1, we have $q_0 = \frac{-1}{n+1}\sum_{k=1}^n e^{j\theta_k}$. Substituting this into (4), we have

$$q_i = \frac{-1}{n+1}\sum_{k=1}^{n} e^{j\theta_k} + e^{j\theta_i}. \qquad (5)$$

Equation (5) defines a coordinate transformation between the canonical system $\langle q_i\rangle_{i=0}^n$ and the new coordinates $\langle \theta_i\rangle_{i=1}^n$. In the new coordinate system, the $\mathcal{G}_0$ formation constraint and the align condition are implicitly encoded. Hence, for simplicity we will work in the new coordinate system in the rest of the paper.

We now derive the expression of the energy J in the new coordinate system. Differentiating (5) and summing the norm square of the result for each i, after some computation we have

$$J=\int_0^{t_f}\left(\sum_{i=0}^{n}\|\dot{q}_i\|^2\right)dt=\int_0^{t_f}\left(\sum_{i,j=1}^{n}\Delta_{ij}\cos(\theta_i-\theta_j)\dot{\theta}_i\dot{\theta}_j\right)dt, \tag{6}$$

where the constants Δ_{ij}, $1\le i,j\le n$, are defined as

$$\Delta_{ij}=\begin{cases}\frac{n}{n+1}, & \text{if } i=j,\\ \frac{-1}{n+1}, & \text{if } j\neq i.\end{cases} \tag{7}$$

Define the (column) vector $\theta=(\theta_1,\dots,\theta_n)\in\mathbb{R}^n$ and the matrix

$$G(\theta)\triangleq[\Delta_{ij}\cos(\theta_i-\theta_j)]_{1\le i,j\le n}\in\mathbb{R}^{n\times n}. \tag{8}$$

Then (6) can be simplified as

$$J=\int_0^{t_f}\dot{\theta}^TG(\theta)\dot{\theta}\,dt. \tag{9}$$

Thus the OFC problem is reduced to the following optimal control problem:

$$\begin{gathered}\text{Minimize } \int_0^{t_f}\frac{1}{2}u^T(t)G(\theta)u(t)\,dt\\ \text{subject to } u(t)=\dot{\theta}(t), t\in[0,t_f],\ \theta(0)=\theta_0,\ \theta(t_f)=\theta_f.\end{gathered} \tag{10}$$

Here $\theta_0,\theta_f\in\mathbb{R}^n$ are chosen to match the initial position $\langle a_i\rangle_{i=0}^n$ and the final position $\langle b_i\rangle_{i=0}^n$, respectively.

B. First Variation and a Natural Candidate Solution

We now solve the optimal control problem (10). Define the Hamiltonian

$$H=\frac{1}{2}u^TG(\theta)u+\lambda^Tu,$$

where $\lambda\in\mathbb{R}^n$ is the co-state. Then by the Maximum Principle, the optimal u is determined by

$$u=\text{argmin}_uH \quad\Rightarrow\quad G(\theta)u+\lambda=0,$$

while the dynamics of λ is given by

$$\dot{\lambda}=-\frac{\partial H}{\partial\theta}=-\frac{1}{2}\frac{\partial}{\partial\theta}\left[u^TG(\theta)u\right].$$

Since $u=\dot{\theta}$, combining the above two equations, we obtain

$$G(\theta)\ddot{\theta}=\frac{1}{2}\frac{\partial}{\partial\theta}\left[\dot{\theta}^TG(\theta)\dot{\theta}\right]-\frac{d}{dt}G(\theta)\cdot\dot{\theta}. \tag{11}$$

Equation (11) is called the *geodesic equation* and its solutions are called geodesics, as the optimal control problem (10) under study is an instance of the shortest distance problem under a suitable Riemannian metric [13].

Evaluating the k-th component of the above vector equation (11), we obtain

$$\left[G(\theta)\ddot{\theta}\right]_k=-\sum_{1\le j\le n}\Delta_{kj}\sin(\theta_k-\theta_j)\dot{\theta}_j^2,\ k=1,\dots,n. \tag{12}$$

Instead of studying the general solutions to the geodesic equation (12), we focus on an important candidate solution θ^* defined as

$$\theta^*=\theta_0+t\cdot\mathbf{1}, \tag{13}$$

where $\theta_0=(\theta_1^0,\dots,\theta_n^0)$ is the new coordinate corresponding to the initial position $\langle a_i\rangle_{i=0}^n$, and $\mathbf{1}\in\mathbb{R}^n$ is the vector whose components are all 1's. In the canonical coordinate system, θ^* corresponds to the rotation of the $n+1$ agents at unit angular velocity counterclockwise around the origin from $\langle a_i\rangle_{i=0}^n$ at time 0 to $\langle b_i\rangle_{i=0}^n$ at time t_f.

According to the initial positions (2) and the transformation (5), we have $\theta_k^0=k\cdot\phi_n$, $k=1,\dots,n$. Then it can be easily verified that θ^* satisfies the geodesic equation (12), and thus is a solution to our particular OFC problem for sufficiently small t_f. However, as the time horizon t_f increases, the optimality of θ^* may be lost due to various reasons. In next subsection, we will study one of them, namely, the occurrence of conjugate points.

C. Second Order Variation and the Jacobi Equation

In order to characterize the conjugate points along θ^*, we need to find the variation of θ in equation (12) around the nominal solution θ^*. To this purpose, let $\delta\theta(t)\in\mathbb{R}^n$ be a variation of $\theta^*(t)$ for $t\in[0,t_f]$. Since the two end points of θ^* are fixed, the variation must be proper, i.e., $\delta\theta(0)=\delta\theta(t_f)=0$. Taking the variation $\delta\theta$ in (12) along the solution θ^*, and using the fact that $\ddot{\theta}^*\equiv0$, $\dot{\theta}_j^*\equiv1$, and $\theta_k^*-\theta_j^*\equiv\theta_k^0-\theta_j^0$, we obtain

$$\ddot{\delta\theta}=-2G(\theta^*)^{-1}G_s(\theta^*)\,\dot{\delta\theta}-\left[G(\theta^*)^{-1}\Lambda-I\right]\delta\theta. \tag{14}$$

Here the matrices Λ and $G_s(\theta^*)\in\mathbb{R}^{n\times n}$ are defined as

$$\Lambda=\text{diag}\,(\mu_1,\dots,\mu_n), \tag{15}$$

$$G_s(\theta^*)=\left[\Delta_{ij}\sin(\theta_i^0-\theta_j^0)\right]_{1\le i,j\le n}, \tag{16}$$

where

$$\mu_k=\sum_{1\le j\le n}\Delta_{kj}\cos(\theta_k^0-\theta_j^0),\quad k=1,\dots,n. \tag{17}$$

Equation (14) is called the *Jacobi equation*. A conjugate point along θ^* (or along q^* in the canonical coordinates) occurs at time τ if there is a nontrivial solution to the Jacobi equation that vanishes at both time 0 and time τ, i.e., if there is a solution $\delta\theta(t)$ not identically zero for $t\in[0,\tau]$ with $\delta\theta(0)=\delta\theta(\tau)=0$. By the standard result of optimal control theory, once $t_f>\tau$, θ^* will no longer be optimal, or equivalently, the corresponding q^* will no longer be the optimal coordinated motion of the $n+1$ agents from $\langle a_i\rangle_{i=0}^n$ at time 0 to $\langle b_i\rangle_{i=0}^n$ at time t_f.

In the next section, we will find the conjugate points along θ^* by solving the Jacobi equation. From the above discussions, this will give us upper bounds on t_f for the optimality of θ^*.

IV. Conjugate Points along θ^*

To solve the Jacobi equation (14), we first write it in the generic form

$$L_2\ddot{x} + L_1\dot{x} + L_0 x = 0, \tag{18}$$

where $x = \delta\theta$, and L_0, L_1, and L_2 are constant matrices defined by

$$L_2 = I, \quad L_1 = 2G(\theta^*)^{-1}G_s(\theta^*), \quad L_0 = G(\theta^*)^{-1}\Lambda - I.$$

A. General Matrix Differential Equations

The equation (18) is a second order homogeneous matrix differential equation (MDE) with constant coefficient matrices $L_0, L_1, L_2 \in \mathbb{R}^{n\times n}$. A standard way of solving a general MDE is described in [14] and can be adopted to solve our problem. We now review some of the important results. The interested reader can refer to [14] for proofs and other details.

Definition 1: Consider the constant-coefficient MDE of order l given by

$$L_l x^{(l)}(t) + \cdots + L_1 x^{(1)}(t) + L_0 x(t) = 0, \tag{19}$$

where $L_0, \ldots, L_l \in \mathbb{C}^{n\times n}$, $\det(L_l) \neq 0$, and $x : \mathbb{R}_+ \to \mathbb{C}^n$ is l-time differentiable. Define the *matrix polynomial* as: $L(\lambda) = \sum_{i=0}^{l} \lambda^i L_i$ for $\lambda \in \mathbb{C}$.

1) λ_0 is a *latent root* of $L(\lambda)$ if $\det(L(\lambda_0)) = 0$;
2) A nonzero vector $v_0 \in \mathbb{C}^n$ is a *latent vector* of $L(\lambda)$ associated with a latent root λ_0 if $L(\lambda_0)v_0 = 0$;
3) A sequence of vectors, $v_0, \ldots, v_{k-1} \in \mathbb{C}^n$ with $v_0 \neq 0$, is a *Jordan chain* of length k for $L(\lambda)$ corresponding to the latent root λ_0 if for each $m = 0, \ldots, k-1$, we have

$$\sum_{i=0}^{m} \frac{L^{(m)}(\lambda_0)v_0}{m!} = 0. \tag{20}$$

Here $L^{(m)}(\lambda)$ denotes the m-th order derivative of $L(\lambda)$ with respect to λ.

The above definition for a matrix polynomial is conceptually similar to the eigenpairs and the Jordan chains of a numerical matrix except that the number of the latent roots of $L(\lambda)$ (counting multiplicity) is $l \cdot n$, and that the latent vectors associated with different latent roots are not necessarily linearly independent. Vectors in the same Jordan chain of $L(\lambda)$ could also be linearly dependent. On the other hand, most other facts about the Jordan chains of numerical matrices still apply here. For example, if λ_i is a latent root of $L(\lambda)$ with algebraic multiplicity n_i and geometric multiplicity m_i, then there are m_i sets of Jordan chains associated with λ_i and the numbers of vectors in these Jordan chains sum up to n_i. See [14] for more details.

With the above definitions, the following theorem describes the relationship between the Jordan chains of $L(\lambda)$ and the primitive solutions of the MDE (19).

Theorem 1: Let $L(\lambda)$ be the matrix polynomial associated with MDE (19).

1) If v_0 and v_1 are two latent vectors of $L(\lambda)$ associated with latent roots λ_0 and λ_1 ($\lambda_0 \neq \lambda_1$), respectively, then $x_0(t) = v_0\, e^{\lambda_0 t}$ and $x_1(t) = v_1\, e^{\lambda_1 t}$ are two linearly independent solutions of the MDE (19).
2) If $v_0, v_1, \ldots, v_{k-1}$ is a Jordan chain of length k for $L(\lambda)$ corresponding to the latent root λ_0, then

$$x_0(t) = v_0\, e^{\lambda_0 t}, \quad x_1(t) = (tv_0 + v_1)\, e^{\lambda_0 t}, \ldots$$
$$x_{k-1}(t) = \left(\sum_{j=0}^{k-1} \frac{t^j}{j!} v_{k-1-j}\right) e^{\lambda_0 t} \tag{21}$$

are k linearly independent solutions of the MDE (19).
3) Solutions of the form (21) but belonging to different Jordan chains of $L(\lambda)$ are linearly independent.

MDE (19) has an $l \cdot n$-dimensional solution space. By Theorem 1, a Jordan chain of length k for $L(\lambda)$ can provide exactly k independent solutions. Thus the whole solution space of the MDE (19) is fully characterized by the Jordan chains of the corresponding matrix polynomial $L(\lambda)$.

B. An Analytical Solution

The conjugate points along θ^* can be located by finding particular solutions $x(t)$ to the equation (18) that start from zero and come back to zero at a later time. Since equation (18) is just a second order MDE, by Theorem 1, its $2n$-dimensional solution space is spanned by linearly independent vector functions defined in terms of the latent roots and the corresponding Jordan chains of $L(\lambda)$. Therefore, the problem of finding the conjugate points of θ^* can be transformed to computing the latent roots and the Jordan chains of $L(\lambda)$.

1) Computation of the Coefficient Matrices: To simplify the expressions of the matrices $G(\theta^*)^{-1}$ and Λ in (14), a standard result from discrete Fourier transform is introduced in the following lemma.

Lemma 2: Let m be an integer that is not an integer multiple of n, i.e., $m \neq l \cdot n$, $\forall l \in \mathbb{Z}$. Let $\beta \in \mathbb{R}$ be arbitrary. Then the following relations hold:

$$\sum_{k=0}^{n-1} \cos(km\phi_n + \beta) = 0, \quad \sum_{k=0}^{n-1} \sin(km\phi_n + \beta) = 0.$$

Using Lemma 2, the inverse of $G(\theta^*)$ defined in (16), denoted by $G(\theta^*)^{-1} = [g^{ij}]_{1\le i,j\le n}$, is given by

$$g^{ij} = \begin{cases} \frac{n+4}{n+2}, & i = j, \\ \frac{2}{n+2}\cos\left((i-j)\phi_n\right), & i \neq j, \end{cases} \tag{22}$$

for $n \geq 3$.

Remark 2: The result in equation (22) can be verified using Lemma 2 with $m = 2$. Thus (22) is valid only when $n \geq 3$ as can be seen from the condition of Lemma 2. In the subsequent discussions, we shall assume $n \geq 3$ temporarily, and then deal with the $n = 2$ case separately at the end of this section.

Furthermore, using Lemma 2, it can be easily verified that μ_k defined in (17) is

$$\mu_k = \sum_{j\neq k} \frac{-\cos((k-j)\phi_n)}{n+1} + \frac{n}{n+1} = 1.$$

Therefore, $\Lambda = \text{diag}\,(\mu_1, \ldots, \mu_n) = I$ is simply the identity matrix.

Given the simplified $G(\theta^*)^{-1}$ and Λ, the coefficient matrices in equation (18) can now be computed as

$$L_2 = I, \qquad [L_0]_{ij} = \frac{2}{n+2}\cos((i-j)\phi_n),$$
$$\text{and} \qquad [L_1]_{ij} = \frac{-4}{n+2}\sin((i-j)\phi_n).$$

As a result, the matrix polynomial associated with the MDE (18) becomes, for $1 \leq i, j \leq n$,

$$[L(\lambda)]_{ij} = \left[\sum_{k=0}^{2} \lambda^k L_k\right]_{ij}$$
$$= \begin{cases} \lambda^2 + \frac{2}{n+2}, & i = j, \\ \frac{-4}{n+2}\sin((i-j)\phi_n)\lambda + \frac{2}{n+2}\cos((i-j)\phi_n), & i \neq j. \end{cases} \tag{23}$$

2) Latent Roots and Latent Vectors of $L(\lambda)$: By Theorem 1, solutions to equation (18) can be expressed in terms of the latent roots and the Jordan chains of $L(\lambda)$. We now compute these for the matrix polynomial $L(\lambda)$ defined in (23). We will show that $L(\lambda)$ has a zero latent root with multiplicity $2(n-2)$, associated with which there are $n-2$ Jordan chains of length 2, as well as four distinct nonzero latent roots of multiplicity one.

Lemma 3: Define a set of vectors $v_k^c, v_k^s \in \mathbb{R}^n$, $k = 0, \ldots, n$, as

$$v_k^c = [cos(k\phi_n), \cos(2k\phi_n), \ldots, \cos(nk\phi_n)]^T,$$
$$v_k^s = [sin(k\phi_n), \sin(2k\phi_n), \ldots, \sin(nk\phi_n)]^T.$$

Let $V \triangleq V^c \cup V^s \triangleq \{v_k^c : 0 \leq k \leq [\frac{n}{2}]\} \cup \{v_k^s : 1 \leq k < \frac{n}{2}\}$, where $[\cdot]$ denotes the integer part of a real number. Then V is an orthogonal basis of $\mathbb{R}^n$.

The proof of Lemma 3 is a simple application of Lemma 2.

Remark 3: v_0^s and $v_{\frac{n}{2}}^s$ (when n is even) are zero vectors; thus they are not included in V^s. It is easily seen that V defined above contains exactly n linearly independent vectors of $\mathbb{R}^n$, for any $n \geq 2$.

Using Lemma 3, we are able to characterize all the Jordan chains of $L(\lambda)$ associated with $\lambda = 0$.

Proposition 1: $\lambda_0 = 0$ is a latent root of the matrix polynomial $L(\lambda)$ defined in (23), and for each $v \in V \setminus \{v_1^c, v_1^s\}$, the pair of vectors $\{v, v\}$ constitute a Jordan chain of $L(\lambda)$ corresponding to λ_0 of length 2.

To prove Proposition 1, one needs to verify that $L(0)v = 0$ and that $L(0)v + L^{(1)}(0)v = 0$ for each $v \in V \setminus \{v_1^c, v_1^s\}$, which is again a trivial application of Lemma 2.

Since $V \setminus \{v_1^c, v_1^s\}$ has $n-2$ vectors, Proposition 1 describes $n-2$ Jordan chains, each of which is of length 2. Together, by Theorem 1, these Jordan chains characterize $2n-4$ independent solutions of the Jacobi equation (18). Let u_i, $i = 1, \ldots, n-2$, be an enumeration of the vectors in $V \setminus \{v_1^c, v_1^s\}$, and define

$$U_0(t) \triangleq [u_1, tu_1, \ldots, u_{n-2}, tu_{n-2}]. \tag{24}$$

Any linear combination of the columns of $U_0(t)$ is a solution to equation (18). On the other hand, equation (18) should have $2n$ independent solutions in total. It turns out that the four missing independent solutions are provided by the latent vectors associated with the nonzero latent roots of $L(\lambda)$.

Proposition 2: Define the complex vector $v_1 = v_1^c + jv_1^s$, where $j = \sqrt{-1}$. Its conjugate is $\bar{v}_1 = v_1^c - jv_1^s$. Define

$$\omega_1 = \frac{-n + \sqrt{2n(n+1)}}{n+2}, \quad \omega_2 = \frac{-n - \sqrt{2n(n+1)}}{n+2}. \tag{25}$$

Then the nonzero latent roots of $L(\lambda)$ and their corresponding latent vectors can be characterized as follows:

1) $\lambda_1 = j\omega_1$ and $\lambda_2 = -j\omega_1$ are two latent roots of $L(\lambda)$ with latent vectors v_1 and $\bar{v}_1$, respectively;
2) $\lambda_3 = j\omega_2$ and $\lambda_4 = -j\omega_2$ are two latent roots of $L(\lambda)$ with latent vectors v_1 and $\bar{v}_1$, respectively.

The above results can be proved using Definition 1 and Lemma 2. Refer to [13] for a complete proof.

By Theorem 1, there are four independent solutions to the MDE (18) corresponding to the four distinct nonzero latent roots λ_i, $i = 1, \ldots, 4$. We arrange these solutions into the columns of a matrix defined by

$$\begin{aligned} U_{\bar{0}}(t) &= [e^{j\omega_1 t}v_1, e^{-j\omega_1 t}\bar{v}_1, e^{j\omega_2 t}v_1, e^{-j\omega_2 t}\bar{v}_1] \\ &= V_1 \cdot \begin{bmatrix} e^{\Omega_1 t} & e^{\Omega_2 t} \end{bmatrix}, \end{aligned} \tag{26}$$

where V_1, Ω_1, and Ω_2 are complex matrices defined by

$$V_1 = \begin{bmatrix} v_1 & \bar{v}_1 \end{bmatrix}, \Omega_1 = \begin{bmatrix} j\omega_1 & 0 \\ 0 & -j\omega_1 \end{bmatrix}, \Omega_2 = \begin{bmatrix} j\omega_2 & 0 \\ 0 & -j\omega_2 \end{bmatrix}. \tag{27}$$

Any complex linear combination of the columns of $U_{\bar{0}}(t)$ of the form $x(t) = U_{\bar{0}}(t)z$ for some $z = [z_1, z_2, z_3, z_4]^T$ in $\mathbb{C}^4$ is a solution to equation (18). However, we are only interested in those real solutions. For this purpose, z must be chosen so that $z_2 = \bar{z}_1$ and $z_4 = \bar{z}_3$. Plugging into $x(t) = U_{\bar{0}}(t)z$, we conclude that the four dimensional real solution space of equation (18) corresponding to the nonzero latent roots of $L(\lambda)$ is spanned by the columns of two real matrices in $\mathbb{R}^{n\times 2}$:

$$U_1^{Re}(t) = \begin{bmatrix} v_1^c & v_1^s \end{bmatrix} \begin{bmatrix} \cos(\omega_1 t) & \sin(\omega_1 t) \\ -\sin(\omega_1 t) & \cos(\omega_1 t) \end{bmatrix}, \tag{28}$$

$$U_2^{Re}(t) = \begin{bmatrix} v_1^c & v_1^s \end{bmatrix} \begin{bmatrix} \cos(\omega_2 t) & \sin(\omega_2 t) \\ -\sin(\omega_2 t) & \cos(\omega_2 t) \end{bmatrix}. \tag{29}$$

To sum up the results in this section, we can now characterize all the real solutions to equation (18).

Proposition 3: Every real solution to the MDE (18) is of the form

$$x(t) = U_0(t)c_0 + U_1^{Re}(t)c_1 + U_2^{Re}(t)c_2, \tag{30}$$

for some constants $c_0 \in \mathbb{R}^{2(n-2)}$, $c_1 \in \mathbb{R}^2$, and $c_2 \in \mathbb{R}^2$. Here the matrices $U_0(t)$, $U_1^{Re}(t)$ and $U_2^{Re}(t)$ are defined in (24), (28), and (29), respectively.

3) Conjugate Points Along θ^*: To find the conjugate points along θ^*, we need to look for those nontrivial real solutions $x(t)$ to the Jacobi equation (18) that start from 0 and return to 0 at some positive time τ, i.e., $x(0) = 0$, and $x(\tau) = 0$. Let $x(t)$ be one such solution. By Proposition 3, $x(t)$ is of the form $x(t) = U_0(t)c_0 + U_1^{Re}(t)c_1 + U_2^{Re}(t)c_2$ for some constants c_0, c_1 and c_2. Observe that the first term $U_0(t)c_0$ is an affine function of time t, and always lies in the subspace spanned by the $n-2$ basis vectors in $V \setminus \{v_1^c, v_1^s\}$, where V is the set of bases given in Lemma 3. On the other hand, the second and the third terms $U_1^{Re}(t)c_1$ and $U_2^{Re}(t)c_2$ always lie in the 2-dimensional subspace spanned by the other two basis vectors v_1^c and v_1^s. Hence, in order for $x(0) = x(\tau) = 0$ to hold for some $\tau > 0$, we must have $c_0 = 0$. Thus $x(t) = U_1^{Re}(t)c_1 + U_2^{Re}(t)c_2$.

By (28) and (29), at time 0, $U_1^{Re}(0) = U_2^{Re}(0) = [v_1^c, v_1^s]$. Hence to satisfy $x(0) = 0$, we must have

$$x(0) = U_1^{Re}(0)c_1 + U_2^{Re}(0)c_2 = \begin{bmatrix} v_1^c & v_1^s \end{bmatrix} (c_1 + c_2) = 0,$$

which implies that $c_2 = -c_1$ by the linear independence of v_1^c and v_1^s. Thus, if we write $c_1 = [a, b]^T$, then

$$\begin{aligned} x(t) &= U_1^{Re}(t)c_1 + U_2^{Re}(t))(-c_1) = [U_1^{Re}(t) - U_2^{Re}(t)]c_1 \\ &= 2\sin(\omega_- t) \begin{bmatrix} v_1^c \ v_1^s \end{bmatrix} \begin{bmatrix} -a\sin(\omega_+ t) + b\cos(\omega_+ t) \\ -a\cos(\omega_+ t) - b\sin(\omega_+ t) \end{bmatrix}, \end{aligned} \quad (31)$$

where ω_+ and ω_- are constants defined by

$$\omega_+ = \tfrac{\omega_1+\omega_2}{2} = -\tfrac{n}{n+2}, \ \omega_- = \tfrac{\omega_1-\omega_2}{2} = \tfrac{\sqrt{2n(n+1)}}{n+2}. \quad (32)$$

Note that a and b can not be zero at the same time (otherwise $x(t) \equiv 0$ is trivial). Under this constraint, it can be easily checked that the two entries of the last factor in (31), $-a\sin(\omega_+ t) + b\cos(\omega_+ t)$ and $-a\cos(\omega_+ t) - b\sin(\omega_+ t)$, can not be zero at the same time. Thus, in order to satisfy $x(\tau) = 0$ for some $\tau > 0$, we must have $\sin(\omega_- \tau) = 0$, i.e., $\tau = \frac{k\pi}{\omega_-}$ for some $k = 1, 2, \ldots$. This gives us the times when conjugate points along θ^* are encountered.

Theorem 2: For the particular OFC problem studied in Section III with $n \geq 3$, the set of conjugate points along the candidate solution θ^* is given by

$$\left\{ \theta^*(\tau) : \tau = \frac{k(n+2)\pi}{\sqrt{2n(n+1)}}, \ k = 1, 2, \ldots \right\}.$$

As a result, the first conjugate point along θ^* occurs at time

$$\tau_n = \frac{\pi}{\omega_-} = \frac{(n+2)\pi}{\sqrt{2n(n+1)}}. \quad (33)$$

From the expression (33), we can see that τ_n decreases as n increases, and $\tau_n \to \frac{\pi}{\sqrt{2}}$ as $n \to \infty$.

Since a geodesic is no longer distance-minimizing beyond its first conjugate point, we have

Theorem 3: θ^* is not an optimal solution to the OFC problem if $t_f > \tau_n$.

As mentioned in Remark 2, the above derivations are valid only under the condition $n \geq 3$. When $n = 2$, the coefficient matrices L_0, L_1, and L_2 in equation (18) are all 2×2. Hence, analytical solution can be obtained much more easily compared with the general n case. After some careful computation, the first conjugate point along θ^* occurs at the time $\tau_2 = \frac{\pi}{\sqrt{2}}$ for $n = 2$, which, interestingly, is exactly the limit of τ_n as $n \to \infty$.

V. Conclusion

The OFC problem is studied for a particular formation constraint, where a number of agents must keep a constant distance from the central agent during their coordinated motions. A candidate solution satisfying the first-order optimality condition is proposed. Conjugate points along the solution are characterized analytically. The approaches adopted in this paper are general enough to make the results meaningful in a variety of applications involving optimal multi-agent coordination. As extensions, our future research will focus on the OFC problem with general formation constraints that are not necessarily described by trees.

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