How should a snake turn on ice: A case study of the asymptotic isoholonomic problem

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***Abstract*— It is a classical result that solutions to the isoperimetric problem, i.e., finding the planar curves with a fixed length that enclose the largest area, are circles. As a generalization, we study an asymptotic version of the dual isoholonomic problem in a Euclidean space with a co-dimension one distribution. We propose the concepts of asymptotic rank and efficiency, and compute these quantities as well as the efficiency-achieving curves in several special cases. In particular, an example of a snake moving on ice is worked out in detail to illustrate the results.**

I. Introduction

As the dual to the isoperimetric problem, the isoholonomic problem has application in a variety of fields, for example, control theory [3], the falling cat problem [6], the swimming microorganism at low Reynolds number [9], and the Berry phase in quantum mechanics [7], etc. Another reference can be found in [1]. In this paper, we will formulate and study an asymptotic version of the isoholonomic problem for which analytic solutions are available in certain cases. To define the problem, we shall start from a motivating example, which can be thought of as the planar version of the falling cat problem.

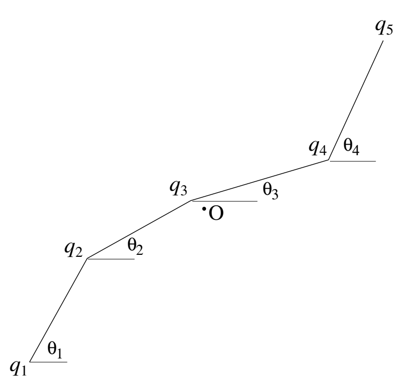

Fig. 1. A snake.

A. Motivating Example: Snake on Ice

Consider the following model of a snake moving on a horizontal plane. The snake consists of $n+2$ unit point masses (nodes) whose positions are denoted by $q_1, \ldots, q_{n+2} \in \mathbb{R}^2$, respectively. These nodes are then connected subsequently by $n+1$ rigid bars of unit length and zero weight, forming a kinematic chain. Figure 1 shows an example when $n = 3$. Suppose that the plane is ideal ice (i.e., frictionless) so that the snake cannot "push off" the ground to gain locomotion. Then since the snake is a closed mechanical system, its total linear momentum and total angular momentum are both conserved. So for an initially stationary snake, we must have

$$\sum_{i=1}^{n+2} \dot{q}_i \quad \equiv \quad 0, \tag{1}$$

$$\sum_{i=1}^{n+2} q_i \times \dot{q}_i \quad \equiv \quad 0. \tag{2}$$

Without loss of generality, we assume that the snake is initially centered at the origin, i.e., $\sum_{i=1}^{n+2} q_i(0) = 0$. Condition (1) then implies that $\sum_{i=1}^{n+2} q_i \equiv 0$.

The configuration of the snake is uniquely determined by the angles $\theta_1, \ldots, \theta_{n+1}$, where θ_i is the angle $q_{i+1} - q_i$ makes with the positive x-axis for $i = 1, \ldots, n+1$. Each θ_i takes values in $\mathbb{R}$ modulo 2π, namely, the 1-torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, so $(\theta_1, \ldots, \theta_{n+1})$ takes values in the $(n+1)$-torus $\mathbb{T}^{n+1}$, which is the *configuration space* of the snake. For given $\theta_1, \ldots, \theta_{n+1}$, $q_1, \ldots, q_{n+2}$ can be recovered by

$$q_1 \quad = \quad -\frac{1}{n+2} \sum_{j=1}^{n+1} (n+2-j)(\cos\theta_j, \sin\theta_j)^t, \tag{3}$$

$$q_i \quad = \quad q_1 + \sum_{j=1}^{i-1} (\cos\theta_j, \sin\theta_j)^t, \; i = 2, \ldots, n+2, \tag{4}$$

Equation (3) and (4) together define an embedding of the configuration space $\mathbb{T}^{n+1}$ into $\mathbb{R}^{2n+4}$. Thus $\mathbb{T}^{n+1}$ inherits isometrically via this embedding a riemannian metric from the standard metric on $\mathbb{R}^{2n+4}$. After some calculation, this metric $\langle \cdot, \cdot \rangle$ can be determined as

$$g_{ij} \triangleq \langle \frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j} \rangle = \Delta_{ij} \cos(\theta_i - \theta_j), \; 1 \le i, j \le n+1, \tag{5}$$

where Δ_{ij} are constants defined by

$$\Delta_{ij} = \begin{cases} \frac{i(n+2-j)}{n+2}, & \text{if } i < j, \\ \frac{(n+2-i)j}{n+2}, & \text{if } i \ge j. \end{cases}$$

Suppose that the trajectory of the snake over an interval $I = [0, 1]$ is given by the curve γ in $\mathbb{T}^{n+1}$. Unless otherwise stated, we assume that all the curves in this paper are defined on I. Define $L(\gamma) = \int_0^1 \|\dot{\gamma}\| \, dt$ and $E(\gamma) = \int_0^1 \|\dot{\gamma}\|^2 \, dt$ as the *length* and the *energy* of γ, respectively, where $\|\cdot\|$ is the norm corresponding to $\langle \cdot, \cdot \rangle$. From the definition of $\langle \cdot, \cdot \rangle$, if the positions of the nodes of the snake corresponding to γ are given by $q_1, \ldots, q_{n+2}$, respectively, then

$$L(\gamma) = \int_0^1 \left(\sum_{i=1}^{n+2} \dot{q}_i^2 \right)^{1/2} dt, \quad E(\gamma) = \int_0^1 \sum_{i=1}^{n+2} \dot{q}_i^2 \, dt.$$

The problem we study in this paper can be roughly stated as: *how can the snake turn most efficiently*? One possible formulation is described in the following. Suppose that the snake starts from configuration $(\theta_1^0, \ldots, \theta_{n+1}^0)$ at time 0, and wishes to retain the shape but with different orientation, for example, it tries to reach configuration $(\theta_1^0+\theta, \ldots, \theta_{n+1}^0+\theta)$ at time 1. Among all such trajectories γ, we want to find the one with minimal length $L(\gamma)$ (or energy $E(\gamma)$, which are equivalent), subject to the constraint (2) that the total angular momentum are zero at all time.

B. Solutions as Sub-Riemannian Geodesics

Without the constraint (2), the above problem becomes finding geodesics in $\mathbb{T}^{n+1}$ with the riemannian metric $\langle\cdot,\cdot\rangle$, which is studied in [5]. However, with the addition of the constraint (2), the problem becomes one of finding sub-riemannian geodesics in a certain sub-riemannian geometry. In fact, let $\gamma = (\theta_1, \ldots, \theta_{n+1})$ be a curve in $\mathbb{T}^{n+1}$, and let $q_1, \ldots, q_{n+2}$ be the corresponding positions of its nodes. Then careful calculations show that (2) is equivalent to

$$\sum_{i,j=1}^{n+1} \Delta_{ij} \cos(\theta_i - \theta_j)\dot{\theta}_j = 0. \tag{6}$$

In other words, if we define a one-form on $\mathbb{T}^{n+1}$ by

$$\alpha = \sum_{i,j=1}^{n+1} \Delta_{ij} \cos(\theta_i - \theta_j) d\theta_j, \tag{7}$$

then condition (2) is equivalent to $\alpha(\dot{\gamma}) = 0$, i.e., $\dot{\gamma} \in \ker\alpha$. Note that $\mathcal{H} = \ker\alpha$ is a co-dimension one distribution on $\mathbb{T}^{n+1}$. Hence γ must be a horizontal curve for this distribution. Moreover, the restriction of $\langle\cdot,\cdot\rangle$ to $\mathcal{H}$ defines a sub-riemannian metric $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. In this sub-riemannian geometry, the sub-riemannian length of the horizontal curve γ coincides with its riemannian length $L(\gamma)$.

Therefore, the problem stated above is to find the shortest horizontal curves in $\mathbb{T}^{n+1}$ connecting $(\theta_1^0, \ldots, \theta_{n+1}^0)$ to $(\theta_1^0+\theta, \ldots, \theta_{n+1}^0+\theta)$, which is a distance-minimizing sub-riemannian geodesic.

Compared with general sub-riemannian geometries, however, the one defined above belongs to a very special category. In fact, $\mathbb{T}^{n+1}$ is a principal $\mathbb{T}$-bundle over $\mathbb{T}^n$ with distribution $\mathcal{H}$ and sub-riemannian metric $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ that are compatible with the action of the structure group $\mathbb{T}$. To see this, consider the following action R of $\mathbb{T}$ on $\mathbb{T}^{n+1}$: for each $\theta \in \mathbb{T}$, $R_\theta : \mathbb{T}^{n+1} \to \mathbb{T}^{n+1}$ is defined by

$$R_\theta(\theta_1, \ldots, \theta_{n+1}) = (\theta_1+\theta, \ldots, \theta_{n+1}+\theta).$$

The configuration of the snake corresponding to $R_\theta(\theta_1, \ldots, \theta_{n+1})$ is obtained from that corresponding to $(\theta_1, \ldots, \theta_{n+1})$ by a rotation of θ counterclockwise. In fact, the set of all shapes of the snake corresponds in a one-to-one way to the set of $\mathbb{T}$-orbits of $\mathbb{T}^{n+1}$, which can be identified as $\mathbb{T}^n \triangleq \{(\theta_1, \ldots, \theta_{n+1}) \in \mathbb{T}^{n+1} : \theta_1 + \ldots + \theta_{n+1} = 0 \mod 2\pi\}$. We shall call $\mathbb{T}^n$ the *shape space*. As the notation suggests, $\mathbb{T}^n$ is topologically an n-torus.

There is a natural projection $\pi : \mathbb{T}^{n+1} \to \mathbb{T}^n$ defined by $\pi(\theta_1, \ldots, \theta_{n+1}) = (\theta_1 - \frac{\sum_{i=1}^{n+1}\theta_i}{n+1}, \ldots, \theta_{n+1} - \frac{\sum_{i=1}^{n+1}\theta_i}{n+1})$, such that for each $(\theta_1, \ldots, \theta_{n+1}) \in \mathbb{T}^n$ its inverse image under π is exactly the $\mathbb{T}$-orbit in $\mathbb{T}^{n+1}$ passing through it. In the terminology of principal bundles, π makes $\mathbb{T}^{n+1}$ into a principal bundle with base space $\mathbb{T}^n$ and structure group $\mathbb{T}$. Each fiber of this bundle consists of all configurations of the snake with a fixed shape but different orientations.

Note that in the definitions (5) and (7), the terms involving θ_i's are of the form $\theta_i - \theta_j$, which remain unchanged under the action R. Hence the horizontal distribution $\mathcal{H}$ and the sub-riemannian metric $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ are both invariant under the action R. Such distributions and sub-riemannian metrics are called *compatible* (with the action of the structure group $\mathbb{T}$).

In this perspective, the problem is to determine the shortest horizontal curve from a configuration $(\theta_1, \ldots, \theta_{n+1})$ to a new configuration $R_\theta(\theta_1, \ldots, \theta_{n+1})$ in the same fiber.

C. Asymptotic Problem

Due to the global nature of the problem proposed above, its solutions are usually impossible to obtain analytically. In this paper, we shall study an asymptotic version of the problem: what is the most efficient way of turning if the snake can only exert an increasingly smaller amount of energy? Besides giving approximate solutions to the global problem when the snake is confined to a neighborhood of the current configuration, solutions to the asymptotic problem can be employed repetitively for the snake to turn a significant angle, which makes sense if the snake has to take a breath at its original shape from time to time.

The paper is organized as follows. First, some notions in sub-riemannian geometry are reviewed in Section II. In Section III, we formulate the problem of asymptotic isoholonomy by proposing the concepts of rank and efficiency. The rank two case is solved in Section IV, and the rank three case with $n = 2$ in Section V. The results are then illustrated in Section VI for the snake example.

II. Basic Setup

Since we are concerned with local solutions, we consider $\mathbb{R}^{n+1}$ instead of $\mathbb{T}^{n+1}$. The projection $\pi : (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mapsto (x_1, \ldots, x_n) \in \mathbb{R}^n$ defines $\mathbb{R}^{n+1}$ as a bundle over $\mathbb{R}^n$ whose fiber over each point $m \in \mathbb{R}^n$ is given by $\pi^{-1}(m) \simeq \mathbb{R}$, making $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ a principal $\mathbb{R}$-bundle.

A. Co-Dimension One Distribution on $\mathbb{R}^{n+1}$

A co-dimension one distribution $\mathcal{H}$ on $\mathbb{R}^{n+1}$ is defined by the vanishing of a one-form

$$\alpha = \sum_{i=1}^{n} \alpha_i dx_i - dx_{n+1}, \tag{8}$$

where $\alpha_1, \ldots, \alpha_n$ are C^∞ functions on $\mathbb{R}^{n+1}$. At each point $q \in \mathbb{R}^{n+1}$, the horizontal space $\mathcal{H}_q$ is the kernel of

α_q in $T_q\mathbb{R}^{n+1} \simeq \mathbb{R}^{n+1}$, thought of as an n-dimensional subspace of $\mathbb{R}^{n+1}$, namely, $\mathcal{H}_q = \{(v_1, \ldots, v_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^n \alpha_i(q)v_i - v_{n+1} = 0\}$.

A *horizontal curve* γ in $\mathbb{R}^{n+1}$ is an absolute continuous curve in $\mathbb{R}^{n+1}$ whose tangent vector $\dot{\gamma}(t)$ wherever it exists belongs to $\mathcal{H}_{\gamma(t)}$. Write $\gamma = (\gamma_1, \ldots, \gamma_{n+1})$ in coordinates and note that $\mathcal{H} = \ker\alpha$, we have that γ is horizontal if and only if $\dot{\gamma}_{n+1} = \sum_{i=1}^n \alpha_i\dot{\gamma}_i$, $a.e.$ For a curve c in $\mathbb{R}^n$ starting from m, let $q = (m, h) \in \pi^{-1}(m)$ be arbitrary. The *horizontal lift* of c based at q is the unique horizontal curve γ in $\mathbb{R}^{n+1}$ starting from q and satisfying $\pi(\gamma) = c$ at all time. If in particular c is a loop in $\mathbb{R}^n$ based at m, then its horizontal lift γ must start and end in the same fiber $\pi^{-1}(m)$, i.e., the end point of γ has the same $x_1, \ldots, x_n$ coordinates as m. The difference in their x_{n+1} coordinates is called the *holonomy* of the loop c (based at q), which in general depends on the choice of $q \in \pi^{-1}(m)$.

B. Compatible Distribution

The distribution $\mathcal{H}$ is called *compatible* (with the bundle structure $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$) if it is invariant under the action of the structure group $\mathbb{R}$, namely, translations along the x_{n+1}-axis. In other words, $\mathcal{H}$ is compatible if and only if the horizontal spaces $\mathcal{H}_q$, thought of as n-dimensional subspaces in $\mathbb{R}^{n+1}$, are the same for q in the same fiber $\pi^{-1}(m)$. In terms of equation (8), this is equivalent to

$$\text{the functions } \alpha_1, \ldots, \alpha_{n+1} \text{ are independent on } x_{n+1}. \quad (9)$$

Because of (9), we can defined a one-form on $\mathbb{R}^n$ as

$$\Theta = \sum_{i=1}^n \alpha_i dx_i, \quad (10)$$

which is called the *connection form* of $\mathcal{H}$. The *curvature form* of $\mathcal{H}$ is the two-form on $\mathbb{R}^n$ defined as

$$\Omega = d\Theta. \quad (11)$$

An important implication of compatible distributions is that the holonomy of a loop c in $\mathbb{R}^n$ based at m is independent of the starting point $q \in \pi^{-1}(m)$ of its horizontal lift, thus can be simply denoted by $h(c)$.

An alternative interpretation of $h(c)$ is the following. Let $c : I \to \mathbb{R}^n$ be a loop based at m, and γ be its horizontal lift based at an arbitrary point in $\pi^{-1}(m)$. Then $h(c) = \gamma_{n+1}(1) - \gamma_{n+1}(0) = \int_0^1 \dot{\gamma}_{n+1}\, dt = \int_0^1 \sum_{i=1}^n \alpha_i\dot{\gamma}_i\, dt = \int_c \Theta$. Now we find a two-dimensional submanifold S immersed in $\mathbb{R}^n$ whose boundary ∂S is exactly c under the canonical orientation of $\mathbb{R}^n$. Then by the Stokes equations,

$$h(c) = \int_{\partial S} \Theta = \int_S d\Theta = \int_S \Omega. \quad (12)$$

Equation (12) expresses the holonomy $h(c)$ as an integral of the curvature form Ω over an arbitrary surface encircled by c. This relation will be useful later.

C. Sub-Riemannian Metric

A sub-riemannian metric on $\mathcal{H}$ is an assignment of inner products to horizontal spaces $\mathcal{H}_q$ that varies smoothly with $q \in \mathbb{R}^{n+1}$. One often denotes these inner products by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and their corresponding norms by $\| \cdot \|_{\mathcal{H}}$. The existence of a sub-riemannian metric enables one to measure the length of horizontal curves: for a horizontal curve γ in $\mathbb{R}^{n+1}$, its length is given by $L(\gamma) = \int_0^1 \|\dot{\gamma}\|\, dt$. It should be emphasized here that the length of a non-horizontal curve is in general not defined. The sub-riemannian distance between two arbitrary points q_1 and q_2 in $\mathbb{R}^{n+1}$ is the infimum of the length of all horizontal curves connecting them. With this distance, the distribution $\mathcal{H}$ and the sub-riemannian metric $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ together specify a *sub-riemannian geometry* on $\mathbb{R}^{n+1}$.

D. Compatible Metric

For a compatible distribution $\mathcal{H}$, a sub-riemannian metric is called *compatible* (with the bundle structure $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$) if it is invariant under the action of the structure group $\mathbb{R}$. Hence translations along the x_{n+1}-axis are isometries for the sub-riemannian geometry.

Compatible sub-riemannian metrics on $\mathcal{H}$ corresponds in a one-to-one way to riemannian metrics on the base space $\mathbb{R}^n$. To see this, we first define the horizontal lift operator. For each pair of $m \in \mathbb{R}^n$ and $q \in \pi^{-1}(m)$, the horizontal lift $h_q : T_m\mathbb{R}^n \to \mathcal{H}_q$ is a linear map that maps $v \in T_m\mathbb{R}^n$ to the unique $u \in \mathcal{H}_q$ satisfying $d\pi_q(u) = v$. Thus h_q is the inverse of the map $d\pi_q$ restricted on $\mathcal{H}_q$, which is an isomorphism by our choice of α in (8). Now starting from a compatible sub-riemannian metric $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, there is a unique riemannian metric $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ on $\mathbb{R}^n$ that makes all horizontal lifts isometries. In fact, $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ is defined by

$$\langle u, v \rangle_{\mathbb{R}^n} = \langle h_q(u), h_q(v) \rangle_{\mathcal{H}}, \quad \forall u, v \in T_m\mathbb{R}^n, \quad (13)$$

which is independent of the choice of $q \in \pi^{-1}(m)$ since $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is compatible. Conversely, a riemannian metric $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ on $\mathbb{R}^n$ induces a compatible sub-riemannian metric on $\mathcal{H}$ as

$$\langle u, v \rangle_{\mathcal{H}} = \langle d\pi_q(u), d\pi_q(v) \rangle_{\mathbb{R}^n}, \quad \forall u, v \in \mathcal{H}_q. \quad (14)$$

III. Problem Formulation

A. Asymptotic Holonomy

Suppose that $\mathcal{H} = \ker\alpha$ is a co-dimension one distribution on $\mathbb{R}^{n+1}$, where α is given in (8), and $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is a sub-riemannian metric on $\mathcal{H}$. Fix a pair (m, q), where $m \in \mathbb{R}^n$, $q \in \mathbb{R}^{n+1}$, and $q \in \pi^{-1}(m)$. Let c be a non-trivial loop in $\mathbb{R}^n$ based at m, and let γ be its horizontal lift in $\mathbb{R}^{n+1}$ based at q. Denote by $h(c)$ the holonomy of c based at q and, with some abuse of notation, by $L(c)$ the length of γ (the length of c is in general not defined). Note that $L(c) > 0$ since c is non-trivial. For each $\epsilon > 0$, denote by ϵc the loop in $\mathbb{R}^n$ obtained by scaling c by a factor of ϵ towards m (a more accurate notation should be $m + \epsilon(c - m)$). Thus $h(\epsilon c)$ and $L(\epsilon c)$ are similarly defined. It is easy to show that as $\epsilon \to 0$,

$L(\epsilon c)$ is of the order of ϵ, while $|h(\epsilon c)|$ is of the order of ϵ^r for some integer $r \geq 1$. We call this integer the *rank* of the loop c, and denote it by $r_q(c)$. The *(asymptotic) efficiency* of c is defined as

$$\eta_q(c) \triangleq \lim_{\epsilon \to 0} \frac{|h(\epsilon c)|}{L^r(\epsilon c)}. \tag{15}$$

The *rank* and the *(asymptotic) efficiency* at q are defined as

$$\begin{aligned} r(q) &\triangleq \min_c \; r_q(c), \\ \eta(q) &\triangleq \sup\{\eta_q(c) : c \text{ such that } r_q(c) = r(q)\}. \end{aligned} \tag{16}$$

Remark 1 *Both $r_q(c)$ and $\eta_q(c)$ depend on $q \in \pi^{-1}(m)$ since $h(c)$ and $L(c)$ in general vary with q. However, if the distribution $\mathcal{H}$ and the sub-riemannian metric $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ are both compatible, then $h(c)$ and $L(c)$ are the same as q varies in the fiber $\pi^{-1}(m)$. Therefore, in this case one can write $r_m(c)$, $\eta_m(c)$, $r(m)$, and $\eta(m)$ instead. We shall simply call $r(m)$ and $\eta(m)$ the rank and the efficiency at m, respectively.*

Example 1 (Heisenberg Geometry) *Consider the following compatible distribution $\mathcal{H}$ and sub-riemannian metric $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ on $\mathbb{R}^3$. Suppose that $\mathcal{H}$ is given by the vanishing of $\alpha = \frac{1}{2}(x_1 dx_2 - x_2 dx_1) - dx_3$, and that $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ is induced by the canonical riemannian metric on $\mathbb{R}^2$. Then the holonomy $h(c)$ of a loop c in $\mathbb{R}^2$ is the signed area it encloses, and $L(c)$ is simply its length in $\mathbb{R}^2$. Thus by the well-known isoperimetric theorem, the rank at any $m \in \mathbb{R}^2$ is two, and the efficiency $\eta(m) = 1/4\pi$, both realized when c is a circle of arbitrary radius passing through m.*

The following lemmas are direct consequence of the above definitions. We omit their proofs here.

Lemma 1 *For a loop c in $\mathbb{R}^n$ based at m, and $q \in \pi^{-1}(m)$,*

- ***(invariance to scaling)*** *$r_q(c) = r_q(\lambda c)$ and $\eta_q(c) = \eta_q(\lambda c)$ for any $\lambda > 0$;*
- ***(invariance to reparameterization)*** *$r_q(c \circ \rho) = r_q(c)$ and $\eta_q(c \circ \rho) = \eta_q(c)$ for any diffeomorphism $\rho : I \to I$.*

Obviously, the rank at $q \in \mathbb{R}^{n+1}$ depends only on the horizontal distribution $\mathcal{H}$ near q, and is independent of the sub-riemannian metric $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. For compatible $\mathcal{H}$, we have

Lemma 2 *Suppose $\mathcal{H}$ is compatible with connection form Θ and curvature form Ω, respectively. Write Ω in coordinates as $\Omega = \sum_{1\leq i,j\leq n} \Omega_{ij}\, dx_i \wedge dx_j$, where $\Omega_{ij} = -\Omega_{ji}$. Then at any $m \in \mathbb{R}^n$, $r(m) - 2$ is equal to the smallest integer $k \geq 0$ for which there exist some i_0, j_0 such that at least one k-th order partial derivative of $\Omega_{i_0 j_0}$ at m is nonzero.*

On the other hand, although $\eta(q)$ does depend on $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, due to the smoothly varying nature of $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ near q, we have

Lemma 3 *$\eta(q)$ depends on the sub-riemannian metric $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ only through its restriction on $\mathcal{H}_q$.*

By Lemma 3, to the effect of studying $\eta(q)$, we can modify $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ on other horizontal spaces so that for any $q' \in \mathbb{R}^{n+1}$,

$$d\pi_{q'} : \mathcal{H}_{q'} \to T_{\pi(q')}\mathbb{R}^n \text{ is an isometry,} \tag{17}$$

where the metrics on all $T_{\pi(q')}\mathbb{R}^n$ are given by the same positive definite n-by-n matrix A_0. For compatible $\mathcal{H}$, $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ thus chosen is also compatible. By a transformation of coordinates within $\mathbb{R}^n$, in the rest of the paper *we shall assume that* (17) *holds with $A_0 = I$*. Hence $L(c)$ is simply the arc length of c as a curve in $\mathbb{R}^n$ with the canonical metric.

B. Equations of Efficiency-Achieving Loops

To find the efficiency $\eta(q)$ at $q \in \mathbb{R}^{n+1}$ and the (family of) loops c based at $m = \pi(q)$ for which $\eta_q(c) = \eta(q)$, one needs to solve the following variational problem.

Problem 1 *Let c be an arbitrary loop in $\mathbb{R}^n$ based at m, and let $h(c)$ be its holonomy. Then solving for $\eta(q)$ is equivalent to one of the following problems:*

1) *minimize $L(c) = \int_0^1 \|\dot{c}\|\, dt$, subject to $h(c) = 1$;*
2) *maximize $h(c)$, subject to $L(c) = 1$;*
3) *minimize $E(c) = \int_0^1 \|\dot{c}\|^2\, dt$, subject to $h(c) = 1$;*
4) *maximize $h(c)$ subject to $E(c) = 1$.*

Formulation 3 is adopted here since it avoids the ambiguity of reparameterizations (solutions in formulations 3 are solutions in formulation 1 parameterized with unit speed). Using the Lagrangian multiplier approach [8], solutions are given by:

$$\ddot{c} = -\lambda\, i_{\dot{c}}\Omega. \tag{18}$$

Here, $i_{\dot{c}}\Omega = \Omega(\dot{c},\cdot)$ is a one-form on $\mathbb{R}^n$ that we identify as a vector via the canonical metric on $\mathbb{R}^n$. We are interested only in those solutions to (18) that start and end in the origin. The constraint that $h(c) = 1$ can be ignored if one is only interested in the shape of such solutions. Equation (18) describes the motions of a particle of unit mass and unit charge moving in a magnetic field on $\mathbb{R}^n$ given by Ω in the case of $n = 2, 3$, and in general admits no analytic solution.

IV. Rank Two Case

Let $\mathcal{H}$ be a compatible co-dimension one distribution on $\mathbb{R}^n$ with connection form Θ and curvature form Ω, and let $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ be a riemannian metric on $\mathcal{H}$ obtained by lifting the canonical metric on $\mathbb{R}^n$. In this section, we try to determine $\eta(m)$ at a point $m \in \mathbb{R}^n$ with rank $r(m) = 2$. Without loss we assume that $m = 0 \in \mathbb{R}^n$ and $q = 0 \in \mathbb{R}^{n+1}$.

By Lemma 2, if $\Omega = \sum_{1\leq i,j\leq n} \Omega_{ij}\, dx_i \wedge dx_j$ in coordinates, then $\Omega_{ij}(0) \neq 0$ for at least some i, j. Define $\Omega^0 \triangleq \sum_{1\leq i,j\leq n} \Omega_{ij}(0)\, dx_i \wedge dx_j$. The for any two dimensional submanifold S in $\mathbb{R}^n$ encircled by a loop c based at 0, we have $h(\epsilon c) = \int_{\epsilon S} \Omega \sim \int_{\epsilon S} \Omega^0 = \epsilon^2 \int_S \Omega^0$, as $\epsilon \to 0$. Hence in determining the efficiency at 0, we may as well assume that $\mathcal{H}$ has the curvature form Ω^0. Under this assumption, $\eta_0(c) = \lim_{\epsilon\to 0} \frac{|h(\epsilon c)|}{L^2(\epsilon c)} = \lim_{\epsilon\to 0} \frac{\epsilon^2 \left| \int_S \Omega^0 \right|}{\epsilon^2 L^2(c)} = \frac{\left| \int_S \Omega^0 \right|}{L^2(c)}$, so

$$\eta(0) = \sup_c \eta_0(c) = \sup_c \frac{\left| \int_S \Omega^0 \right|}{L^2(c)} = \sup_c \frac{\left| \int_S \Omega^0 \right|}{E(c)}. \quad (19)$$

Define an n-by-n skew-symmetric matrix

$$Z = [\Omega_{ij}(0)]_{1\le i,j\le n}. \quad (20)$$

Z can be transformed into the following standard form:

$$Z = Q \cdot \mathrm{diag}\left(\begin{bmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & -\sigma_l \\ \sigma_l & 0 \end{bmatrix}, 0, \ldots, 0 \right) \cdot Q^t,$$

where $Q \in \mathbb{R}^{n\times n}$ is orthonormal, $l > 0$ is an integer with $2l \le n$, and $\sigma_1 \ge \cdots \ge \sigma_l > 0$. In fact, $\sigma_1, \ldots, \sigma_l$ each repeated twice are the nonzero singular values of Z.

Now perform the following coordinate transformation:

$$(y_1, \ldots, y_n) = (x_1, \ldots, x_n)Q. \quad (21)$$

Then $\Omega^0 = 2(\sigma_1 dy_1 \wedge dy_2 + \cdots + \sigma_l dy_{2l-1} \wedge dy_{2l})$. So for S in $\mathbb{R}^n$ encircled by a loop c based at 0 with coordinates $y_1, \ldots, y_n$, we have $\int_S \Omega^0 = 2\sum_{i=1}^l \sigma_i \int_S dy_{2i-1} \wedge dy_{2i}$. Note that $\int_S dy_{2i-1} \wedge dy_{2i}$ is the area of the projection of S onto the plane Π_i spanned by the y_{2i-1} and y_{2i} axes. By the classical isoperimetric theorem [4], $|\int_S dy_{2i-1} \wedge dy_{2i}| \le \frac{1}{4\pi}\int_0^1 (\dot{y}_{2i-1}^2 + \dot{y}_{2i}^2)\, dt$, with equality if and only if (y_{2i-1}, y_{2i}) for $t \in I$ draws a circle of arbitrary radius through the origin in Π_i. Summing up, we have $|\int_S \Omega^0| \le 2\sum_{i=1}^l \sigma_i |\int_S dy_{2i-1} \wedge dy_{2i}| \le 2\sigma_1 \sum_{i=1}^l \frac{1}{4\pi}\int_0^1 (\dot{y}_{2i-1}^2 + \dot{y}_{2i}^2)\, dt \le \frac{\sigma_1}{2\pi} E(c)$, where the equality holds if and only if for all $i = 1, \ldots, l$ such that $\sigma_i = \max_{1\le i\le l} \sigma_i$, the projection of c onto Π_i is a circle of arbitrary radius through 0, and the projection of c to all other axes are all zero. Therefore,

Theorem 1 *Suppose that $\mathcal{H}$ is a compatible distribution on $\mathbb{R}^n$ with curvature form Ω, and $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ is a sub-riemannian metric on $\mathcal{H}$ such that $d\pi$ is an isometry from $\mathcal{H}_0$ to $T_0\mathbb{R}^n$ with the canonical metric. Let $\sigma_1 \ge \cdots \ge \sigma_l$ each repeated twice be the nonzero singular values of the matrix Z defined in* (20). *Then the efficiency at* 0 *is*

$$\eta(0) = \frac{\sigma_1}{2\pi}. \quad (22)$$

Moreover, let Π_i be the subspace spanned by the y_{2i-1} and y_{2i} axes under the coordinate transformation (21). *Then for a loop c in $\mathbb{R}^n$ based at* 0 *to have efficiency $\eta(0)$, its projection onto each plane Π_i with $\sigma_i = \sigma_1$ must be a circle, and its projections on all other axes are zero.*

V. Rank Three Case

In this section, we study the efficiency at point $m = 0 \in \mathbb{R}^n$ for a compatible horizontal distribution $\mathcal{H}$ on $\mathbb{R}^{n+1}$ with rank $r(0) = 3$. So the curvature form Ω satisfies $\Omega^0 = 0$ and $\Omega^1 \triangleq \sum_{1\le i,j,k\le n} \frac{\partial \Omega_{ij}(0)}{\partial x_k} x_k\, dx_i \wedge dx_j \ne 0$. By our previous discussions, we can replace Ω by Ω^1.

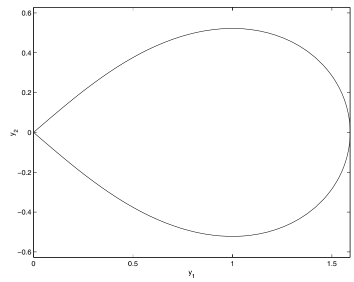

Fig. 2. A solution curve to problem (23).

We study the simplest case $n = 2$ only. So $\Omega^1 = 2(\frac{\partial\Omega_{12}(0)}{\partial x_1} x_1 + \frac{\partial\Omega_{12}(0)}{\partial x_2} x_2) dx_1 \wedge dx_2$. Define $\kappa = \sqrt{[\frac{\partial\Omega_{12}(0)}{\partial x_1}]^2 + [\frac{\partial\Omega_{12}(0)}{\partial x_2}]^2}$, which is nonzero by assumption. Perform the coordinate transformation $(y_1, y_2) = (x_1, x_2)Q$, where Q is the orthonormal matrix defined by

$$Q = \frac{1}{\kappa}\begin{bmatrix} \frac{\partial\Omega_{12}(0)}{\partial x_1} & \frac{\partial\Omega_{12}(0)}{\partial x_2} \\ -\frac{\partial\Omega_{12}(0)}{\partial x_2} & \frac{\partial\Omega_{12}(0)}{\partial x_1} \end{bmatrix}.$$

Then Ω^1 is now $\Omega^1 = 2\kappa y_1\, dy_1 \wedge dy_2$. For a nontrivial loop c in $\mathbb{R}^2$ based at 0 enclosing the (oriented) area S, we have

$$h(\epsilon c) \sim \int_{\epsilon S} \Omega^1 = \epsilon^3 \int_S \Omega^1 = \epsilon^3 \int_c \kappa y_1^2 dy_2.$$

Hence $\eta_0(c) = \lim_{\epsilon\to 0} \frac{h(\epsilon c)}{L^3(\epsilon c)} = \frac{\kappa \int_c y_1^2 dy_2}{L^3(c)}$, and

$$\eta(0) = \sup_c \eta_0(c) = \sup_c \frac{\kappa \int_c y_1^2 dy_2}{L^3(c)} = \kappa \cdot \sup_c \frac{\int_c y_1^2 dy_2}{E^{3/2}(c)}.$$

To determine the c for which $\eta(0)$ is achieved, one needs to solve the following variational problem:

$$\text{find } c \text{ that minimizes } E(c) \text{ such that } \int_c y_1^2 dy_2 = 1. \quad (23)$$

Problem (23) can be solved using the Maximal Principle. The details are omitted here, and can be found in an upcoming paper. Figure 2 plots one solution curve (up to a scaling) to problem (23), which should be parameterized so that it proceeds counterclockwise with constant speed.

Remark 2 *The efficiency-achieving loop calculated in this section is part of the trajectory of a charged particle moving in a magnetic field $\mathbf{B}$ with linear components on $\mathbb{R}^2$, referred to as a* grad **B** drift *since it exhibits an overall drift orthogonal to the direction grad $|\mathbf{B}|$ ([2]).*

VI. Back to the Snake

Suppose that in the snake example we have $n = 2$. So the snake consists of three rigid segments, whose orientations are given by the angles θ_i, $i = 1, 2, 3$. The configuration space is $\mathbb{T}^3$, with a Riemannian metric given by (5), where $\Delta = (\Delta_{ij})_{1\le i,j\le 3} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. By (7), the co-dimension one distribution $\mathcal{H}$ is given by the vanishing of the one-form $\alpha = \sum_{i,j=1}^{3} \Delta_{ij} \cos(\theta_i - \theta_j) d\theta_j$.

Suppose now that the snake is at the initial configuration q corresponding to $\theta_1 = \theta_2 = \theta_3 = 0$, i.e. the three segments of the snake are all aligned in the positive horizontal direction. To compute the rank at q, we perform the following coordinate transformation in a neighborhood of q:

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{5}}{3} & \frac{2\sqrt{5}}{3} & -\frac{\sqrt{5}}{3} \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}, \tag{24}$$

The choice of such a transformation serves several purposes. First, $\phi_3 = \theta_1 + \theta_2 + \theta_3$ is the direction along the fibers of $\mathbb{T}^3$ under the action of $\mathbb{T}$ as described in Section I. Second, the plane Π spanned by the ϕ_1 and ϕ_2 axes is transversal to the ϕ_3 axis, hence can be regarded as the base space for the principal bundle $\mathbb{T}^3$, at least locally around the origin. Third, the projection $d\pi : \mathcal{H}_q \to T_{(0,0)}\Pi$ is an isometry if Π is equipped with the canonical Euclidean metric. Thus condition (17) is satisfied with $A_0 = I$.

In the new coordinates, q corresponds to the origin $\phi_1 = \phi_2 = \phi_3 = 0$, and the condition that $\alpha = 0$ is equivalent to the vanishing of a suitable connection form Θ defined on $\Pi \simeq \mathbb{R}^2$. After a careful calculation, the curvature form Ω is $\Omega = d\Theta = f(\phi_1, \phi_2) d\phi_1 \wedge d\phi_2$, where $f(\phi_1, \phi_2)$ is given by

$$\begin{aligned} f(\phi_1, \phi_2) &= \frac{3\sin(\frac{3\sqrt{5}}{10}\phi_1)}{5 f_1(\phi_1, \phi_2)} \left\{ 4\cos(\frac{1}{2}\phi_2) + \frac{f_2(\phi_1, \phi_2)}{f_1(\phi_1, \phi_2)} \right\}, \\ f_1(\phi_1, \phi_2) &= 5 + 4\cos(\frac{3\sqrt{5}}{10}\phi_1)\cos(\frac{1}{2}\phi_2) + \cos\phi_2, \\ f_2(\phi_1, \phi_2) &= \cos(\frac{3\sqrt{5}}{10}\phi_1)[10\sin^2(\frac{1}{2}\phi_2) - 6\cos^2(\frac{1}{2}\phi_2)] \\ &\quad + 4\sin^2(\frac{1}{2}\phi_2)\cos(\frac{1}{2}\phi_2). \end{aligned}$$

One can verify that $f(0,0) = 0$, $\frac{\partial f}{\partial \phi_1}(0,0) \neq 0$, $\frac{\partial f}{\partial \phi_2}(0,0) = 0$. As a result of Proposition 2, the rank at q is three. To compute the efficiency at q, we can replace Ω by its first order approximate $\Omega^1 = \frac{51}{250} d\phi_1 \wedge d\phi_2$. In particular, maximum efficiency at q is achieved by the horizontal lifting of a curve c in the (ϕ_1, ϕ_2) plane plotted in Figure 2. Horizontally lifting c to a curve γ in (ϕ_1, ϕ_2, ϕ_3) coordinates, transforming γ back to the $(\theta_1, \theta_2, \theta_3)$ coordinates using (24), and finally using equation (3), we obtain the motions for the snake to turn with the least energy asymptotically starting from the aligned initial position. Figure 3 shows the snapshots of the motions at different time instances obtained numerically.

Remark 3 *By equation* (VI)*, in a neighborhood of the origin in the* (ϕ_1, ϕ_2) *coordinates, the zero-th order approximate of the curvature form* Ω *vanishes if and only if* $\phi_1 = 0$*, i.e., if and only if* $\theta_1 + \theta_3 - 2\theta_2 = 0$*. So the rank is three at points satisfying this condition and two otherwise. As a result, for snake starting from a shape close to the aligned one, it is more difficult for the snake to turn if its initial shape satisfies the condition* $\theta_1 + \theta_3 - 2\theta_2 = 0$.

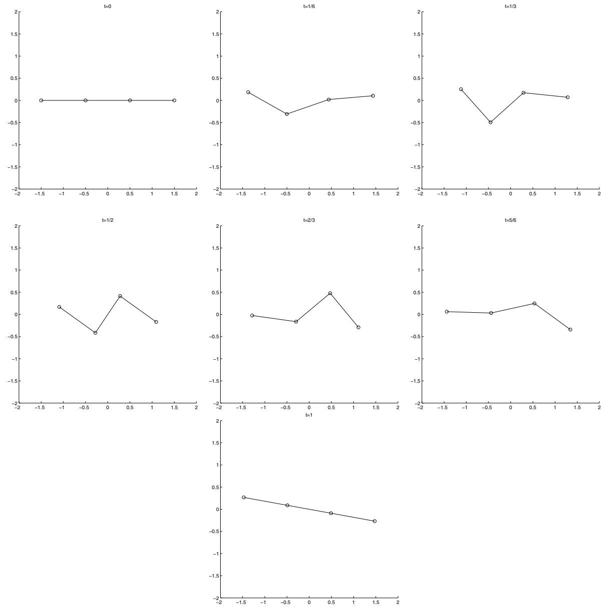

Fig. 3. Snap shots of snake turning.

VII. REFERENCES

[1] A. A. Agrachev and J.-P. Gauthier. On the dido problem and plane isoperimetric problems. *Acta Appl. Math.*, 57:287–338, 1999.

[2] H. Alfven. *Cosmical Electrodynamics*. Oxford University Press, 1950.

[3] R. W. Brockett. Control theory and singular riemannian geometry. In P. J. Hilton and G. S Young, editors, *New Directions in Applied Mathematics*, pages 11–27. Springer-Verlag, 1982.

[4] H. Howards, M. Hutchings, and F. Morgan. The isoperimetric problem on surfaces. *Amer. Math. Monthly*, 106(5):430–439, 1999.

[5] J. Hu, M. Prandini, and S. Sastry. Optimal collision avoidance and formation switching on Riemannian manifolds. *submitted to SIAM J. on Control and Optimization*, 2003.

[6] T. R. Kane and M. P. Scher. A dynamical explanation of the falling cat phenomenon. *International J. Solids and Structures*, (5):663–670, 1969.

[7] R. Montgomery. The isoholonomic problem and some applications. *Comm. Math. Phys.*, (128):565–592, 1990.

[8] R. Montgomery. *A Tour of Subriemannian Geometries, Their Geodesics and Applications*. Amer. Math. Society, Providence, RI, 2002.

[9] A. Shapere and F. Wilczek. Self-propulsion at low reynolds number. *Phys. Rev. Lett.*, (58):2051–2054, 1987.